SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of Kee-Wai Lau, Hong Kong, China from the list of solvers of 3073.

3085. [2005 : 459, 462] *Proposed by Neven Jurič, Zagreb, Croatia.*

A *magic square of order* n is an $n \times n$ array containing the integers from 1 to n^2 such that the sum of the elements in each row, in each column, and on each of the two diagonals is the same.

Let $\mathfrak{M}$ be a magic square of odd order $n \geq 3$. Increase the values of all the entries in $\mathfrak{M}$ by $2n+2$ to get a new $n \times n$ array, say M_1. Place M_1 in the interior of an $(n+2) \times (n+2)$ array M'. Show that the border rows and columns of this can be filled in with the unused integers between 1 and $(n+2)^2$ to create a new magic square $\mathfrak{M}'$ of order $n+2$.

Solution by Michel Bataille, Rouen, France.

We will describe a process given in René Descombes, *Les carrés magiques*, Vuibert, 2000.

If $\mathfrak{M}$ is a magic square, we denote by $S(\mathfrak{M})$ the sum of the elements in each row, column, or diagonal. For a magic square of order n, we have

$$nS(\mathfrak{M}) \;=\; 1+2+\cdots+n^2 \;=\; \tfrac{1}{2}n^2(n^2+1)\,;$$

hence, $S(\mathfrak{M}) = \frac{1}{2}n(n^2+1)$. For the desired magic square $\mathfrak{M}'$ of order $n+2$, we must have $S(\mathfrak{M}') = \frac{1}{2}(n+2)(n^2+4h+5)$. In M_1, the sum of the elements in each row, column, or diagonal, is

$$S(\mathfrak{M}) + n(2n+2) \;=\; \tfrac{1}{2}n(n^2+4n+5)\,.$$

Thus, in each row, column, or diagonal of M', we must add two elements whose sum is

$$k \;=\; \tfrac{1}{2}(n+2)(n^2+4n+5) - \tfrac{1}{2}n(n^2+4n+5) \;=\; n^2+4n+5 \;=\; (n+2)^2+1\,.$$

These two elements are to be taken from the missing numbers

$$1,\ 2,\ \ldots,\ 2n+2,\ k-1,\ k-2,\ \ldots,\ k-(2n+2).$$

Clearly, it is sufficient to indicate the locations of $1, 2, \ldots, 2n+2$ in the border rows and columns.

Now, set $n = 2p - 1$ and label the blank squares of M' as follows:

$$\begin{array}{cccc}
(1,1) & (1,2) & \cdots & (1,2p+1) \\
(2,1) & & & (2,2p+1) \\
\vdots & & & \vdots \\
(2p,1) & & & (2p,2p+1) \\
(2p+1,1) & (2p+1,2) & \cdots & (2p+1,2p+1)
\end{array}$$

In these blank squares, we distribute the numbers $1, 2, \ldots, 2n+2$ as follows:

$$\begin{array}{rcl}
4p & \longrightarrow & (1,p) \\
4p-2 & \longrightarrow & (1,p-1) \\
& \vdots & \\
2p+2 & \longrightarrow & (1,1) \\
2p & \longrightarrow & (1,2p+1) \\
2p-2 & \longrightarrow & (2p+1,p+2) \\
2p-4 & \longrightarrow & (2p+1,p+3) \\
& \vdots & \\
2 & \longrightarrow & (2p+1,2p)
\end{array}
\qquad
\begin{array}{rcl}
4p-1 & \longrightarrow & (p,1) \\
4p-3 & \longrightarrow & (p-1,1) \\
& \vdots & \\
2p+3 & \longrightarrow & (2,1) \\
2p+1 & \longrightarrow & (p+1,1) \\
2p-1 & \longrightarrow & (2p+1,p+1) \\
2p-3 & \longrightarrow & (p+2,2p+1) \\
2p-5 & \longrightarrow & (p+3,2p+1) \\
& \vdots & \\
1 & \longrightarrow & (2p,2p+1)
\end{array}$$

In this way, each row, column, and diagonal of M_1 receives exactly one extra number, and we can complete the new magic square.

For example, starting with the magic square of order 3:

4	9	2
3	5	7
8	1	6

we first obtain

6	8			4
7	12	17	10	
5	11	13	15	
	16	9	14	1
		3	2	

and finally the magic square of order 5:

6	8	23	24	4
7	12	17	10	19
5	11	13	15	21
25	16	9	14	1
22	18	3	2	20

With the same method, the latter yields the following magic square of order 7:

8	10	12	45	46	48	6
9	18	20	35	36	16	41
11	19	24	29	22	31	39
7	17	23	25	27	33	43
47	37	28	21	26	13	3
49	34	30	15	14	32	1
44	40	38	5	4	2	42

Also solved by the proposer.

3087. [2005 : 459, 462] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle with sides a, b, c opposite the angles A, B, C, respectively. If R is the circumradius and r the inradius of $\triangle ABC$, prove that:

(a) $\dfrac{3R}{r} \geq \dfrac{a+c}{b} + \dfrac{b+a}{c} + \dfrac{c+b}{a} \geq 6$;

(b) $\left(\dfrac{R}{r}\right)^3 \geq \left(\dfrac{a}{b}+\dfrac{b}{a}\right)\left(\dfrac{b}{c}+\dfrac{c}{b}\right)\left(\dfrac{a}{c}+\dfrac{c}{a}\right) \geq 8$.

(Both (a) and (b) are refinements of Euler's Inequality, $R \geq 2r$.)

Similar solutions by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; and Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

(a) The right inequality follows by adding the easy to prove inequalities $\frac{a}{b}+\frac{b}{a} \geq 2$, $\frac{b}{c}+\frac{c}{b} \geq 2$, and $\frac{c}{a}+\frac{a}{c} \geq 2$. The left inequality follows in the same way from the known inequality $\frac{R}{r} \geq \frac{b}{c}+\frac{c}{b}$ (item 5.30 in [1]) and the cyclic versions of it, namely $\frac{R}{r} \geq \frac{c}{a}+\frac{a}{c}$ and $\frac{R}{r} \geq \frac{a}{b}+\frac{b}{a}$.

(b) The same argument works for the second part of the problem upon substituting "adding" by "multiplying".

References

[1] D.S. Mitrinović et al., *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (second solution); MICHEL BATAILLE, Rouen, France; SCOTT BROWN, Auburn University, Montgomery, AL, USA; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RONGZHENG JIAO, Yangzhou University, Yangzhou, China; KEE-WAI LAU, Hong Kong, China; VEDULA N. MURTY, Dover, PA, USA; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; TITU ZVONARU, Comănești, Romania; and the proposer.

3088. [2005 : 543, 546] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle and P a point in the plane of this triangle, not lying on any of the three lines determined by its sides. Let AD, BE, and CF be the Cevians through the point P. The lines through A parallel to BE and CF meet the line BC at L and L', respectively. Points M, M', N, and N' are similarly defined. Prove that L, L', M, M', N, N' all lie on a conic.

Solution by John G. Heuver, Grande Prairie, AB.

We will apply the converse of Carnot's Theorem to the triangle ABC: if, on the lines BC, CA, AB, pairs of points L and L', M and M', N and N', respectively, are taken such that

$$\frac{AN}{BN}\cdot\frac{AN'}{BN'}\cdot\frac{BL}{CL}\cdot\frac{BL'}{CL'}\cdot\frac{CM}{AM}\cdot\frac{CM'}{AM'} \;=\; 1\,,$$

and if no three of the points L, L', M, M', N, N' are collinear, then these six points lie on a non-degenerate conic. [A proof is given below; for further details see Howard Eves, *A Survey of Geometry*, Revised Edition (Allyn and Bacon, 1972), pages 256, 262, and 414.] The given parallelisms imply the following six equalities:

$$\begin{array}{lll} \dfrac{AN}{BN} = \dfrac{DC}{BC}\,, & \dfrac{AN'}{BN'} = \dfrac{AC}{EC}\,, & \dfrac{BL}{CL} = \dfrac{EA}{CA}\,, \\ \dfrac{BL'}{CL'} = \dfrac{BA}{FA}\,, & \dfrac{CM}{AM} = \dfrac{FB}{AB}\,, & \dfrac{CM'}{AM'} = \dfrac{CB}{DB}\,. \end{array}$$

Thus,

$$\begin{aligned} &\frac{AN}{BN}\cdot\frac{AN'}{BN'}\cdot\frac{BL}{CL}\cdot\frac{BL'}{CL'}\cdot\frac{CM}{AM}\cdot\frac{CM'}{AM'} \\ &\qquad = \frac{DC}{BC}\cdot\frac{AC}{EC}\cdot\frac{EA}{CA}\cdot\frac{BA}{FA}\cdot\frac{FB}{AB}\cdot\frac{CB}{DB} \\ &\qquad = \frac{AE}{EC}\cdot\frac{CD}{DB}\cdot\frac{BF}{FA} \;=\; 1\,, \end{aligned}$$

where the last equality holds by Ceva's Theorem. The converse of Carnot's Theorem then gives the desired result. [*Ed:* Should three of the given points be collinear, then the conic would degenerate into a pair of lines.]

Also solved by MICHEL BATAILLE, Rouen, France; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

Bataille's approach was essentially the same as that of our featured solution. As part of his solution he proved the converse of Carnot's Theorem. He used areal coordinates relative to triangle ABC to show that the points $(0,1,\lambda)$, $(0,1,\lambda')$, $(\mu,0,1)$, $(\mu',0,1)$, $(1,\nu,0)$, $(1,\nu',0)$ on the sides of the reference triangle all lie on a conic (or more precisely, their coordinates satisfy a second degree equation), if $\lambda\lambda'\mu\mu'\nu\nu' = 1$. His proof: Let r, s, t, u, v, w be such that $\lambda = r/s$, $\lambda' = s/t$, $\mu = t/u$, $\mu' = u/v$, $\nu = v/w$, $\nu' = w/r$. Then one checks easily that the coordinates of all six points satisfy the equation

$$vx^2 + ry^2 + tz^2 - \frac{s^2+rt}{s}yz - \frac{u^2+tv}{u}zx - \frac{w^2+rv}{w}xy \;=\; 0\,.$$

(Remark. *Carnot's Theorem says that $\lambda\lambda'\mu\mu'\nu\nu' = 1$ is a necessary condition for the six points to lie on the same conic.*)

3089. [2005 : 543, 546] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let ABC be a triangle and P a point in the plane of this triangle, not lying on any of the three lines determined by its sides. Let AD, BE, and CF be the Cevians through the point P. The lines through E and F parallel to AD meet the line BC at L and L', respectively. Points M, M', N, and N' are similarly defined. Prove that L, L', M, M', N, N' all lie on a conic.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Using
$$\frac{BL'}{BD} = \frac{BF}{BA} \quad \text{and} \quad \frac{BD}{BC} = \frac{BN}{BF},$$
we see that
$$\frac{BL'}{BC} = \frac{BL'}{BD}\cdot\frac{BD}{BC} = \frac{BF}{BA}\cdot\frac{BN}{BF} = \frac{BN}{BA}.$$
It follows that $NL' \parallel AC$. Likewise, $LM' \parallel BA$ and $MN' \parallel CB$. Therefore, opposite sides of the hexagon $L'LM'MN'N$ are parallel and thus, its vertices lie on a conic by the converse of Pascal's Theorem.

Also solved by MICHEL BATAILLE, Rouen, France; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was one incorrect submission.

3090. [2005 : 543, 546] *Proposed by Arkady Alt, San Jose, CA, USA.*

Find all non-negative real solutions (x,y,z) to the following system of inequalities:
$$\begin{aligned} 2x(3-4y) &\geq z^2+1\,,\\ 2y(3-4z) &\geq x^2+1\,,\\ 2z(3-4x) &\geq y^2+1\,. \end{aligned}$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Without loss of generality, we may assume that $x \leq \min\{y,z\}$. From the first equation, we have
$$(3x-1)^2+8x(y-x)+(z^2-x^2) = z^2+1-2x(3-4y) \leq 0.$$
Each term of the sum on the left is non-negative; hence, $x=y=z=\frac{1}{3}$.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.

3091. [2005 : 543, 546] *Proposed by Mihály Bencze and Marian Dinca, Brasov, Romania.*

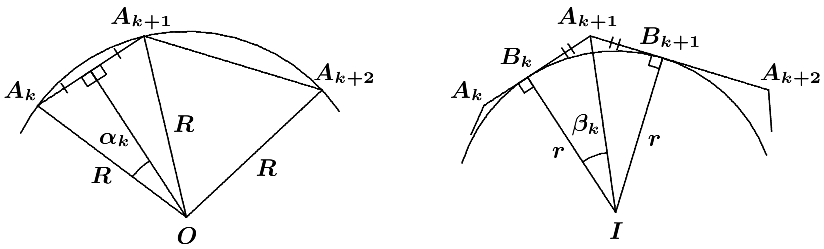
Let $A_1A_2\cdots A_n$ be a convex polygon which has both an inscribed circle and a circumscribed circle. Let B_1, B_2, $\ldots$, B_n denote the points of tangency of the incircle with sides A_1A_2, A_2A_3, $\ldots$, A_nA_1, respectively. Prove that

$$\frac{2sr}{R} \;\le\; \sum_{k=1}^{n} B_kB_{k+1} \;\le\; 2s\cos\left(\frac{\pi}{n}\right)\,,$$

where R is the radius of the circumscribed circle, r is the radius of the inscribed circle, s is the semiperimeter of the polygon $A_1A_2\cdots A_n$, and $B_{n+1} = B_1$.

Solution by Joel Schlosberg, Bayside, NY, USA.

Take all subscripts modulo n. Let O and I be the circumcentre and incentre, respectively, of $A_1A_2\ldots A_n$.



Let $\alpha_k = \frac{1}{2}\angle A_kOA_{k+1}$ and $\beta_k = \frac{1}{2}\angle B_kIB_{k+1}$. Note that $0 < \alpha_k < \pi$ since $0 < \angle A_kOA_{k+1} < 2\pi$. Note also that

$$\sum_{k=1}^{n} \beta_k \;=\; \tfrac{1}{2}\sum_{k=1}^{n} \angle B_kIB_{k+1} \;=\; \pi\,,$$

and since the angles of quadrilateral $B_kIB_{k+1}A_{k+1}$ sum to 2π,

$$\begin{aligned} 2\pi &= \angle B_kIB_{k+1} + \angle B_kA_{k+1}B_{k+1} + \angle IB_kA_{k+1} + \angle IB_{k+1}A_{k+1} \\ &= 2\beta_k + \angle A_kA_{k+1}A_{k+2} + (\pi/2) + (\pi/2)\,, \end{aligned}$$

so that $\beta_k = \frac{1}{2}(\pi - \angle A_kA_{k+1}A_{k+2})$. Clearly, $0 < \beta_k < \pi/2$.

Triangle A_kOA_{k+1} gives $A_kA_{k+1} = 2R\sin\alpha_k$, so that

$$2s \;=\; \sum_{k=1}^{n} A_kA_{k+1} \;=\; 2R\sum_{k=1}^{n}\sin\alpha_k\,. \tag{1}$$

Similarly, triangle B_kIB_{k+1} gives $B_kB_{k+1} = 2r\sin\beta_k$, and therefore,

$$\sum_{k=1}^{n} B_kB_{k+1} \;=\; 2r\sum_{k=1}^{n}\sin\beta_k\,. \tag{2}$$

Finally, triangles B_kIA_{k+1} and $A_{k+1}IB_{k+1}$ give

$$B_kA_{k+1} + A_{k+1}B_{k+1} \;=\; 2r\tan\beta_k\,,$$

so that

$$2s \;=\; \sum_{k=1}^{n} A_kA_{k+1} \;=\; \sum_{k=1}^{n} (B_kA_{k+1} + A_{k+1}B_{k+1}) \;=\; 2r \sum_{k=1}^{n} \tan\beta_k\,. \quad (3)$$

If A_{k+1} and O are on the same side of A_kA_{k+2}, then

$$\begin{aligned}\angle A_kA_{k+1}A_{k+2} &= \tfrac{1}{2}\angle A_kOA_{k+2} \;=\; \tfrac{1}{2}(2\pi - 2\alpha_k - 2\alpha_{k+1}) \\ &= \pi - \alpha_k - \alpha_{k+1}\,.\end{aligned}$$

If A_{k+1} and O are on opposite sides of A_kA_{k+2}, then

$$\begin{aligned}\angle A_kA_{k+1}A_{k+2} &= \pi - \tfrac{1}{2}\angle A_kOA_{k+2} \;=\; \pi - \tfrac{1}{2}(2\alpha_k + 2\alpha_{k+1}) \\ &= \pi - \alpha_k - \alpha_{k+1}\,.\end{aligned}$$

Hence, $\beta_k = \frac{1}{2}(\pi - \angle A_kA_{k+1}A_{k+2}) = \frac{1}{2}(\alpha_k + \alpha_{k+1})$.

Since the function $\sin x$ is concave for $x \in (0, \pi)$, we have

$$\tfrac{1}{2}(\sin\alpha_k + \sin\alpha_{k+1}) \;\le\; \sin\tfrac{1}{2}(\alpha_k + \alpha_{k+1}) \;=\; \sin\beta_k\,.$$

Then

$$\sum_{k=1}^{n} \sin\alpha_k \;=\; \sum_{k=1}^{n} \tfrac{1}{2}(\sin\alpha_k + \sin\alpha_{k+1}) \;\le\; \sum_{k=1}^{n} \sin\beta_k\,.$$

Therefore, using (1) and (2), we see that

$$2s\frac{r}{R} \;=\; \frac{r}{R}2R\sum_{k=1}^{n} \sin\alpha_k \;\le\; 2r\sum_{k=1}^{n} \sin\beta_k \;=\; \sum_{k=1}^{n} B_kB_{k+1}\,,$$

which completes the proof of the left inequality.

Since the function $\sin x$ is concave for $x \in (0, \pi/2)$, we have

$$\sum_{k=1}^{n} \sin\beta_k \;\le\; n\sin\frac{\beta_1 + \cdots + \beta_n}{n} \;=\; n\sin\frac{\pi}{n}\,,$$

and since the function $\tan x$ is convex for $x \in (0, \pi/2)$, we get

$$\sum_{k=1}^{n} \tan\beta_k \;\ge\; n\tan\frac{\beta_1 + \cdots + \beta_n}{n} \;=\; n\tan\frac{\pi}{n}\,,$$

so that

$$\sum_{k=1}^{n} \sin\beta_k \;\le\; n\sin\frac{\pi}{n} \;=\; n\tan\frac{\pi}{n}\cos\frac{\pi}{n} \;\le\; \cos\frac{\pi}{n}\textstyle\sum_{k=1}^{n}\tan\beta_k\,. \quad (4)$$

Applying (2), (3) and (4), we have

$$\sum_{k=1}^{n} B_kB_{k+1} \;=\; 2r\sum_{k=1}^{n} \sin\beta_k \;\le\; 2r\cos\frac{\pi}{n}\sum_{k=1}^{n} \tan\beta_k \;=\; 2s\cos\frac{\pi}{n}\,,$$

which completes the proof of the right inequality.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

3092. [2005 : 544, 546] *Proposed by Vedula N. Murty, Dover, PA, USA.*

(a) Let a, b, and c be positive real numbers such that $a+b+c=abc$. Find the minimum value of $\sqrt{1+a^2}+\sqrt{1+b^2}+\sqrt{1+c^2}$.

[Compare with ***CRUX with MAYHEM*** problem 2814 [2003 : 110; 2004 : 112].]

(b) Let a, b, and c be positive real numbers such that $a+b+c=1$. Find the minimum value of

$$\frac{1}{\sqrt{abc}}+\sum_{\text{cyclic}}\sqrt{\frac{bc}{a}}\,.$$

Solution by Kee-Wai Lau, Hong Kong, China.

(a) By the AM–GM Inequality, we have

$$\sqrt[3]{a+b+c} \;=\; \sqrt[3]{abc} \;\le\; \frac{a+b+c}{3}\,,$$

which implies that $a+b+c \ge 3\sqrt{3}$.

For $x>0$, let $f(x)=\sqrt{1+x^2}$. Then $f'(x)=\dfrac{x}{\sqrt{1+x^2}}>0$ and $f''(x)=\dfrac{1}{(1+x^2)^{3/2}}>0$. Hence,

$$f(a)+f(b)+f(c) \;\ge\; 3f\left(\frac{a+b+c}{3}\right) \;\ge\; 3f\left(\sqrt{3}\right) \;=\; 6\,,$$

and equality holds when $a=b=c=\sqrt{3}$. This shows that the required minimum is 6.

(b) Since $a+b+c=1$, we obtain

$$\begin{aligned}\frac{1}{\sqrt{abc}}+\sum_{\text{cyclic}}\sqrt{\frac{bc}{a}} &= \frac{1+ab+bc+ca}{\sqrt{abc}}\\ &= \frac{1}{9}\left(\frac{3a+1}{\sqrt{a}}\right)\left(\frac{3b+1}{\sqrt{b}}\right)\left(\frac{3c+1}{\sqrt{c}}\right)+\frac{5-27abc}{9\sqrt{abc}}\,.\end{aligned}$$

Since

$$\begin{aligned}abc &= \left(\sqrt[3]{abc}\right)^3 \;\le\; \left(\frac{a+b+c}{3}\right)^3 \;=\; \frac{1}{27}\,,\\ \frac{3a+1}{\sqrt{a}} &= \left(\sqrt{3}\,\sqrt[4]{a}-\frac{1}{\sqrt[4]{a}}\right)^2+2\sqrt{3} \;\ge 2\sqrt{3}\,,\end{aligned}$$

and similarly, $\dfrac{3b+1}{\sqrt{b}}\ge 2\sqrt{3}$ and $\dfrac{3c+1}{\sqrt{c}}\ge 2\sqrt{3}$, we see that

$$\frac{1}{9}\left(\frac{3a+1}{\sqrt{a}}\right)\left(\frac{3b+1}{\sqrt{b}}\right)\left(\frac{3c+1}{\sqrt{c}}\right)+\frac{5-27abc}{9\sqrt{abc}} \;\ge\; 4\sqrt{3}\,,$$

and equality holds for $a=b=c=\frac{1}{3}$. This shows that the required minimum is $4\sqrt{3}$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; JOHN G. HEUVER, Grande Prairie, AB; JOEL SCHLOSBERG, Bayside, NY, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

There was quite a wide variety of methods used by our solvers for this problem. That is what made it a good problem.

3093. [2005 : 544, 547] *Proposed by Mihály Bencze, Brasov, Romania.*

Let p_k be the k^{th} prime. Show that the following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n(p_{n+1}-p_n)} .$$

I. *Solution by Mohammed Aassila, Strasbourg, France, modified slightly by the editor.*

A well-known consequence of the Prime Number Theorem states that $p_n \sim n\ln n$, which implies that $p_{2N} < 3N\ln(2N)$ for sufficiently large N. Thus, $P_{2N} < 3N(\ln 2 + \ln N) < 6N\ln N$.

Using the AM–HM Inequality, we then have

$$\begin{aligned} \sum_{n=N}^{2N-1} \frac{1}{n(p_{n+1}-p_n)} &\geq \frac{1}{2N} \sum_{n=N}^{2N-1} \frac{1}{p_{n+1}-p_n} \\ &\geq \frac{1}{2N} \frac{N^2}{\sum_{n=N}^{2N-1}(p_{n+1}-p_n)} \\ &= \frac{N}{2(p_{n+1}-p_n)} > \frac{N}{2p_{2N}} > \frac{1}{12\ln N} . \end{aligned}$$

Summing over all N of the form 2^k for sufficiently large k, we get

$$\sum_{n=1}^{\infty} \frac{1}{n(p_{n+1}-p_n)} > \sum_k \frac{1}{12\ln(2^k)} = \frac{1}{12(\ln 2)} \sum_k \frac{1}{k} .$$

Since the harmonic series $\sum \frac{1}{k}$ is divergent, the conclusion follows.

II. *Essentially the same solution by Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

It is known (see, for example, Problem 3.2.71 on p. 84 of *Problems in Mathematical Analysis I*, AMS (1996) by W.J. Kaczor and M.T. Nowak) that if $\{a_n\}$ is a monotonically increasing sequence of positive numbers such that

$\sum_{n=1}^{\infty} \frac{1}{a_n}$ is divergent, then $\sum_{n=1}^{\infty} \frac{1}{(n+1)a_{n+1} - na_n}$ also diverges. Since

$$\frac{1}{n(p_{n+1} - p_n)} > \frac{1}{(n+1)p_{n+1} - np_n},$$

and since $\sum_{n=1}^{\infty} \frac{1}{p_n}$ is well known to be divergent, the result follows immediately by letting $a_n = p_n$.

Also solved by OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOM LEONG, Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; and the proposer.

Janous pointed out that the result has been known for a long time and gave the following reference: L. Panaitopol, Sur la suite des differences des nombres premiers consecutives, (Romanian) Gaz. Mat., Bucur.*, Ser. A 79 (1974), 238–242.*

Leong actually gave a proof for the result quoted in solution II above.

3094. [2005 : 544, 547] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers, where $n \geq 3$. Let $S = \sum_{k=1}^{n} x_k$ and $P = \prod_{k=1}^{n} (1 + x_k^2)$. Prove that

(a) $P \leq \max_{1 \leq k \leq n} \left\{ \left(1 + \frac{S^2}{k^2}\right)^k \right\}$;

(b) $P \leq \left(1 + \frac{S^2}{n^2}\right)^n$ if $S > 2\sqrt{2}(n-1)$;

(c) $P \leq 1 + S^2$ if $S \leq 2\sqrt{2}$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

In fact, all three parts are valid for $n \geq 1$, and the bound on S in (b) can be improved. For $n = 1$, the inequalities are all trivial. We assume $n \geq 2$ in our proofs.

(a) First consider the case $n = 2$. If $P \leq 1 + S^2$, then we are done. Otherwise, we have $(1+x_1^2)(1+x_2^2) = P > 1+S^2 = 1+(x_1+x_2)^2$, which gives $x_1x_2 > 2$; then

$$\left(1 + \frac{S^2}{2^2}\right)^2 - P = \frac{(x_1 - x_2)^2(x_1^2 + 6x_1x_2 + x_2^2 - 8)}{16} \geq 0.$$

Hence, the inequality is true for $n = 2$.

Assume, as an induction hypothesis, that the inequality is true for $n-1$ variables, for some $n \geq 3$. Consider the product

$$P(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} (1 + x_k^2).$$

For $1 \le i \le n$, let $x_{i,0} = x_i$ and $x_{i,m} = \frac{1}{2}(x_{i,m-1} + x_{i+1,m-1})$ for $m \ge 1$, where the first subscripts are taken modulo n. Note that $\sum_{i=1}^{n} x_{i,m} = S$ for all $m \ge 1$. If, for some i, we have $(1+x_{i,0}^2)(1+x_{i+1,0}^2) \le 1 + (x_{i,0}+x_{i+1,0})^2$, then the number of variables may be reduced to $n-1$, and we are done. On the other hand, if $(1+x_{i,0}^2)(1+x_{i+1,0}^2) > 1 + (x_{i,0}+x_{i+1,0})^2$ for all i, then, by the proven case $n = 2$, we have

$$(1+x_{i,0}^2)(1+x_{i+1,0}^2) \;\le\; \left[1 + \left(\frac{x_{i,0}+x_{i+1,0}}{2}\right)^2\right]^2$$

for all i, and thus, $P(x_{1,0}, x_{2,0}, \ldots, x_{n,0}) \le P(x_{1,1}, x_{2,1}, \ldots, x_{n,1})$.

Now we iterate. If $(1+x_{i,m}^2)(1+x_{i+1,m}^2) \le 1 + (x_{i,m}+x_{i+1,m})^2$ for some i at some stage m, then we can reduce to the case of $n-1$ variables. Otherwise, we obtain an infinite sequence $\{(x_{1,m}, x_{2,m}, \ldots, x_{n,m})\}_{m=0}^{\infty}$ such that $P(x_{1,m-1}, x_{2,m-1}, \ldots, x_{n,m-1}) \le P(x_{1,m}, x_{2,m}, \ldots, x_{n,m})$ for all $m \ge 1$. This sequence converges to $(S/n, S/n, \ldots, S/n)$ (see [1]). The continuity of P implies that $P(x_1, x_2, \ldots, x_n) \le P(S/n, S/n, \ldots, S/n)$, completing the proof.

(b) In the case $n = 2$, if $S \ge 2\sqrt{2}$, then

$$\left(1 + \frac{S^2}{2^2}\right)^2 - P \;=\; \frac{(x_1 - x_2)^2(4x_1x_2 + S^2 - 8)}{16} \;\ge\; 0\,.$$

Now consider the case $n = 3$. By solving polynomial equations, we find that $1 + S^2 \le (1 + \frac{1}{9}S^2)^3$ for $S \ge \frac{3}{2}\sqrt{2(\sqrt{33}-3)} \approx 3.514$ and $(1 + \frac{1}{4}S^2)^2 \le (1 + \frac{1}{9}S^2)^3$ for $S \ge a = \frac{3}{2}\sqrt{2(5\sqrt{105}+33)} \approx 4.867$. By part (a), we have $P \le (1 + \frac{1}{9}S^2)^3$ for $S \ge a$. This is an improvement of the given bound, since $4\sqrt{2} > a$.

Finally, suppose that $n \ge 4$. Let $f(x) = \ln(1+x^2)$ for $x \ge 0$. Then

$$f'(x) \;=\; \frac{2x}{1+x^2} \qquad \text{and} \qquad f''(x) \;=\; \frac{2(1-x^2)}{(1+x^2)^2}\,.$$

Note that the equation $f'(x) = f(x)/x$ has a unique positive root $r \approx 1.98$. The tangent line to the graph of f at the point where $x = r$ passes through the origin. Let $y = T(x)$ be the equation of this tangent line. Define

$$g(x) \;=\; \begin{cases} T(x) & \text{if } 0 \le x < r, \\ f(x) & \text{if } r \le x. \end{cases}$$

Then g is concave, and $g(x) \ge f(x)$ for all $x \ge 0$. Using Jensen's Inequality, we get

$$\ln P \;=\; \sum_{k=1}^{n} f(x_k) \;\le\; \sum_{k=1}^{n} g(x_k) \;\le\; ng(S/n) \;=\; nf(S/n)$$

if $S \geq rn$. Thus, $\ln P \leq n \ln\left(1 + (S/n)^2\right)$ for $S \geq rn$, and therefore the inequality in (b) holds for $S \geq rn$. The bound on S here is an improvement of the given bound, since $2\sqrt{2}(n-1)/n \geq 3\sqrt{2}/2 > 2 > r$ for $n \geq 4$.

(c) By the AM–GM Inequality, $x_1 x_2 \leq \left(\frac{x_1 + x_2}{2}\right)^2 \leq \left(\frac{S}{2}\right)^2 \leq 2$. Hence,

$$(1 + x_1^2)(1 + x_2^2) - \left[1 + (x_1 + x_2)^2\right] = x_1 x_2 (x_1 x_2 - 2) \leq 0\,.$$

An easy induction completes the proof.

Reference:

[1] G.-Z. Chang and T.W. Sederberg, "Over and Over Again", MAA, 1997, 29–31.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (c) only); and the proposer.

3095. [2005 : 544, 547] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let a, b, c, p, and q be natural numbers. Using $\lfloor x \rfloor$ to denote the integer part of x, prove that

$$\min\left\{a, \left\lfloor \frac{c + pb}{q} \right\rfloor\right\} \leq \left\lfloor \frac{c + p(a+b)}{p+q} \right\rfloor\,.$$

Solution by Joel Schlosberg, Bayside, NY, USA.

We have

$$\begin{aligned}
\frac{c + p(a+b)}{p+q} &= \frac{pa}{p+q} + \frac{c+pb}{p+q} = \frac{p}{p+q}a + \frac{q}{p+q}\frac{c+pb}{q} \\
&\geq \frac{p}{p+q}a + \frac{q}{p+q}\left\lfloor \frac{c+pb}{q} \right\rfloor \\
&\geq \frac{p}{p+q}\min\left\{a, \left\lfloor \frac{c+pb}{q} \right\rfloor\right\} + \frac{q}{p+q}\min\left\{a, \left\lfloor \frac{c+pb}{q} \right\rfloor\right\} \\
&= \min\left\{a, \left\lfloor \frac{c+pb}{q} \right\rfloor\right\}\,.
\end{aligned}$$

Since $\min\left\{a, \left\lfloor \frac{c+pb}{q} \right\rfloor\right\}$ is an integer,

$$\left\lfloor \frac{c + p(a+b)}{p+q} \right\rfloor \geq \min\left\{a, \left\lfloor \frac{c+pb}{q} \right\rfloor\right\}\,.$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3096. [2005 : 544, 547] *Proposed by Arkady Alt, San Jose, CA, USA.*

Let ABC be a triangle with sides a, b, c opposite the angles A, B, C, respectively. Prove that

$$\sum_{\text{cyclic}} \frac{bc}{b+c}\sin^2\left(\frac{A}{2}\right) \;\le\; \frac{a+b+c}{8}\,.$$

Similar solutions by Vedula N. Murty, Dover, PA, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since $\frac{bc}{b+c} \le \frac{b+c}{4}$, $\sin^2\frac{A}{2} = \frac{1-\cos A}{2}$, and $a = b\cos C + c\cos B$, we obtain

$$\begin{aligned} 8\sum_{\text{cyclic}} \frac{bc}{b+c}\sin^2\frac{A}{2} - \sum_{\text{cyclic}} a \;&\le\; \sum_{\text{cyclic}}[(b+c)(1-\cos A)] - \sum_{\text{cyclic}} a \\ &=\; \sum_{\text{cyclic}} a - \sum_{\text{cyclic}}(b\cos C + c\cos B) \;=\; 0\,, \end{aligned}$$

which yields the desired inequality.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer.

3097. [2005 : 545, 547] *Proposed by Mihály Bencze, Brasov, Romania.*

Let a and b be two positive real numbers such that $a < b$. Define $A(a,b) = \frac{a+b}{2}$ and $L(a,b) = \frac{b-a}{\ln b - \ln a}$. Prove that

$$L(a,b) \;<\; L\left(\tfrac{a+b}{2}, \sqrt{ab}\right) \;<\; \left(A(\sqrt{a},\sqrt{b})\right)^2 \;<\; A(a,b)\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

The inequality $\left(A(\sqrt{a},\sqrt{b})\right)^2 < A(a,b)$ is simply the Power–Mean Inequality. Applying the Hadamard's Inequality to the convex function $f(x) = 1/x$, we get

$$\begin{aligned} \frac{1}{L\left(\frac{1}{2}(a+b), \sqrt{ab}\right)} \;&=\; \frac{1}{\frac{1}{2}(a+b) - \sqrt{ab}} \int_{\sqrt{ab}}^{\frac{1}{2}(a+b)} f(x)\,dx \\ &>\; f\left(\frac{\frac{1}{2}(a+b) + \sqrt{ab}}{2}\right) \;=\; \frac{1}{\left(A(\sqrt{a},\sqrt{b})\right)^2}\,. \end{aligned}$$

This gives the inequality $L\left(\frac{1}{2}(a+b),\sqrt{ab}\right) < \left(A(\sqrt{a},\sqrt{b})\right)^2$.

The inequality $L(a,b) < L\left(\frac{1}{2}(a+b),\sqrt{ab}\right)$ transforms successively into

$$\begin{aligned}
(b-a)\ln\left(\tfrac{1}{2}(a+b)\right) &< \left(\sqrt{ab}-a\right)\ln a + \left(b-\sqrt{ab}\right)\ln b\,,\\
\left(\tfrac{1}{2}(a+b)\right)^{\sqrt{a}+\sqrt{b}} &< a^{\sqrt{a}}b^{\sqrt{b}}\,,\\
\left(\tfrac{1}{2}(1+b/a)\right)\left(\tfrac{1}{2}(1+a/b)\right)\sqrt{b/a} &< 1\,,\\
\left(\tfrac{1}{2}(1+x^2)\right)\left(\frac{x^2+1}{2x^2}\right)^x &< 1\,,
\end{aligned}$$

with $x = \sqrt{b/a} > 1$. Now, for $x > 1$, let

$$f(x) = \ln\left(\tfrac{1}{2}(x^2+1)\right) + x\ln\left(\frac{x^2+1}{2x^2}\right).$$

Then

$$\begin{aligned}
f'(x) &= \frac{2(x-1)}{x^2+1} + \ln\left(\frac{x^2+1}{2x^2}\right)\\
\text{and}\qquad f''(x) &= \frac{-2(x-1)^2(x+1)}{x(x^2+1)^2} < 0\,.
\end{aligned}$$

Hence, f is strictly concave. Since $f(1) = 0$ and $f'(1) = 0$, we conclude that $f(x) < 0$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

3098★. [2005 : 545, 548] *Proposed by D.Z. Djokovic, University of Waterloo, Waterloo, ON; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON.*

Let n and k be any positive integers such that $k \leq n$. Let S denote the sequence of length $2n$ obtained by interlacing the two sequences $n, n-1, \ldots, 2, 1$ and $-1, -2, \ldots, -(n-1), -n$, and let $\mathcal{F}$ be the set of all $\binom{2n-k+1}{k}$ subsequences K of S which have length k and do not contain any pair of consecutive terms of S. Prove that

$$\sum_{K\in\mathcal{F}} P(K) = 0\,,$$

where $P(K)$ is the product of all k terms of the sequence K.

For example, if $n = 3$ and $k = 2$, then $S = 3,\ -1,\ 2,\ -2,\ 1,\ -3$, and

$$\begin{aligned}\mathcal{F} &= \big\{\{3,\,2\},\,\{3,\,-2\},\,\{3,\,1\},\,\{3,\,-3\},\,\{-1,\,-2\},\,\{-1,\,1\},\\ &\qquad \{-1,\,-3\},\,\{2,\,1\},\,\{2,\,-3\},\,\{-2,\,-3\}\big\}\,;\end{aligned}$$

hence,

$$\sum_{K\in\mathcal{F}} P(K) \;=\; 6-6+3-9+2-1+3+2-6+6 \;=\; 0\,.$$

[This result was obtained as a by-product of some research in Lie Algebra. The proposers have a proof for odd k. They hope that an elementary proof can be found.]

Solution by Tom Leong, Brooklyn, NY, USA.

Let S be the sequence $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$, where $a_m = n-m+1$ and $b_m = -m$ for $m = 1,\,2,\,\ldots,\,n$. Let $s_k(x) = \sum P(K)$, where the sum is over all subsequences $K \in \mathcal{F}$ whose first term is x. For instance, in the example given in the statement of the problem,

$$s_2(-1) \;=\; (-1)(-2)+(-1)(1)+(-1)(-3) \;=\; 4\,.$$

Note first that if $K \in \mathcal{F}$ begins with a_m or b_m, then we must have $k-1 \le n-m$ or $m \le n-k+1$. Thus, there are no subsequences in $\mathcal{F}$ beginning with either a_m or b_m if $m > n-k+1$.

We now prove by induction on k that

$$s_k(a_m) \;=\; (-1)^{k+1}k!\binom{n-m+1}{k}\binom{m+k-2}{k-1} \qquad (1)$$

$$\text{and}\quad s_k(b_m) \;=\; (-1)^{k}k!\binom{m+k-1}{k}\binom{n-m}{k-1} \qquad (2)$$

for $m = 1,\,2,\,\ldots,\,n-k+1$.

If $k = 1$, then equations (1) and (2) reduce to $s_1(a_m) = n-m+1$ and $s_1(b_m) = -m$, which are obviously true. Applying the inductive hypothesis, we have

$$\begin{aligned}&\sum_{i=m+1}^{n-k+1}\big(s_k(a_i)+s_k(b_i)\big)\\ &= (-1)^k k!\sum_{i=m+1}^{n-k+1}\left[-\binom{n-i+1}{k}\binom{i+k-2}{k-1}+\binom{i+k-1}{k}\binom{n-i}{k-1}\right]\\ &= (-1)^k k!\binom{n-m}{k}\binom{m+k-1}{k} \qquad (3)\end{aligned}$$

where (3) follows from (4) given below which we will prove later. Assuming

(3) holds for the moment, we have

$$\begin{aligned} s_{k+1}(a_m) &= a_m \sum_{i=m+1}^{n-k+1} (s_k(a_i) + s_k(b_i)) \\ &= (n-m+1)(-1)^k k! \binom{n-m}{k} \binom{m+k-1}{k} \\ &= (-1)^k (k+1)! \binom{n-m+1}{k+1} \binom{m+k-1}{k}. \end{aligned}$$

Similarly, using (3) and the inductive hypothesis again, we have

$$\begin{aligned} s_{k+1}(b_m) &= b_m \left(\sum_{i=m+2}^{n-k+1} s_k(a_i) + \sum_{i=m+1}^{n-k+1} s_k(b_i) \right) \\ &= -m \left(\sum_{i=m+1}^{n-k+1} (s_k(a_i) + s_k(b_i)) - s_k(a_{m+1}) \right) \\ &= -m(-1)^k k! \left[\binom{n-m}{k} \binom{m+k-1}{k} \right. \\ &\qquad \left. + \binom{n-m}{k} \binom{m+k-1}{k-1} \right] \\ &= -m(-1)^k k! \binom{n-m}{k} \left[\binom{m+k-1}{k} + \binom{m+k-1}{k-1} \right] \\ &= -m(-1)^k k! \binom{n-m}{k} \binom{m+k}{k} \\ &= (-1)^{k+1} (k+1)! \binom{n-m}{k} \binom{m+k}{k+1}. \end{aligned}$$

This completes the induction.

To complete the solution, we prove the identity required for (3), namely

$$\sum_{i=m+1}^{n-k+1} \binom{i+k-1}{k} \binom{n-i}{k-1} - \sum_{i=m+1}^{n-k+1} \binom{n-i+1}{k} \binom{i+k-2}{k-1} = \binom{n-m}{k} \binom{m+k-1}{k}. \qquad (4)$$

We give a combinatorial argument. Let $T = \{1, 2, \ldots, n+k\}$. The term $\binom{i+k-1}{k}\binom{n-i}{k-1}$ counts the number of $2k$-element subsets of T whose $(k+1)^{\text{st}}$ largest element is $i+k$. Thus, the first sum on the left side counts the number of $2k$-element subsets of T whose $(k+1)^{\text{st}}$ largest element is greater than $m+k$. Similarly, the second sum on the left side counts the number of $2k$-element subsets of T whose k^{th} largest element is greater than $m+k-1$. Their difference counts the number of $2k$-element subsets of T whose k smallest elements are all less than $m+k$ and whose k largest elements are greater than $m+k$. Clearly, the right side also counts this number.

There was also one incorrect submission.

For the truth of the result when k is odd, Leong gave the following simple argument: Let S be the sequence $x_1, x_2, \ldots, x_{2n}$. For a subsequence $K = x_{i_1}, x_{i_2}, \ldots, x_{i_k}$, let $K' = x_{2n-i_k+1}, \ldots, x_{2n-i_2+1}, x_{2n-i_1+1}$. Note that $K \in \mathcal{F}$ if and only if $K' \in \mathcal{F}$. Since $x_i = -x_{2n-i+1}$, we have $P(K) = -P(K')$, and consequently, $\sum\limits_{K\in\mathcal{F}} P(K) = 0$.

3099. [2005 : 545, 548] *Proposed by Mihály Bencze, Brasov, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n} \ln(1+a_k) \;\le\; \left(\ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right)^n .$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let $f(x) = \ln\bigl(\ln(1+e^x)\bigr)$. Then, by straightforward computations, we find that

$$\begin{aligned} f'(x) &= \frac{e^x}{(1+e^x)\ln(1+e^x)} \\ \text{and}\qquad f''(x) &= \frac{e^x(1+e^x)\ln(1+e^x) - e^x\bigl(e^x\ln(1+e^x)+e^x\bigr)}{\bigl((1+e^x)\ln(1+e^x)\bigr)^2} \\ &= \frac{e^x g(x)}{\bigl((1+e^x)\ln(1+e^x)\bigr)^2}, \end{aligned}$$

where $g(x) = \ln(1+e^x) - e^x$.

Since $g'(x) = \dfrac{e^x}{1+e^x} - e^x = \dfrac{-e^{2x}}{1+e^x} < 0$ and $\lim\limits_{x\to-\infty} g(x) = 0$, we deduce that $g(x) < 0$ and $f''(x) < 0$ for all real x.

Hence, f is strictly concave and, by Jensen's Inequality, we have

$$\exp\left(\frac{1}{n}\sum_{k=1}^{n} f(\ln a_k)\right) \;\le\; \exp\left(f\left(\frac{1}{n}\sum_{k=1}^{n}\ln a_k\right)\right). \tag{1}$$

Note that

$$\frac{1}{n}\sum_{k=1}^{n} f(\ln a_k) \;=\; \frac{1}{n}\sum_{k=1}^{n}\ln\bigl(\ln(1+a_k)\bigr) \;=\; \sum_{k=1}^{n}\ln\left(\bigl(\ln(1+a_k)\bigr)^{1/n}\right). \tag{2}$$

and

$$\begin{aligned} f\left(\frac{1}{n}\sum_{k=1}^{n}\ln a_k\right) &= \ln\left(\ln\left(1+\exp\left(\frac{1}{n}\sum_{k=1}^{n}\ln a_k\right)\right)\right) \\ &= \ln\left(\ln\left(1+\prod_{k=1}^{n} a_k^{1/n}\right)\right) \\ &= \ln\left(\ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right)\right). \qquad (3) \end{aligned}$$

From (1), (2), and (3), we conclude that

$$\prod_{k=1}^{n} \ln(1+a_k)^{1/n} \;\le\; \ln\left(1+\sqrt[n]{\prod_{k=1}^{n} a_k}\right),$$

from which the result follows immediately.

Equality holds if and only if $a_1 = a_2 = \cdots = a_n$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; BIN ZHAO, YunYuan HuaZhong University of Technology and Science, Wuhan, Hubei, China; and the proposer. All the solutions are essentially the same as the one featured above.

3100. [2005 : 545, 548] *Proposed by Michel Bataille, Rouen, France.*

Let $f:\mathbb{R}\to\mathbb{R}$ satisfy $f\big(xf(y)\big) = yf(x)$ for all real numbers x and y.

(a) Show that f is an odd function.

(b) Determine f, given that f has exactly one discontinuity.

(a) *Solution by Joel Schlosberg, Bayside, NY, USA.*

Setting $x = y = 0$ in the given condition shows that $f(0) = 0$. Note that the zero function, namely $f(x) = 0$ for all x, satisfies the functional equation; it is odd, as desired (and continuous on $\mathbb{R}$). We now suppose that f is a non-zero function; therefore, we assume that there is some $a \in \mathbb{R}$ with $f(a) \neq 0$. If $f(y) = f(z)$, then

$$yf(a) \;=\; f\big(af(y)\big) \;=\; f\big(af(z)\big) \;=\; z\big(f(a)\big).$$

Cancelling $f(a)$ from both sides shows that $y = z$. Thus, f is one-to-one. Choosing any $z \neq 0$, we have

$$f\big(zf(xy)\big) \;=\; xyf(z) \;=\; xf\big(zf(y)\big) \;=\; f\big(zf(y)f(x)\big).$$

Since f is one-to-one, we see that $zf(xy) = zf(x)f(y)$, and

$$f(xy) \;=\; f(x)f(y). \tag{1}$$

Substituting $x = y = 1$ in (1), we get $f(1) = \big(f(1)\big)^2$. Since f is one-to-one, we must have $f(1) \neq f(0) = 0$; thus, by cancellation,

$$f(1) \;=\; 1.$$

Substituting $x = y = -1$ in (1), we find $1 = f(1) = \big(f(-1)\big)^2$; hence, $f(-1) = \pm 1$. Since f is one-to-one, we must have $f(-1) \neq f(1) = 1$. Therefore,

$$f(-1) \;=\; -1.$$

Finally, we substitute $y = -1$ in (1) to get

$$f(-x) \;=\; -f(x)\,;$$

thus, f is an odd function, proving (a).

(b) *Solution by Chip Curtis, Missouri Southern State University, Joplin, MO, USA.*

Using $f(1) = 1$ from part (a) and setting $x = 1$ in the given condition, we deduce that $f\bigl(f(y)\bigr) = yf(1) = y$ for all y; that is, f is its own inverse. We now assume that f has exactly one discontinuity. Since f is odd, this discontinuity must be at 0. Since f is one-to-one and continuous on $(0,\infty)$, it is either strictly increasing or strictly decreasing on that half line.

Suppose first that f is strictly increasing on $(0,\infty)$. Let $x \in (0,\infty)$. If $f(x) > x$, then $f\bigl(f(x)\bigr) > f(x)$; since f is its own inverse, this implies that $x > f(x)$, a contradiction. On the other hand, if $f(x) < x$, then $f\bigl(f(x)\bigr) < f(x)$, implying that $x < f(x)$, again a contradiction. Thus, for all $x \in (0,\infty)$, we must have $f(x) = x$. But since f is odd, this implies that $f(x) = x$ for all $x \in \mathbb{R}$, a contradiction since f has a discontinuity at $x = 0$. Hence, f is not strictly increasing on $(0,\infty)$.

Finally, suppose f is strictly decreasing on $(0,\infty)$. We will show that $f(x) = 1/x$ for all $x \in (0,\infty)$. Let $x \in (0,\infty)$ and set $y = f(x)$. Note that $f(xy) = f\bigl(xf(x)\bigr) = xf(x) = xy$. If $y < 1/x$, then $xy < 1$, which implies that $xy = f(xy) > f(1) = 1$, a contradiction. If $y > 1/x$, then $xy > 1$, which implies that $xy = f(xy) < f(1) = 1$, again a contradiction. Thus, $y = 1/x$; whence, $f(x) = 1/x$ for all $x \in (0,\infty)$. Since f is odd and $f(0) = 0$, we have

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; CHARLES R. DIMINNIE and ROGER ZARNOWSKI, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Curtis also solved part (a) *and Schlosberg also solved part* (b).

Part (b) *follows quickly from known results. Diminnie and Zarnowski referred to* $f(xy) = f(x)f(y)$ *(which appears in our solution to part* (a)) *as* Cauchy's Power Equation*; they referenced* [1] *for the fact that, when* $f(x)$ *is continuous for* $x > 0$*, the only solutions are* $f(x) = 0$ *and* $f(x) = x^c$*. Janous stated that it is well known that if* f *is any odd involution which is continuous on the positive reals, then necessarily* $f(x) = 1/x$ *or* $f(x) = -(1/x)$*; he provided reference* [2].

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge University Press, 1989, pp. 29–30.

[2] J. Dhombres, *Some Aspects of Functional Equations*, Chulalongkorn University Press, Bangkok, 1979.