THE OLYMPIAD CORNER

No. 258

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For our problems in this issue we give a first installment of the short-listed problems from the 44th IMO in Japan. My thanks go to Andy Liu, Canadian Team Leader to the IMO, for collecting them for our use.

44th INTERNATIONAL MATHEMATICAL OLYMPIAD
Short-listed Problems
Algebra

A1. Let a_{ij} ($i = 1, 2, 3$; $j = 1, 2, 3$) be real numbers such that a_{ij} is positive for $i = j$ and negative for $i \neq j$. Prove that there exist positive real numbers c_1, c_2, c_3 such that the numbers $a_{11}c_1 + a_{12}c_2 + a_{13}c_3$, $a_{21}c_1 + a_{22}c_2 + a_{23}c_3$, and $a_{31}c_1 + a_{32}c_2 + a_{33}c_3$ are all negative, all positive, or all zero.

A2. Find all non-decreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that

(a) $f(0) = 0$, $f(1) = 1$;

(b) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a and b such that $a < 1 < b$.

A3. Consider pairs of sequences of positive real numbers

$$a_1 \geq a_2 \geq a_3 \geq \cdots \quad \text{and} \quad b_1 \geq b_2 \geq b_3 \geq \cdots .$$

For any such pair, define $c_i = \min\{a_i, b_i\}$ for $i = 1, 2, 3, \ldots$. For $n = 1, 2, 3, \ldots$, define the sums

$$A_n = a_1 + \cdots + a_n , \quad B_n = b_1 + \cdots + b_n , \quad C_n = c_1 + \cdots + c_n .$$

(a) Does there exist a pair $(a_i)_{i\geq 1}$, $(b_i)_{i\geq 1}$ such that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are unbounded while the sequence $(C_n)_{n\geq 1}$ is bounded?

(b) Does the answer to question (a) change if the additional assumption is made that $b_i = 1/i$, for $i = 1, 2, \ldots$?

Justify your answer.

Combinatorics

C1. Let A be a 101-element subset of the set $S = \{1, 2, \ldots, 1000000\}$. Prove that there exist numbers $t_1, t_2, \ldots, t_{100}$ in S such that the sets

$$A_j = \{x + t_j \mid x \in A\} , \quad j = 1, 2, \ldots, 100$$

are pairwise disjoint.

C2. Let $D_1, \ldots, D_n$ be closed discs in the plane. (A closed disc is the region bounded by a circle, taken jointly with this circle.) Suppose that every point in the plane is contained in at most 2003 discs D_i. Prove that there exists a disc D_k which intersects at most $7 \cdot 2003 - 1$ other discs D_i.

C3. Let $n \geq 5$ be a given integer. Determine the greatest integer k for which there exists a polygon with n vertices (convex or not, with a boundary which is not self-intersecting) having k internal right angles.

Geometry

G1. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

G2. Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing through A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segments PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and the line AC does not depend on the choice of Γ.

G3. Let ABC be a triangle, and let P be a point in its interior. Denote by D, E, F the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Suppose that $AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$. Denote by I_A, I_B, I_C the excentres of the triangle ABC. Prove that P is the circumcentre of the triangle $I_A I_B I_C$.

G4. Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1 and Γ_3 are externally tangent at P, and Γ_2 and Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2 meet at A, Γ_2 and Γ_3 meet at B, Γ_3 and Γ_4 meet at C, and Γ_4 and Γ_1 meet at D, where the points A, B, C, D are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

Number Theory

N1. Let m be a fixed integer greater than 1. The sequence $x_0, x_1, x_2, \ldots$ is defined as follows:

$$x_i = \begin{cases} 2^i & \text{if } 0 \leq i \leq m-1, \\ \sum\limits_{j=1}^{m} x_{i-j} & \text{if } i \geq m. \end{cases}$$

Find the greatest k for which the sequence contains k consecutive terms divisible by m.

N2. Each positive integer a undergoes the following procedure in order to obtain the number $d = d(a)$:

(i) Move the last digit of a to the first position to obtain the number b;

(ii) Square b to obtain the number c;

(iii) Move the first digit of c to the end to obtain the number d.

(All numbers in the problem are considered to be represented in base 10.) For example, for $a = 2003$, we obtain $b = 3200$, $c = 10240000$, and $d = 02400001 = 2400001 = d(2003)$.

Find all numbers a for which $d(a) = a^2$.

N3. Determine all pairs of positive integers (a, b) such that

$$\frac{a^2}{2ab^2 - b^3 + 1}$$

is a positive integer.

N4. Let b be an integer greater than 5. For each positive integer n, consider the number

$$x_n = \underbrace{1\,1\,\cdots\,1}_{n-1}\underbrace{2\,2\,\cdots\,2}_{n}\,5\,,$$

written in base b.

Prove that the following condition holds if and ony if $b = 10$: there exists a positive integer M such that for any integer n greater than M, the number x_n is a perfect square.

A package arrived recently from Ioannis Katsikis, Athens, Greece containing a number of solutions to problems whose solutions have already been given in the *Corner*. In the package were solutions to the following:

- Singapore Mathematical Olympiad 2002, Open Section Part A [2005 : 215], [2006 : 378–383], Problems 1, 3, 4, 5, 6, 7, 9, and 10; and Part B [2005 : 216], [2006 : 384–386], Problems 3 and 4;

- XVIII Italian Mathematical Olympiad [2005 : 217], [2006 : 386–388], Problems 1, 3, and 4;

- 2001–2002 British Mathematical Olympiad, Round 1 [2005 : 287], [2006 : 423–426], Problems 1, 2, 3, 4, and 5; and Round 2 [2005 : 288], [2006 : 426–428], Problems 1, 2, and 3;

- the 15$^{\text{th}}$ Korean Mathematical Olympiad [2005 : 288–289], [2006 : 429–432], Problems 2 and 5.

Since we have not previously published a solution to Problem 5 of the 15$^{\text{th}}$ Korean Mathematical Olympiad, we now give the solution of Katsikis.

5. Let ABC be an acute triangle, and let O be its circumcircle. Let the perpendicular line from A to BC meet O at D. Let P be a point on O, and let Q be the foot of the perpendicular line from P to the line AB. Prove that if Q is on the outside of O and $2\angle QPB = \angle PBC$, then D, P, Q are collinear.

Solution by Ioannis Katsikis, Athens, Greece, modified by the editor.

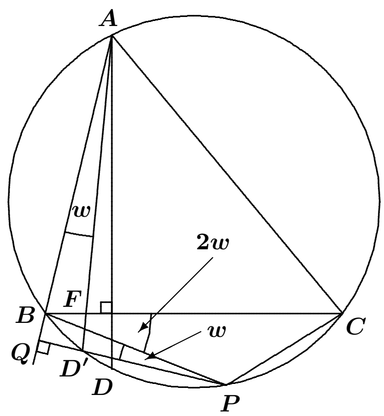
Let $w = \angle QPB$. Then $\angle PBC = 2w$ and $\angle QBP = 90^\circ - w = \angle ABC$.

Case 1. The line PQ intersects the circle O at a point D' between P and Q.

Let F be the intersection of AD' and BC. We have $\angle BAD' = \angle BPD'$ (since these angles subtend the same arc), and hence $\angle BAD' = \angle BPQ = w$. Then

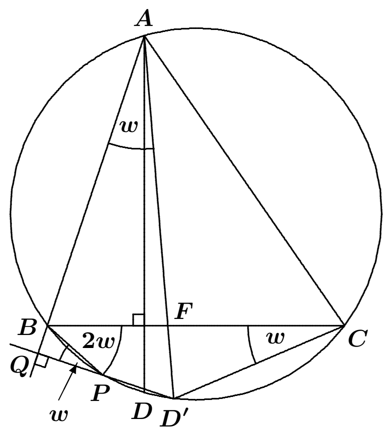
$$\begin{aligned} \angle AFB &= 180^\circ - \angle BAD' - \angle ABC \\ &= 180^\circ - w - (90^\circ - w) \\ &= 90^\circ . \end{aligned}$$

Then $D' = D$. Thus, D, P, Q are collinear.



Case 2. The line PQ intersects the circle O at a point D' such that P is between D' and Q.

Let F be the intersection of AD' and BC. Since quadrilateral $BPD'C$ is inscribed in the circle O, we see that $\angle BCD' = 180^\circ - \angle BPD' = w$. Hence, $\angle BAD' = w$. In the triangle ABF, we have $\angle BAF = \angle BAD' = w$ and $\angle ABF = \angle ABC = 90^\circ - w$. Therefore, $\angle AFB = 90^\circ$. Then $D' = D$. Thus, D, P, Q are collinear.



Case 3. The line PQ is tangent to the circle O at P.

The proof is similar to Case 2. Quadrilateral $BPD'C$ now degenerates into triangle BPC. Then $\angle BCP = w$ by the Tangent-Chord Theorem, and the rest of Case 2 follows.

Next we consider readers' solutions to the Yugoslav Qualification for IMO 2002 First Round and Second Round given in [2005 : 373–374].

First Round

1. A man standing at the point $(1,1)$ in the coordinate plane wants to find an object that lies at some point (α,β), where $\alpha \in \{1,\,2,\,\ldots,\,m\}$, and $\beta \in \{1,\,2,\,\ldots,\,n\}$. After finding the object, he will return to the starting point. Find the minimal worst case time needed for doing this job, if he does not know exactly at which point the object lies, and if he can move in any direction with velocity not greater than one.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The minimal worst case time is equal to the minimal worst case length of a path needed for doing the job.

Any chosen closed path has to go through each of the points (α,β), where $\alpha \in \{1,\,2,\,\ldots,\,m\}$ and $\beta \in \{1,\,2,\,\ldots,\,n\}$; otherwise, the man cannot be sure of finding the object. Therefore, any chosen closed path must have length at least mn.

If $m = 1$, it is easy to see that the minimal length of a closed path going through each point is $2n$. In the same way, if $n = 1$, the minimal length is $2m$.

If m and n are both greater than 1, then any closed path must have even length because the number of left moves is equal to the number of right moves, and the number of up moves is equal to the number of down moves.

Case 1. mn is even, say m even.

There exists a closed path with length mn, going through each point and back to $(1,1)$. We may use:

$$\begin{array}{rcl}
(1,1) & \to & (2,1) \to \cdots \to (m,1) \to \\
\to (m,2) & \to & (m,3) \to \cdots \to (m,n) \to \\
\to (m-1,n) & \to & (m-1,n-1) \to \cdots \to (m-1,2) \to \\
\to (m-2,2) & \to & (m-2,3) \to \cdots \to (m-2,n) \to \\
& \vdots & \\
\to (1,n) & \to & (1,n-1) \to \cdots \to (1,1)\,.
\end{array}$$

Case 2. mn is odd.

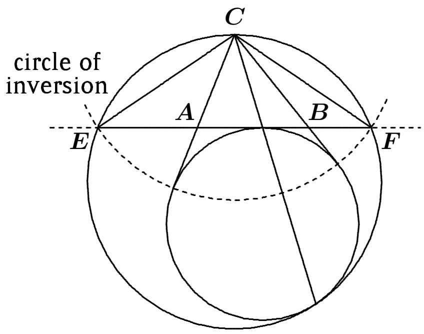
Then the minimal length is at least $mn+1$. Conversely, there exists a closed path with length $mn+1$, going through each point and returning back to $(1,1)$. We may use the same path as above until we reach $(3,n)$ and then use:

$$\begin{array}{rcl}
(3,n) & \to & (2,n) \to (1,n) \to (1,n-1) \to (2,n-1) \to \\
\to (2,n-2) & \to & (1,n-2) \to \cdots \to (2,2) \to (2,1) \to (1,1)\,.
\end{array}$$

2. Let p be the semiperimeter of the triangle ABC. Let the points E and F lie on the line AB such that $CE = CF = p$. Prove that the circumcircle of the triangle EFC and the circle that touches the side AB and the extension of the sides AC and BC of the triangle ABC meet in one point.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Tangency invites inversion; thus, we invert in the circle with centre C and radius $p = CE = CF$.



The circumcircle of $\triangle EFC$ becomes the straight line through E and F.

Since the tangents from C (or any other vertex) to the excircle beyond the opposite side are of length p, the excircle that touches the side AB is invariant under this inversion.

Tangency is preserved under inversion; therefore, the circumcircle of $\triangle EFC$ and the excircle beyond AB meet in one point.

3. Let $\{x_n\}_{n\geq 2}$, be a sequence such that $x_2 = 1$, $x_3 = 1$, and, for $n \geq 3$,

$$(n+1)(n-2)x_{n+1} \;=\; n(n^2-n-1)x_n - (n-1)^3 x_{n-1}\,.$$

Prove that x_n is an integer if and only if n is a prime.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's write-up.

Let $y_n = nx_n - 1$. Then $y_2 = 1$, $y_3 = 2$ and, for $n \geq 3$,

$$(n-2)y_{n+1} \;=\; (n^2-n-1)y_n - (n-1)^2 y_{n-1}\,;$$

thus,

$$(n-2)(y_{n+1}-y_n) \;=\; (n-1)^2(y_n - y_{n-1})\,.$$

Let $z_n = y_n - y_{n-1}$. Then $z_3 = 1$ and $z_{n+1} = \dfrac{(n-1)^2}{n-2} z_n$ for $n \geq 3$. It follows easily that $z_n = (n-2)\cdot(n-2)!$ for $n \geq 3$. Then

$$\begin{aligned} y_n &= (y_n - y_{n-1}) + (y_{n-1} - y_{n-2}) + \cdots + (y_3 - y_2) + y_2 \\ &= z_n + z_{n-1} + \cdots + z_3 + y_2 \;=\; 1 + \sum_{k=1}^{n-2} k\cdot k! \;=\; (n-1)! \end{aligned}$$

(This last equality follows by an easy induction.) Therefore, for $n \geq 2$, we have $x_n = \dfrac{(n-1)!+1}{n}$.

Thus, x_n is an integer if and only if n divides $(n-1)!+1$, which is equivalent to n being prime, according to Wilson's Theorem.

Second Round

1. What is the maximal value of the expression $a + b + c + abc$, if a, b, c are non-negative numbers such that $a^2 + b^2 + c^2 + abc \leq 4$?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's write-up.

Let $f(a, b, c) = a + b + c + abc$ and $g(a, b, c) = a^2 + b^2 + c^2 + abc$. Let S be the set of the triples (a, b, c) of non-negative numbers such that $g(a, b, c) \leq 4$.

First, it is clear that $S \subset [0, 2]^3$ and that S is a closed set. Since f is continuous, the desired maximum exists. Let M denote this maximum. Since $f(1, 1, 1) = 4$, we must have

$$M \geq 4 . \tag{1}$$

Let $(x, y, z) \in S$ such that $f(x, y, z) = M$. If one of x, y, z is 2, say $x = 2$, then $y = z = 0$ and $f(x, y, z) = 2$, which is a contradiction since $f(x, y, z) = M \geq 4$. If one of x, y, z is 0, say $x = 0$, then $y^2 + z^2 \leq 4$ and, using the inequality between the arithmetic and quadratic means, we have

$$f(x, y, z) = y + z \leq 2\sqrt{\frac{y^2 + z^2}{2}} \leq 2\sqrt{2} < M ,$$

a contradiction. Thus,

$$0 < x, y, z < 2 . \tag{2}$$

Now we will prove that, if $x \neq y$, then $\left(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z\right) \in S$ and $f\left(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z\right) > f(x, y, z)$.

We have

$$\begin{aligned} & g\left(\tfrac{1}{2}(x + y), \tfrac{1}{2}(x + y), z\right) \\ &\qquad = \left(\tfrac{1}{2}(x + y)\right)^2 + \left(\tfrac{1}{2}(x + y)\right)^2 + z^2 + \left(\tfrac{1}{2}(x + y)\right)^2 z \\ &\qquad = x^2 + y^2 + z^2 + xyz - \tfrac{1}{4}(x - y)^2(2 - z) \\ &\qquad = g(x, y, z) - \tfrac{1}{4}(x - y)^2(2 - z) \leq g(x, y, z) , \end{aligned}$$

since $z < 2$. This proves that $\left(\frac{1}{2}(x + y), \frac{1}{2}(x + y), z\right) \in S$. Moreover,

$$f\left(\tfrac{1}{2}(x + y), \tfrac{1}{2}(x + y), z\right) - f(x, y, z) = \left[\left(\tfrac{1}{2}(x + y)\right)^2 - xy\right] z > 0 ,$$

by the AM–GM Inequality and (2). Thus,

$$f\left(\tfrac{1}{2}(x + y), \tfrac{1}{2}(x + y), z\right) > f(x, y, z) ,$$

as claimed.

Since $f(x, y, z) = M$ is the maximal value of f, this proves that $x = y$. We can prove in the same way that $x = z$, so that $x = y = z$. Let t be this common value. Then $3t^2 + t^3 \leq 4$, which forces $t \leq 1$. On the other hand, for $t \leq 1$, we clearly have $f(t, t, t) = 3t + t^3 \leq 4$. Then $M \leq 4$.

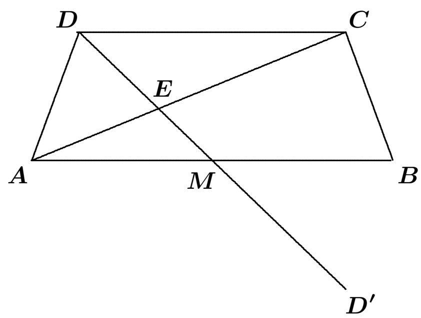
In view of (1), it follows that $M = 4$.

Next we look at readers' solutions to problems of the 27ième Olympiade Mathématique Belge, Midi Finale, given in [2005 : 374–375].

1. Soit M le milieu de la base $[AB]$ d'un trapèze isocèle $ABCD$ et E le point d'intersection de MD et de AC. Si M est le centre du cercle circonscrit au trapèze et si $[AD]$ et $[DE]$ ont la même longueur, déterminer l'amplitude de l'angle $\angle DAB$.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Let Γ denote the circumcircle of $ABCD$. We assume that Γ has radius 1 unit. We will use the notation $\widehat{XY}$ for the length of the minor arc between points X and Y on Γ.



Since $AD = DE$, we have

$$\begin{aligned} \angle DAC &= \angle DEA \\ &= \angle DCE + \angle CDE \\ &= \angle DCA + \angle CDD', \end{aligned}$$

where DD' is a diameter of Γ. Note that $\angle DAC = \frac{1}{2}\widehat{DC}$, $\angle DCA = \frac{1}{2}\widehat{AD}$, and $\angle CDD' = \frac{1}{2}\widehat{CD'}$. Thus,

$$\widehat{DC} \;=\; \widehat{AD} + \widehat{CD'} \;=\; \widehat{AD} + 180^\circ - \widehat{DC}\,,$$

or

$$2\widehat{DC} \;=\; \widehat{AD} + 180^\circ\,. \tag{1}$$

We also have $\widehat{AD} + \widehat{DC} + \widehat{CB} = 180^\circ$. Since $\widehat{AD} = \widehat{CB}$, this gives

$$2\widehat{AD} + \widehat{DC} \;=\; 180^\circ\,. \tag{2}$$

Solving (1) and (2), we get $\widehat{AD} = 36^\circ$ and $\widehat{DC} = 108^\circ$. Finally,

$$\angle DAB \;=\; \tfrac{1}{2}\widehat{DB} \;=\; \tfrac{1}{2}\big(\widehat{DC} + \widehat{CB}\big) \;=\; \tfrac{1}{2}(108^\circ + 36^\circ) \;=\; 72^\circ\,.$$

2. La somme de quatre nombres réels est nulle ; la somme de leurs cubes est également nulle. Est-il vrai qu'alors deux de ces quatre nombres sont nécessairement opposés ?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's write-up.

Oui. Soient x, y, z, t ces quatre réels.

Par l'absurde : supposons que la somme de deux quelconques d'entre eux n'est jamais nulle. Puisque $x + y = -(z + t)$, on a

$$\begin{aligned} x^3 + y^3 + z^3 + t^3 &= (x + y)(x^2 - xy + y^2) + (z + t)(z^2 - zt + t^2) \\ &= (x + y)\big[(x^2 - xy + y^2) - (z^2 - zt + t^2)\big]\,, \end{aligned}$$

d'où $x^2 - xy + y^2 = z^2 - zt + t^2$.

De même, $x^2 - xt + t^2 = z^2 - zy + y^2$. En sommant ces deux égalités, il vient $2x^2 - x(y+t) = 2z^2 - z(y+t)$; c'est à dire, $(x-z)(2x+2z-y-t) = 0$ ou encore $3(x-z)(x+z) = 0$ et ainsi $x = z$.

En raisonnant de la même façon, on prouve que $x = y = z = t$. Mais alors, puisque leur somme est nulle, c'est qu'ils sont tous égaux à 0, ce qui contredit notre point de départ.

3. (a) Existe-t-il quatre nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

(b) Existe-t-il cinq nombres naturels distincts non nuls tels que la somme de trois quelconques d'entre eux soit toujours un nombre premier ?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's write-up.

(a) Oui, par exemple 7, 11, 13 et 23.

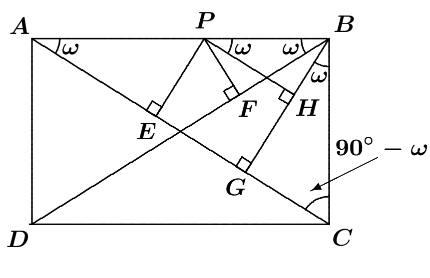
(b) Non. Soient a, b, c, d, e cinq nombres naturels distincts et non nuls. Si trois de ces nombres ont le même reste modulo 3, leur somme est un nombre divisible par 3 et strictement supérieur à 3. Ce ne peut donc pas être un nombre premier. Donc, au plus deux des nombres sont dans une classe donnée modulo 3. Comme il y a 5 nombres à répartir et trois classes possibles, c'est donc que chaque classe contient au moins un nombre. Mais alors, en choisissant un nombre dans chacune des classes, on obtient la même impossibilité. D'où la conclusion.

4. Soit un rectangle $ABCD$, P un point situé sur un des côtés de ce rectangle, E et F les pieds des hauteurs abaissées de P sur les diagonales du rectangle. Démontrer que la somme $|PE| + |PF|$ reste constante lorsque P parcourt le périmètre de $ABCD$.

Solution by Pavlos Maragoudakis, Pireas, Greece.

Let G be the projection of B on AC and H the projection of P on BG.

If $\angle BPH = \omega$, then we have $\angle GBC = \omega$, $\angle BCA = 90^\circ - \omega$, $\angle CAB = \omega$, and $\angle ABD = \omega$. Now $\triangle PFB \cong \triangle PHB$ (AAS), since PB is common, $\angle PFB = \angle PHB = 90^\circ$, and $\angle HPB = \angle FBP = \omega$. Thus, $PF = BH$.



Also, $PE = HG$ from the rectangle $PEGH$.

Finally $PE + PF = HG + BH = BG$, which is constant.

Next we look at readers' solutions to problems of the 27[ième] Olympiad Mathématique Belge, Maxi Finale, given in [2005 : 375].

1. Soit la suite $(a_n)_{n\in\mathbb{N}}$ telle que $a_n = n + \lfloor\sqrt{n}\rfloor$ pour tout $n \in \mathbb{N}$. Déterminer le plus petit entier naturel k pour lequel a_k, a_{k+1}, $\dots$, a_{k+2001} constituent une suite de 2002 entiers consécutifs. (Note : $\lfloor x\rfloor$ désigne le plus grand entier plus petit ou égal à x.)

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We given Bornsztein's write-up.

Let $n \in \mathbb{N}$. There exists a unique non-negative integer k such that $k^2 \le n < (k+1)^2$. Then $k = \lfloor\sqrt{n}\rfloor$ and $a_n = n + k$.

It follows that, for $q = 0, 1, \dots, 2k$, the numbers a_{k^2+q} are $2k+1$ consecutive integers. Moreover, we have $a_{(k+1)^2-1} = (k+1)^2 - 1 + k$ and $a_{(k+1)^2} = (k+1)^2 + (k+1) = a_{(k+1)^2-1} + 2$.

Hence, the sequence $\{a_n\}$ is formed by groups of $1, 3, \dots, 2n+1, \dots$ consecutive integers, with a gap of 2 between consecutive groups. For $n \ge 1$, the first term of the n^{th} group (the one with length $2n-1$) is $a_{(n-1)^2}$. Thus, the first group of at least 2002 consecutive integers is the one with length $2003 = 2 \times 1002 - 1$ and first term a_{1001^2}.

Therefore, the least integer k such that a_k, a_{k+1}, $\dots$, a_{k+2001} are 2002 consecutive integers is $k = 1001^2$.

2. (a) Dans le plan, soient $AB_1C_1D_1$ et $AB_2C_2D_2$ deux carrés ayant un sommet commun (les sommets sont cités dans le même sens). Si B, C et D sont respectivement les milieux des segments $[B_1B_2]$, $[C_1C_2]$ et $[D_1D_2]$, le quadrilatère $ABCD$ est-il aussi un carré ?

(b) Qu'en est-il si les sommets des carrés $AB_1C_1D_1$ et $AB_2C_2D_2$ sont cités en sens opposés ?

Solved by Bruce Crofoot, Thompson Rivers University, Kamloops, BC; and Pavlos Maragoudakis, Pireas, Greece. We present Crofoot's solution.

Place the squares in the complex plane $\mathbb{C}$ with A at the origin. There is some non-zero complex number ζ such that the square $AB_2C_2D_2$ is the image of $AB_1C_1D_1$ under the transformation $z \mapsto \zeta z$ ($z \in \mathbb{C}$). Without loss of generality, assume that the square $AB_1C_1D_1$ has sides of length 1. Place this square so that $B_1 = 1$, $C_1 = 1+i$, and $D_1 = i$.

Note that $B = \frac{1}{2}(B_1 + B_2)$, since B is the mid-point of the segment B_1B_2. Similarly, $C = \frac{1}{2}(C_1 + C_2)$ and $D = \frac{1}{2}(D_1 + D_2)$.

(a) Suppose the vertices A, B_2, C_2, D_2 are in counterclockwise order (the same as the order of A, B_1, C_1, D_1). Then $B_2 = \zeta B_1 = \zeta$, $C_2 = \zeta C_1 = \zeta(1+i)$, and $D_2 = \zeta D_1 = \zeta i$. Hence,

$$B = \tfrac{1}{2}(1+\zeta)\,, \qquad C = \tfrac{1}{2}(1+\zeta)(1+i)\,, \qquad D = \tfrac{1}{2}(1+\zeta)i\,.$$

Therefore, the quadrilateral $ABCD$ is the image of $AB_1C_1D_1$ under the transformation $z \mapsto \frac{1}{2}(1+\zeta)z$ $(z \in \mathbb{C})$. It follows that $ABCD$ is a square unless $\zeta = -1$.

When $\zeta = -1$, the squares $AB_1C_1D_1$ and $AB_2C_2D_2$ are the same size and the angle of rotation from one to the other is 180°. In this case, the points A, B, C, D coincide and there is no quadrilateral $ABCD$.

(b) Suppose the vertices A, B_2, C_2, D_2 are in clockwise order (the opposite of the order of A, B_1, C_1, D_1). Then $B_2 = \zeta D_1 = \zeta i$, $C_2 = \zeta C_1 = \zeta(1+i)$, and $D_2 = \zeta B_1 = \zeta$. Hence,

$$B = \tfrac{1}{2}(1+\zeta i)\,, \qquad C = \tfrac{1}{2}(1+\zeta)(1+i)\,, \qquad D = \tfrac{1}{2}(i+\zeta)\,.$$

Thus, $\overrightarrow{AB} = \frac{1}{2}(1+\zeta i)$ and $\overrightarrow{AD} = \frac{1}{2}(i+\zeta)$.

We have $|\overrightarrow{AB}| = |\overrightarrow{AD}|$ if and only if $|1+\zeta i|^2 = |i+\zeta|^2$; that is,

$$(1+\zeta i)(1-\overline{\zeta} i) = (i+\zeta)(-i+\overline{\zeta})\,,$$

which simplifies to $\zeta = \overline{\zeta}$. Thus, $|\overrightarrow{AB}| = |\overrightarrow{AD}|$ if and only if $\mathfrak{Im}\zeta = 0$.

We have $\overrightarrow{AB} \perp \overrightarrow{AD}$ if and only if $\mathfrak{Re}\{(1+\zeta i)(\overline{i+\zeta})\} = 0$. Since

$$(1+\zeta i)(\overline{i+\zeta}) = (1+\zeta i)(-i+\overline{\zeta}) = 2\,\mathfrak{Re}\zeta + i(|\zeta|^2 - 1)\,,$$

we conclude that $\overrightarrow{AB} \perp \overrightarrow{AD}$ if and only if $\mathfrak{Re}\zeta = 0$.

From these calculations, we see that there is no case where both $|\overrightarrow{AB}| = |\overrightarrow{AD}|$ and $\overrightarrow{AB} \perp \overrightarrow{AD}$. Thus, $ABCD$ is never a square. (However, $ABCD$ is always a parallelogram when it is non-degenerate, as can be easily checked.)

3. Voici une vue partielle d'une table de multiplication dans laquelle un tableau rectangulaire a été sélectionné.

1	2	3	4	5	6	⋯
2	4	6	8	10	12	⋯
3	6	9	12	15	18	⋯
4	8	12	16	20	24	⋯
5	10	15	20	25	30	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋱

Pour chaque tableau dont l'élément du coin supérieur gauche et celui du coin inférieur droit sont respectivement 1 et 2002, on calcule la somme de tous ses éléments. Quelle est la plus petite des sommes ainsi obtenues ?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's write-up.

Soient a et b les nombres situés respectivement dans le coin inférieur gauche et dans le coin supérieur droit du tableau sélectionné. Ainsi on a $ab = 2002$. La somme $S(a,b)$ de tous les nombres du tableau est

$$\begin{aligned} S(a,b) &= \sum_{k=1}^{a}(k + 2k + \cdots + bk) = \left(\sum_{k=1}^{a} k\right)\left(\sum_{k=1}^{b} k\right) \\ &= \frac{a(a+1)b(b+1)}{4} = \frac{2002(2003+a+b)}{4}, \end{aligned}$$

comme $ab = 2002$.

Il s'agit donc de minimiser $a+b$ sous la contrainte $ab = 2002$. Comme $2002 = 2 \times 7 \times 11 \times 13$, le nombre 2002 a 16 diviseurs positifs. Par symétrie des rôles, il suffit d'étudier les cas $a \in \{1,\ 2,\ 7,\ 11,\ 13,\ 14,\ 22,\ 26\}$.

On vérifie à la main que la valeur minimale de $a + b$ est alors $103 = 26 + 77$. La somme minimale cherchée est donc

$$S(26,77) = \frac{2002(2003+103)}{4} = 1054053\,.$$

4. Trouver tous les nombres premiers a et b tels que $a^{a+1} + b^{b+1}$ est aussi un nombre premier.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's write-up.

Soient a et b deux nombres premiers tel que $a^{a+1} + b^{b+1} = p$ est premier. Comme $a,\, b \geq 2$, on a $p > 2$ et donc p est impair. Par suite, parmi a et b l'un est pair et l'autre est impair, disons a pair. Comme a est premier, c'est donc que $a = 2$.

L'équation se réécrit alors $8 + b^{b+1} = p$. Si $b \geq 5$ alors, puisque $b+1$ est pair, on a $b^{b+1} \equiv 1 \pmod 3$ et donc $p \equiv 0 \pmod 3$. Or, $p > 8$, donc p ne peut être premier. Par suite $b = 3$. On a alors $p = 89$, qui est bien premier.

Finalement, à l'ordre près, le seule solution est $(a,b) = (2,3)$.

That completes the *Corner* for this issue. Send me your nice solutions, and soon for problems that have appeared in 2006 numbers of the *Corner*.