THE OLYMPIAD CORNER

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In this issue we present the problems of the three rounds of the Iranian Mathematical Olympiad 2002. Thanks go to Andy Liu, Canadian Team Leader to the IMO 2003 in Japan, for collecting the contests for our use.

IRANIAN MATHEMATICAL OLYMPIAD 2002

First Round

Time: 2×4.5 hours

1. Find all permutations $(a_1, \dots, a_n)$ of $(1, \dots, n)$ which have the property that $i+1$ divides $2(a_1 + \cdots + a_i)$ for every i, $1 \le i \le n$.

2. A rectangle is partitioned into small rectangles so that the edges of the small rectangles are parallel to the edges of the first rectangle. We call a point *a cross point* if it belongs to four different small rectangles. We call a segment *maximal* if there is no other segment containing it.

Show that the number of maximal segments plus the number of cross points is 3 less than the number of small rectangles.

3. In the convex quadrilateral $ABCD$, we have $\angle ABC = \angle ADC = 135^\circ$. There are two points M and N on the rays AB and AD, respectively, such that $\angle MCD = \angle NCB = 90^\circ$. The circumcircles of AMN and ABD intersect at A and K. Prove that $AK \perp KC$.

4. Let A and B be two fixed points in the plane. Let $ABCD$ be a convex quadrilateral such that $AB = BC$, $AD = DC$, and $\angle ADC = 90^\circ$. Prove that there is a fixed point P such that, for every such quadrilateral $ABCD$ on the same side of the line AB, the line DC passes through P.

5. Let δ be a symbol such that $\delta \ne 0$ and $\delta^2 = 0$. Define

$$\begin{aligned} \mathbb{R}[\delta] &= \{a + b\delta \mid a\,,\, b \in \mathbb{R}\} \\ a + b\delta = c + d\delta &\iff a = c \text{ and } b = d\,, \\ (a+b\delta) + (c+d\delta) &= (a+c) + (b+d)\delta\,, \\ (a+b\delta)\cdot(c+d\delta) &= ac + (ad+bc)\delta\,. \end{aligned}$$

Let $P(x)$ be a polynomial with real coefficients. Show that $P(x)$ has a multiple root in $\mathbb{R}$ if and only if $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

6. Let G be a simple graph with 100 edges on 20 vertices. We can choose a pair of disjoint edges in 4050 ways. Prove that G is regular.

Second Round

Time: 2×4.5 hours

1. The sequence $\{a_n\}$ is defined by $a_0 = 2$, $a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$. Show that if p is a prime factor of $a_{2k} - 2$, then p is a factor of $a_{2k+1} - 1$.

2. Let A be a point outside the circle Ω. The tangents from A to Ω touch Ω at B and C. A tangent L to Ω intersects AB and AC at P and Q, respectively. The line parallel to AC passing through P meets BC at R. Prove that as L varies, QR passes through a fixed point.

3. An ant moves on a straight path on the surface of a cube. If the ant reaches an edge, it goes on in such a way that if the cube were opened to make the adjacent faces coplanar, the path would become a straight line. If the ant reaches a vertex, it returns on the same path.

(a) Show that for every starting point of the ant, there are infinitely many directions for the ant to move in a periodic path.

(b) Show that if the ant starts on a fixed face, the periodicity of the path depends only on the direction (not the starting point).

4. Find the smallest positive integer n for which the following condition holds: For every finite set of points in the plane, if, for every n points in this set, there exist two lines covering all n points, then there exist two lines covering all points in the set.

5. Let I be the incentre of triangle ABC. Assume that the incircle touches AB and AC at X and Y, respectively. The line through X and I meets the incircle at M. Let X' be the point of intersection of AB and CM. Point L is on the segment $X'C$ such that $X'L = CM$. Prove that A, L, and Y are collinear if and only if $AB = AC$.

6. Let a, b, and c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that $a + b + c \leq 3$.

Third Round

Time: 2×4.5 hours

1. Find all real polynomials $P(x)$ such that $P(a) \in \mathbb{Z}$ implies that $a \in \mathbb{Z}$.

2. Let E be a fixed ellipse. Let B_1 be an arbitrary point outside E. The tangent from B_1 to E touches E at a point C_1. Let B_2 be a point on the line of B_1C_1 such that $B_1C_1 = C_1B_2$. For each positive integer i, define B_{i+1} in terms of B_i in this manner. Prove that the sequence $\{B_i\}$ is bounded in the plane.

3. In a triangle ABC, define C_a to be the circle tangent to AB, to AC, and to the incircle of the triangle ABC, and let r_a be the radius of C_a. Define r_b and r_c in the same way. Prove that $r_a + r_b + r_c \geq 4r$, where r is the inradius of the triangle ABC.

4. Let n and k be integers such that $2 \leq k \leq n$. Let $\mathcal{F}$ be a subset of $P(\{1, \ldots, n\})$ with the property that, for every F, $G \in \mathcal{F}$, there exists an integer t such that $1 \leq t \leq n$ and $\{t, t+1, \ldots, t+k-1\} \subseteq F \cap G$. Prove that $|\mathcal{F}| \leq 2^{n-k}$.

5. For every real number x define $\langle x \rangle = \min(\{x\}, \{1-x\})$, where $\{x\}$ denotes the fractional part of x. Prove that, for every irrational number α and every positive real number ε, there exists a positive integer n such that $\langle n^2\alpha \rangle < \varepsilon$.

We next give an alternative solution to problem 4 of the Hong Kong (China) Contest, for which we published a solution in the December 2005 number of the *Corner*.

4. [2004 : 84; 2005 : 522–523] *Hong Kong (China) Olympiad 1999.*

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all x, $y \in \mathbb{R}$,

$$f\big(x + yf(x)\big) = f(x) + xf(y) .$$

Alternate solution by B.J. Venkatachala, Indian Institute of Science, Bangalore, India.

The function which is identically 0 clearly satisfies the given condition,

$$f\big(x + yf(x)\big) = f(x) + xf(y) . \qquad (1)$$

Now let f be any other function satisfying this condition. We will show that f must be both additive and multiplicative, which implies that $f(x) = x$ for all $x \in \mathbb{R}$.

Taking $x = 1$ and $y = 0$ in (1), we get $f(0) = 0$. If $f(x) = 0$ for some x, then

$$0 = f(x) = f\big(x + yf(x)\big) = f(x) + xf(y) = xf(y) .$$

Choosing y such that $f(y) \neq 0$, we see that $x = 0$. Thus, $f(x) = 0$ implies $x = 0$.

Putting $x = 1$, we get $f\big(1 + yf(1)\big) = f(1) + f(y)$, for all $y \in \mathbb{R}$. If $f(1) \neq 1$, we may choose $y = 1/\big(1 - f(1)\big)$. This gives $1 + yf(1) = y$; hence, we obtain $f(y) = f\big(1+yf(1)\big) = f(1)+f(y)$ forcing $f(1) = 0$. This leads to the absurdity that $1 = 0$. Hence, $f(1) = 1$. Taking $x = 1$ in (1), we obtain $f(1 + y) = 1 + f(y)$ for all $y \in \mathbb{R}$.

Take any $x \neq 0$. Then $f(x) \neq 0$. Setting $y = 1/f(x)$ in (1), we get

$$f(x+1) \;=\; f(x) + xf\left(\frac{1}{f(x)}\right).$$

We conclude that $f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$ for all $x \neq 0$. Replacing y in (1) by $y/f(x)$ with $x \neq 0$, we get

$$f(x+y) \;=\; f(x) + xf\left(\frac{y}{f(x)}\right), \tag{2}$$

valid for all $x \neq 0$ and $y \in \mathbb{R}$. Replacing x by $1/f(x)$ in (2) gives

$$f\left(\frac{1+yf(x)}{f(x)}\right) \;=\; \frac{1}{x} + \frac{1}{f(x)}f(yx), \tag{3}$$

which is again valid for all $x \neq 0$ and $y \in \mathbb{R}$. Replacing y in (2) by $1 + yf(x)$ and using (3), we obtain

$$f\big(x+1+yf(x)\big) \;=\; f(x) + xf\left(\frac{1+yf(x)}{f(x)}\right) \;=\; f(x) + 1 + \frac{x}{f(x)}f(yx).$$

Since $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$, this simplifies to

$$f\big(x+yf(x)\big) \;=\; f(x) + \frac{x}{f(x)}f(yx).$$

We then use (1) to obtain $xf(y) = \frac{x}{f(x)}f(yx)$. Since $x \neq 0$, we get $f(xy) = f(x)f(y)$. This last equation is valid for $x = 0$, since $f(0) = 0$. Thus, $f(xy) = f(x)f(y)$ for all x, $y \in \mathbb{R}$. Using this in (2), we get additivity:

$$f(x+y) \;=\; f(x) + xf(y)f\big(1/f(x)\big) \;=\; f(x) + f(y),$$

for all x, $y \in \mathbb{R}$. Thus, f is both additive and multiplicative. Since f is not the zero function, it follows that $f(x) = x$ for all $x \in \mathbb{R}$.

We now turn to solutions from our readers to problems of the 2nd Czech-Polish-Slovak Mathematical Competition, written in Zwardoń, Poland, June 2002 and given in [2005 : 152–153].

4. An integer $n > 1$ and a prime p are such that n divides $p-1$, and p divides $n^3 - 1$. Show that $4p - 3$ is the square of an integer.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Since n divides $p-1$, we deduce that $p \equiv 1 \pmod{n}$ and $n \leq p-1$. It follows that $n - 1 < p$. Since p is prime, we have $\gcd(n-1, p) = 1$.

Since p divides $n^3 - 1 = (n-1)(n^2+n+1)$, it follows from Gauss' Theorem that p divides n^2+n+1. Let $n^2+n+1 = kp$, where k is a positive integer. Then $k \equiv kp = n^2+n+1 \equiv 1 \pmod{n}$. Moreover,

$$k \;=\; \frac{n^2+n+1}{p} \;\leq\; \frac{n^2+n+1}{n+1} \;<\; n+1\,.$$

Therefore, $k = 1$ and $p = n^2+n+1$. Then

$$4p-3 \;=\; 4(n^2+n+1)-3 \;=\; (2n+1)^2\,,$$

and we are done.

5. In an acute-angled triangle ABC with circumcentre O, points P and Q lying respectively on sides AC and BC are such that

$$\frac{AP}{PQ} \;=\; \frac{BC}{AB} \quad\text{and}\quad \frac{BQ}{PQ} \;=\; \frac{AC}{AB}\,.$$

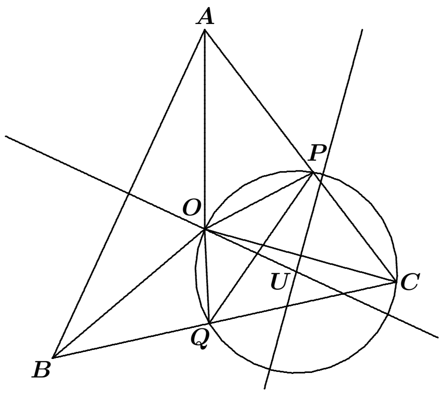
Show that the points O, P, Q, and C are concyclic.

Solution by Michel Bataille, Rouen, France.

We will use standard notation for the sides, angles, and circumradius of $\triangle ABC$. Define $k = AP/a$. Using the given equations, we get

$$k \;=\; \frac{AP}{a} \;=\; \frac{BQ}{b} \;=\; \frac{PQ}{c}\,.$$

Then $CP = b - AP = b - ka$ and $CQ = a - BQ = a - kb$.



The Law of Cosines gives

$$\begin{aligned} k^2c^2 &= PQ^2 \;=\; (a-kb)^2 + (b-ka)^2 - 2(a-kb)(b-ka)\cos C \\ &= (1+k^2)(a^2+b^2-2ab\cos C) - 4kab + 2k(a^2+b^2)\cos C \\ &= (1+k^2)c^2 - 2k\big(2ab - (a^2+b^2)\cos C\big)\,, \end{aligned}$$

and hence,

$$c^2 \;=\; 2k\big(2ab - (a^2+b^2)\cos C\big)\,.$$

But, using the Law of Sines, we have

$$\begin{aligned}
&2ab - (a^2+b^2)\cos C\\
&\qquad = \; 8R^2\sin A\sin B - 4R^2\cos C(\sin^2 A + \sin^2 B)\\
&\qquad = \; 4R^2\big[\cos(A-B) - \cos(A+B) - \cos C(\sin^2 A + \sin^2 B)\big]\\
&\qquad = \; 4R^2\sin^2 C\cos(A-B) \;=\; c^2\cos(A-B)\,,
\end{aligned}$$

where we have used the identity

$$\begin{aligned}
\cos(A-B)\cos(A+B) &= \cos^2 A(1-\sin^2 B) - \sin^2 B(1-\cos^2 A)\\
&= \cos^2 A - \sin^2 B\,.
\end{aligned}$$

It follows that $k = \dfrac{1}{2\cos(A-B)}$.

Since $\angle AOC = 2B$ and $OA = OC$, we have $\angle OAP = 90^\circ - B$. Thus,

$$\begin{aligned}
OP^2 &= OA^2 + AP^2 - 2OA\cdot AP\cos(90^\circ - B)\\
&= R^2 + k^2a^2 - 2kRa\sin B\\
&= 4R^2k^2\left(\frac{1}{4k^2} + \sin^2 A - \frac{1}{k}\sin A\sin B\right)\\
&= 4R^2k^2\big[\cos^2(A-B) + \sin^2 A - 2\cos(A-B)\sin A\sin B\big]\\
&= 4R^2k^2\big[\cos^2(A-B) + \sin^2 A\\
&\qquad\qquad - \cos(A-B)\big(\cos(A-B) - \cos(A+B)\big)\big]\\
&= 4R^2k^2(\sin^2 A + \cos^2 A - \sin^2 B) \;=\; 4R^2k^2\cos^2 B\,.
\end{aligned}$$

Therefore, $OP = 2kR\cos B$. Similarly, $OQ = 2kR\cos A$. Now,

$$\begin{aligned}
&OP\cdot CQ + OQ\cdot CP\\
&\quad = \; 4kR^2(\cos B\sin A - k\sin B\cos B + \cos A\sin B - k\sin A\cos A)\\
&\quad = \; 4kR^2\big[\sin(A+B) - k\sin(A+B)\cos(A-B)\big]\\
&\quad = \; 4kR^2\left(\sin C - \tfrac{1}{2}\sin C\right) \;=\; 2kR^2\sin C \;=\; R\cdot kc \;=\; OC\cdot PQ\,,
\end{aligned}$$

where we have used the identity

$$\sin(A+B)\cos(A-B) \;=\; \sin A\cos A + \sin B\cos B\,.$$

It follows from Ptolemy's Theorem that O, P, Q, C are concyclic.

Note: Since $AP\cdot AC = BQ\cdot BC$, the points A and B have the same power with respect to the circumcircle Γ of $\triangle CPQ$. Thus, letting U be the centre of Γ and ρ be the radius of Γ, we have $UA^2 - \rho^2 = UB^2 - \rho^2$. It follows that $UA = UB$, and U is on the perpendicular bisector of AB. This remark provides an easy construction of points P, Q satisfying the conditions of the problem: draw the perpendicular bisectors of AB and OC, which meet at U. Then the circle with centre U passing through C meets the sides AC and BC again at P and Q, respectively.

Next we move to the May 2005 number of the *Corner* and readers' solutions to problems of the Singapore Mathematical Olympiad 2002, Open Section, Part A, given in [2005 : 215–216].

1. Let $f(x)$ be a function which satisfies

$$f(29+x) \;=\; f(29-x)\,,$$

for all values of x. If $f(x)$ has exactly three real roots α, β, and γ, determine the value of $\alpha+\beta+\gamma$.

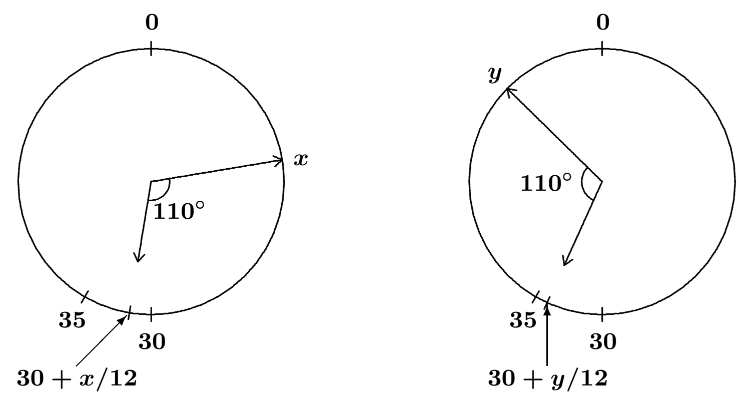
Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We given Krimker's solution.

Since f has exactly three real roots and f has the same value at points symmetric about 29, one of the roots must be 29. Let $\gamma = 29$. The other two roots, α and β, must be symmetric about 29; hence, $\alpha = 29+x$ amd $\beta = 29-x$ for some real number $x \neq 0$. Therefore,

$$\alpha+\beta+\gamma \;=\; (29+x)+(29-x)+29 \;=\; 87\,.$$

2. John left town A at x minutes past 6:00 pm and reached town B at y minutes past 6:00 pm the same day. He noticed that at both the beginning and the end of the trip, the minute hand made the same angle of 110 degrees with the hour hand on his watch. How many minutes did it take John to go from town A to town B?

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Geoffrey A. Kandall, Hamden, CT, USA; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Kandall.



It is implicit that $0 < x < y < 60$. Since an angle of 110° corresponds to $55/3$ minutes, we have the following equations:

$$\left(30 + \frac{x}{12}\right) - x \;=\; \frac{55}{3}\,, \qquad y - \left(30 + \frac{y}{12}\right) \;=\; \frac{55}{3}\,.$$

Hence, $\frac{11}{12}x = 30 - \frac{55}{3}$ and $\frac{11}{12}y = 30 + \frac{55}{3}$, from which we get $y - x = 40$. This is the time of the trip in minutes.

3. Let $x_1 = \dfrac{1}{2002}$. For $n \geq 1$, define $nx_{n+1} = (n+1)x_n + 1$. Find x_{2002}.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the write-up of Krimker.

The sequence $x_n = n/2002 + n - 1$ satisfies the recurrence relation of the problem. Indeed, $x_1 = 1/2002$, and

$$\begin{aligned} nx_{n+1} &= n\left(\frac{n+1}{2002} + n\right) \;=\; \frac{n(n+1)}{2002} + n^2 \\ &= (n+1)\left(\frac{n}{2002} + n - 1\right) + 1 \;=\; (n+1)x_n + 1\,. \end{aligned}$$

Then, $x_{2002} = 2002$.

4. For integers $n \geq 1$, let $a_n = n^2 + 500$ and $d_n = \gcd(a_n, a_{n+1})$. Determine the largest value of d_n.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Wang, modified by the editor.

The answer is 2001, attained when $n = 1000$.

Since d_n divides both a_n and a_{n+1}, it follows that d_n divides

$$a_{n+1} - a_n \;=\; (n+1)^2 + 500 - (n^2 + 500) \;=\; 2n + 1\,.$$

Then d_n divides $n(2n+1) = 2n^2 + n$, and consequently, d_n also divides $2n^2 + n - 2a_n = n - 1000$. Since $2n + 1 - 2(n - 1000) = 2001$, we deduce that $d_n \mid 2001$.

Suppose $d_n = 2001 = 3 \cdot 23 \cdot 29$. Then d_n is divisible by 3, 23, and 29. Since $2n + 1 \equiv 0 \pmod 3$, we have $2n \equiv -1 \equiv 2 \pmod 3$; that is,

$$n \;\equiv\; 1 \pmod 3\,. \tag{1}$$

Similarly, since $2n + 1 \equiv 0 \pmod{23}$, we have $2n \equiv -1 \equiv 22$, or

$$n \;\equiv\; 11 \pmod{23}\,, \tag{2}$$

and since $2n+1 \equiv 0 \pmod{29}$, we have $2n \equiv -1 \equiv 28$, or

$$n \equiv 14 \pmod{29} . \qquad (3)$$

Applying the Chinese Remainder Theorem and using the standard method, we let $M_1 = 23 \cdot 29 = 667$, $M_2 = 3 \cdot 29 = 87$ and $M_3 = 3 \cdot 23 = 69$, and we then solve the following system of congruences:

$$\begin{aligned} 667x &\equiv 1 \pmod{3} , & (4)\\ 87x &\equiv 1 \pmod{23} , & (5)\\ 69x &\equiv 1 \pmod{29} . & (6) \end{aligned}$$

By routine methods, we find the least positive solutions of (4), (5), and (6) to be $x_1 = 1$, $x_2 = 9$, and $x_3 = 8$, respectively. Hence, a solution to the system (1), (2), and (3) is given by

$$n = 1 \cdot 667 \cdot 1 + 11 \cdot 87 \cdot 9 + 14 \cdot 69 \cdot 8 = 17008 \equiv 1000 \pmod{2001} .$$

Conversely, when $n = 1000$, we have

$$\begin{aligned} d_n &= \gcd(1000^2 + 500, 1001^2 + 500) \\ &= \gcd(500 \cdot 2001, 501 \cdot 2001) = 2001 . \end{aligned}$$

Thus, 2001 is a value for d_n (attained when $n = 1000$). There cannot be any larger value for d_n, since d_n divides 2001.

5. It is given that the polynomial $p(x) = x^3 + ax^2 + bx + c$ has three distinct positive integer roots and $p(2002) = 2001$. Let $q(x) = x^2 - 2x + 2002$. It is also given that the polynomial $p\big(q(x)\big)$ has no real roots. Determine the value of a.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Pavlos Maragoudakis, Pireas, Greece. We give the solution by Krimker, modified by the editor.

The polynomial $q(x) = (x-1)^2 + 2001$ takes on every value in the interval $[2001, \infty)$. Since $p\big(q(x)\big)$ has no real roots, all three roots of $p(x)$ must be less than 2001. Denoting the roots of $p(x)$ by x_1, x_2, x_3, we have

$$p(x) = (x - x_1)(x - x_2)(x - x_3) .$$

Then $p(2002) = (2002 - x_1)(2002 - x_2)(2002 - x_3) = 2001 = 3 \cdot 23 \cdot 29$. Since each factor $2002 - x_i$ is a positive integer, we must have

$$\{2002 - x_1,\ 2002 - x_2,\ 2002 - x_3\} = \{3,\ 23,\ 29\} ,$$

and hence $\{x_1, x_2, x_3\} = \{1999, 1979, 1973\}$. Then

$$a = -(x_1 + x_2 + x_3) = -5951 .$$

6. Find the largest positive integer N such that $N!$ ends with exactly twenty-five "zero" digits.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

Let $f(N)$ denote the number of zeroes at the end of $N!$. Then $f(N)$ equals the number of factors of 10 that may be formed in $N!$. Since $10 = 5\times 2$ and there are clearly fewer 5s than 2s, it follows that $f(N)$ is equal to the number of factors of 5 in $1\cdot 2\cdot 3\cdots N$. Then (by a well-known formula)

$$f(N) \;=\; \sum_{k=0}^{\infty} \left\lfloor \frac{N}{5^k} \right\rfloor .$$

By straightforward computations, we find that

$$f(109) \;=\; \left\lfloor \frac{109}{5} \right\rfloor + \left\lfloor \frac{109}{25} \right\rfloor \;=\; 21+4 \;=\; 25\,,$$

while

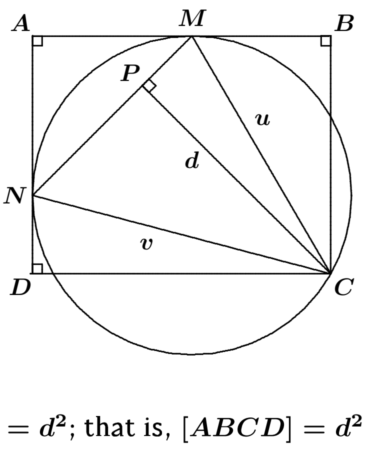
$$f(110) \;=\; \left\lfloor \frac{110}{5} \right\rfloor + \left\lfloor \frac{100}{25} \right\rfloor \;=\; 22+4 \;=\; 26\,.$$

Since f is an increasing function, we see that the required value of N is 109.

7. A circle passes through the vertex C of a rectangle $ABCD$ and touches its sides AB and AD at points M and N, respectively. Suppose the distance from C to MN is 2 cm. Find the area of $ABCD$ in cm^2.

Solved by Bruce Crofoot, Thompson Rivers University, Kamloops, BC; Geoffrey A. Kandall, Hamden, CT, USA; and Pavlos Maragoudakis, Pireas, Greece. We present a composite of the solutions by Crofoot and Kandall.

The answer is $[ABCD] = 4$ cm^2.



More generally, let d be the distance from C to MN. We introduce additional notation as shown in the diagram. Since AB is tangent to the circle at M, we have

$$\angle BMC \;=\; \angle MNC \;=\; \angle PNC\,.$$

Hence, $\triangle BMC$ is similar to $\triangle PNC$. Similarly, $\triangle DNC$ is similar to $\triangle PMC$. Consequently,

$$\frac{BC}{d} \;=\; \frac{u}{v} \quad\text{and}\quad \frac{DC}{d} \;=\; \frac{v}{u}\,.$$

From these two equations, we get $BC\cdot DC = d^2$; that is, $[ABCD] = d^2$.

8. Let $m = 144^{\sin^2 x} + 144^{\cos^2 x}$. How many such m's are integers?

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Wang.

The answer is 122. Indeed, we prove that all the integers from 24 to 145 inclusive are attainable.

Consider the function $f(t) = 144^t + 144^{1-t}$ where $t = \sin^2 x$. Then $0 \le t \le 1$. Since $f'(t) = (\ln 144)(144^t - 144^{1-t})$, we have $f'(t) = 0$ if and only if $t = \frac{1}{2}$. Since $f'(t) > 0$ if and only if $\frac{1}{2} < t < 1$, we see that f is decreasing on $\left(0, \frac{1}{2}\right)$ and increasing on $\left(\frac{1}{2}, 1\right)$. Hence, the absolute minimum of f is $f\left(\frac{1}{2}\right) = 24$ (attained when $x = \frac{\pi}{4}$, for example), and the absolute maximum of f is $f(0) = f(1) = 145$ (attained when $x = 0$, for example).

Since f is a continuous function, the Intermediate Value Theorem then guarantees that every integer between 24 and 145 is also attainable, and our claim follows.

9. Evaluate $\sum\limits_{k=1}^{2002} \frac{k \cdot k!}{2^k} - \sum\limits_{k=1}^{2002} \frac{k!}{2^k} - \frac{2003!}{2^{2002}}$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Krimker's write-up.

The expression we are trying to evaluate may be rewritten as

$$\sum_{k=1}^{n} \frac{(k-1)k!}{2^k} - \frac{(n+1)!}{2^n},$$

where $n = 2002$. We will prove by induction that this is equal to -1 for every positive integer n.

Note that the claim is true for $n = 1$. Suppose now that $n \ge 2$ and that the equality is valid for $n - 1$. We will show that it holds for n. Indeed,

$$\begin{aligned}
\sum_{k=1}^{n} \frac{(k-1)\cdot k!}{2^k} &= \sum_{k=1}^{n-1} \frac{(k-1)\cdot k!}{2^k} + \frac{n!(n-1)}{2^n} \\
&= \frac{n!}{2^{n-1}} - 1 + \frac{n!(n-1)}{2^n} \\
&= \frac{2n! + n!(n-1)}{2^n} - 1 = \frac{n!(2+n-1)}{2^n} - 1 \\
&= \frac{n!(n+1)}{2^n} - 1 = \frac{(n+1)!}{2^n} - 1 .
\end{aligned}$$

Thus, the claim is true for all positive integers n.

10. How many ways are there to arrange 5 identical red, 5 identical blue, and 5 identical green marbles in a straight line such that every marble is adjacent to at least one marble of the same colour as itself?

Solution by Pavlos Maragoudakis, Pireas, Greece.

There are 426 ways.

Each set of 5 marbles of the same colour must remain together or else be separated into two groups, with 2 adjacent marbles and 3 adjacent marbles. Thus, we have the following cases:

(i) The three quintuplets of the same colour remain 'united'.

There are then 3 groups of marbles. We have $3! = 6$ ways to arrange them.

(ii) We choose one colour and 'break' it into the two possible parts, while we leave the other two colours 'united'.

There are then 4 groups of marbles. The two parts of the 'broken' colour should be non-adjacent among the 4 groups; thus, we have 3 choices: (1^{st},3^{rd}), or (1^{st},4^{th}), or (2^{nd},4^{th}). We have $3 \cdot 3 \cdot 2 \cdot 2 = 36$ ways.

(iii) We choose two colours and 'break' them.

There are then 5 groups of marbles. We place the two parts of the 1^{st} 'broken' colour in two non-adjacent spots, and we do the same for the 2^{nd} 'broken' colour. We have 6 choices.

1^{st}colour	2^{nd}colour
(1^{st},3^{rd})	(2^{nd},4^{th})
(1^{st},3^{rd})	(2^{nd},5^{th})
(1^{st},4^{th})	(2^{nd},5^{th})
(1^{st},4^{th})	(3^{rd},5^{th})
(1^{st},5^{th})	(2^{nd},4^{th})
(2^{nd},4^{th})	(3^{rd},5^{th})

We have $6 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 144$ ways.

(iv) We 'break' all colours.

This gives 6 groups of marbles. We have 5 choices:

1^{st}colour	2^{nd}colour	3^{rd}colour
(1^{st},3^{rd})	(2^{nd},5^{th})	(4^{th},6^{th})
(1^{st},4^{th})	(2^{nd},5^{th})	(3^{rd},6^{th})
(1^{st},5^{th})	(2^{nd},4^{th})	(3^{rd},6^{th})
(1^{st},6^{th})	(2^{nd},4^{th})	(3^{rd},5^{th})
(1^{st},4^{th})	(2^{nd},4^{th})	(3^{rd},5^{th})

We have $5 \cdot 3 \cdot 2 \cdot 2 \cdot 2 \cdot = 240$ ways.

Altogether, we have $6 + 36 + 144 + 240 = 426$ ways.

Now we turn to Part B of the Singapore Mathematical Olympiad given in [2005 : 216].

2. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers between 1001 and 2002 inclusive. Suppose $\sum\limits_{i=1}^{n} a_i^2 = \sum\limits_{i=1}^{n} b_i^2$. Prove that

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \;\ge\; \frac{17}{10}\sum_{i=1}^{n} a_i^2\,.$$

Determine when equality holds.

Solution by Pierre Bornsztein, Maisons-Laffitte, France, modified by the editor.

There is a misprint. The given inequality is false if $a_i = b_i$ for each i. The correct inequality is

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \;\le\; \frac{17}{10}\sum_{i=1}^{n} a_i^2\,.$$

We will now prove this inequality.

For each i, we have

$$\frac{1}{2} \;=\; \frac{1001}{2002} \;\le\; \frac{a_i}{b_i} \;\le\; \frac{2002}{1001} \;=\; 2\,,$$

and therefore $(2a_i - b_i)(2b_i - a_i) \ge 0$; that is,

$$5a_ib_i \;\ge\; 2(a_i^2 + b_i^2)\,. \tag{7}$$

Multiplying this inequality by a_i/b_i, we get

$$5a_i^2 \;\ge\; 2\frac{a_i^3}{b_i} + 2a_ib_i\,. \tag{8}$$

From (7), we have $2a_ib_i \ge \frac{4}{5}(a_i^2 + b_i^2)$. Using this inequality in (8), we obtain

$$5a_i^2 \;\ge\; 2\frac{a_i^3}{b_i} + \frac{4}{5}(a_i^2 + b_i^2)\,,$$

which may be rewritten as

$$\frac{a_i^3}{b_i} \;\le\; \frac{21}{10}a_i^2 - \frac{2}{5}b_i^2\,. \tag{9}$$

Note that equality occurs in (9) if and only if $b_i = 2a_i$ or $a_i = 2b_i$; that is, if and only if $(a_i, b_i) = (1001, 2002)$ or $(a_i, b_i) = (2002, 1001)$.

Summing over i in (9) and recalling that $\sum\limits_{i=1}^{n} b_i^2 = \sum\limits_{i=1}^{n} a_i^2$, we get

$$\sum_{i=1}^{n} \frac{a_i^3}{b_i} \;\le\; \frac{21}{10}\sum_{i=1}^{n} a_i^2 - \frac{2}{5}\sum_{i=1}^{n} a_i^2 \;=\; \frac{17}{10}\sum_{i=1}^{n} a_i^2\,,$$

as desired.

Equality occurs if and only if, for each i, either $(a_i, b_i) = (1001, 2002)$ or $(a_i, b_i) = (2002, 1001)$. The condition $\sum\limits_{i=1}^{n} a_i^2 = \sum\limits_{i=1}^{n} b_i^2$ can be rewritten as $1001^2 p + (n-p)2002^2 = 2002^2 p + 1001^2(n-p)$, which is $p = \frac{1}{2}n$. Thus, equality occurs if and only if n is even and $(a_i, b_i) = (1001, 2002)$ for half of the subscripts i while $(a_i, b_i) = (2002, 1001)$ for the other half.

3. Let n be a positive integer. Determine the smallest possible sum

$$a_1 b_1 + a_2 b_2 + \cdots + a_{2n+2} b_{2n+2}\,,$$

where a_1, a_2, …, a_{2n+2} and b_1, b_2, …, b_{2n+2} are rearrangements of the binomial coefficients

$$\binom{2n+1}{0},\quad \binom{2n+1}{1},\quad \ldots,\quad \binom{2n+1}{2n+1}.$$

Justify your answer.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's solution.

According to the rearrangement inequality, the sum is minimized when one of the sequences a_1, a_2, …, a_{2n+2} and b_1, b_2, …, b_{2n+2} is increasing and the other is decreasing. Since the binomial coefficients are increasing from $\binom{2n+1}{0}$ to $\binom{2n+1}{n+1}$ and decreasing from $\binom{2n+1}{n+1}$ to $\binom{2n+1}{2n+1}$, the minimal sum is

$$2\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n+1+k} \;=\; 2\sum_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n-k},$$

where the last step uses the well-known identity $\binom{m}{j} = \binom{m}{m-j}$.

Now consider a group of $2n+1$ boys and $2n+1$ girls. We want to select n persons from this group. There are clearly $\binom{4n+2}{n}$ ways to do that. On the other hand, letting k be the number of boys in the selected group, we see that the total number of ways is also $\sum\limits_{k=0}^{n}\binom{2n+1}{k}\binom{2n+1}{n-k}$. Therefore, the minimal sum is $2\binom{4n+2}{n}$.

4. Find all real-valued functions $f : \mathbb{Q} \longrightarrow \mathbb{R}$ defined on the set of all rational numbers $\mathbb{Q}$ satisfying the conditions

$$f(x+y) \;=\; f(x)+f(y)+2xy\,,$$

for all x, y in $\mathbb{Q}$ and $f(1) = 2002$. Justify your answers.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We present Bataille's solution.

It is readily checked that the function $r \mapsto r(r+2001)$ is a solution. We will show that it is unique. Let f be an arbitrary solution. Denote by $\mathcal{C}$ the given condition $f(x+y) = f(x)+f(y)+2xy$, and fix $z \in \mathbb{Q}$, $z > 0$. It is easily proved that $f(nz) = n\big(f(z)+(n-1)z^2\big)$ for all $n \in \mathbb{N}$ (by induction, using $\mathcal{C}$ with $x = nz$ and $y = z$ for the inductive step). Then, for all $n \in \mathbb{N}$,

$$2002 \;=\; f(1) \;=\; f\left(n \times \frac{1}{n}\right) \;=\; n\left(f\left(\frac{1}{n}\right)+(n-1)\cdot\frac{1}{n^2}\right)\,,$$

and thus, $f\left(\frac{1}{n}\right) = \frac{1}{n}\left(2001+\frac{1}{n}\right)$. Then, for all positive integers m and n,

$$f\left(\frac{m}{n}\right) \;=\; f\left(m \times \frac{1}{n}\right) \;=\; m\left(f\left(\frac{1}{n}\right)+(m-1)\cdot\frac{1}{n^2}\right) \;=\; \frac{m}{n}\left(\frac{m}{n}+2001\right).$$

Thus, $f(r) = r(r+2001)$ holds for all positive $r \in \mathbb{Q}$. Since $f(0) = 0$ (condition $\mathcal{C}$ with $x = y = 0$) and $f(-r) = 2r^2 - f(r)$ (condition $\mathcal{C}$ with $x = r$ and $y = -r$), it can be verified that $f(r) = r(r+2001)$ actually holds for all $r \in \mathbb{Q}$. This completes the proof.

To finish this number of the *Corner*, we give solutions by our readers to problems of the XVIII Italian Mathematical Olympiad, Cesenatico, Italy, May 2002, given in [2005 : 217].

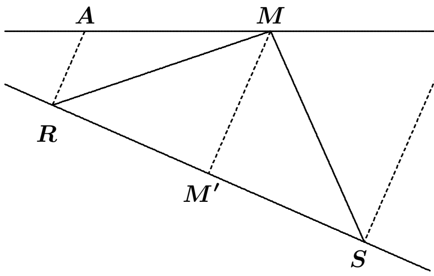
1. Find all 3–digit positive integers that are 34 times the sum of their digits.

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bilinski's write-up.

Let abc be a 3–digit positive integer with the required property (where a, b, and c are the digits). Then $100a+10b+c = 34(a+b+c)$, which simplifies to $22a = 8b+11c$. This equation implies that b is divisible by 11. Since b is a digit, we must have $b = 0$, and then $c = 2a$. This gives us the numbers 102, 204, 306, and 408. It can be verified that each of these is a solution.

3. Let A and B be two points of the plane, and let M be the mid-point of AB. Let r be a line, and let R and S be the projections of A and B onto r. Assuming that A, M, and R are not collinear, prove that the circumcircle of triangle AMR has the same radius as the circumcircle of BSM.

Solved by Michel Bataille, Rouen, France; and Pavlos Maragoudakis, Pireas, Greece. We present Bataille's solution.



Let M' be the projection of M onto r. Then M' is the mid-point of RS (since M is the mid-point of AB) and $MM' \perp RS$. It follows that $\triangle RMS$ is isosceles with

$$RM \;=\; MS\,. \tag{1}$$

Now, let ρ_a and ρ_b be the circumradii of $\triangle AMR$ and $\triangle BMS$, respectively. We have

$$2\rho_a \;=\; \frac{RM}{\sin(\angle RAM)}\,, \quad 2\rho_b \;=\; \frac{SM}{\sin(\angle SBM)}\,. \tag{2}$$

But $\angle RAM + \angle SBM = 180^\circ$ (since $AR \parallel BS$); hence,

$$\sin(\angle RAM) \;=\; \sin(\angle SBM)\,. \tag{3}$$

From (1), (2), and (3), we obtain $\rho_a = \rho_b$.

4. Find all values of n for which all solutions of the equation $x^3 - 3x + n = 0$ are integers.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Pireas, Greece. We give Bornsztein's solution.

If the given equation has an integer root α, then $n = -\alpha^3 + 3\alpha$ is an integer too. Therefore, the values of n that we seek must all be integers.

Let $f(x) = x^3 - 3x$. The given equation is then $f(x) = -n$. Straightforward computations show that f is increasing on $(-\infty, -1]$ and $[1, +\infty)$, and decreasing on $[-1, 1]$. Moreover, $f(-1) = 2$ and $f(1) = -2$. Thus, the equation $f(x) = -n$ has three real roots if and only if $|n| \leq 2$. Therefore, $n \in \{-2,\, -1,\, 0,\, 1,\, 2\}$. Furthermore, one of the integer solutions has to be -1, 0, or 1; thus, $n \in \{f(-1),\, f(0),\, f(1)\} = \{-2,\, 0,\, 2\}$. Direct checking shows that the desired values are $n = -2$ and $n = 2$.

5. Prove that, if $m = 5^n + 3^n + 1$ is prime, then 12 divides n.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Krimker's solution.

By the Division Algorithm, we have $n = 12q + r$ for some integers q and r such that $0 \leq r \leq 11$. In the following three cases we will apply Fermat's Little Theorem and elementary congruence properties.

Case 1. r is odd; that is, $r = 2k+1$. Then

$$m \;=\; 5^{12q+2k+1} + 3^n + 1 \;\equiv\; \left(5^2\right)^{6q}\left(5^2\right)^k 5 + 1 \;\equiv\; 5 + 1 \;\equiv\; 0 \pmod{3}\;.$$

Case 2. $r = 2$, $r = 6$, or $r = 10$; that is, $r = 4k+2$ with $0 \leq k \leq 2$. Then

$$m \;=\; 5^n + 3^{12q+4k+2} + 1 \;\equiv\; \left(3^4\right)^{3q}\left(3^4\right)^k 3^2 + 1 \;\equiv\; 3^2 + 1 \;\equiv\; 0 \pmod{5}\;.$$

Case 3. $r = 4$ or $r = 8$; that is $r = 4k$ with $k = 1$ or $k = 2$. Then

$$\begin{aligned} m \;&=\; 5^{12q+4k} + 3^{12q+4k} + 1 \;=\; \left(5^6\right)^{2q}\left(5^4\right)^k + \left(3^6\right)^{2q}\left(3^4\right)^k + 1 \\ &\equiv\; 2^k + 4^k + 1 \;\equiv\; 0 \pmod{7}\;. \end{aligned}$$

Since m is prime and $m \geq 9$, none of the cases above are possible. Thus, $r = 0$, and 12 divides n.

That completes this number of the *Corner*. This is a call for solutions—readers will have noted that we are rapidly clearing our backlog and will soon be in a position to publish solutions within a year of giving the contests in ***Crux Mathematicorum***. We need your contributions of nice solutions and generalizations, preferably within 8 months of the appearance of the problem.