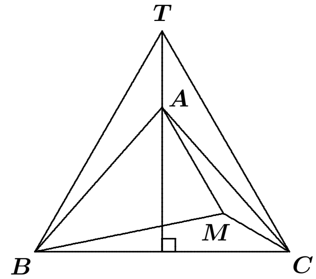
Mayhem Solutions

M188. *Proposed by Charalampos Stergiou, Chalkida, Greece.*

Consider triangle ABC in which $\angle B = \angle C = 35^\circ$. In the interior of the triangle we take a point M such that $\angle MBC = 25^\circ$ and $\angle MCB = 30^\circ$. Prove, without trigonometry, that $\angle AMC = 150^\circ$.

Solution by Titu Zvonaru, Comăneşti, Romania.

We will prove the following more general result: Given $\triangle ABC$ with $\angle B = \angle C = \alpha$, where $30^\circ < \alpha < 60^\circ$, let M be a point in the interior of the triangle such that $\angle MBC = 60^\circ - \alpha$ and $\angle MCB = 30^\circ$. Prove, without the aid of trigonometry, that $\angle AMC = 150^\circ$. (For $\alpha = 35^\circ$, we obtain the given problem.)



Let T be a point on the same side of BC as A such that $\triangle BCT$ is equilateral. Because $\triangle ABC$ and $\triangle BCT$ are both isosceles, AT is the perpendicular bisector of BC. Thus, $\angle BTA = 30^\circ$. In triangles BAT and BMC, we have $\angle ABT = 60^\circ - \alpha = \angle MBC$, $\angle ATB = 30^\circ = \angle MCB$, and $BC = BT$. Hence, $\triangle BAT$ and $\triangle BMC$ are congruent.

It follows that $AB = AM$ and $\triangle ABM$ is isosceles. Then

$$\begin{aligned} \angle ABT &= \angle MBC = 60^\circ - \alpha\,, \\ \angle ABM &= 60^\circ - 2(60^\circ - \alpha) = 2\alpha - 60^\circ\,, \\ \angle BAM &= \frac{180^\circ - \angle ABM}{2} = \frac{180^\circ - (2\alpha - 60^\circ)}{2} = 120^\circ - \alpha\,, \\ \text{and} \quad \angle MAC &= \angle BAC - \angle BAM = (180^\circ - 2\alpha) - (120^\circ - \alpha) \\ &= 60^\circ - \alpha\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \angle AMC &= 180^\circ - \angle ACM - \angle MAC \\ &= 180^\circ - (\alpha - 30^\circ) - (60^\circ - \alpha) = 150^\circ\,. \end{aligned}$$

Also solved by Alper Cay, Uzman Private School, Kayseri, Turkey.

M189. *Proposed by Mihály Bencze, Brasov, Romania.*

Find all real solutions of the following system of equations:

$$\begin{aligned} x + \sqrt{x^2+1} &= 10^{y-x}\,, \\ y + \sqrt{y^2+1} &= 10^{z-y}\,, \\ z + \sqrt{z^2+1} &= 10^{x-z}\,. \end{aligned}$$

Solution by the proposer.

The system can be rewritten as

$$\begin{aligned} x+\log\left(x+\sqrt{x^2+1}\right) &= y\,, \\ y+\log\left(y+\sqrt{y^2+1}\right) &= z\,, \\ z+\log\left(z+\sqrt{z^2+1}\right) &= x\,; \end{aligned}$$

that is, $f(x)=y$, $f(y)=z$, $f(z)=x$, where $f(x)=x+\log\left(x+\sqrt{x^2+1}\right)$. If $x>0$, then $x+\sqrt{x^2+1}>1$, which implies that $f(x)>x$. Similarly, if $x<0$, then $f(x)<x$.

Let (x,y,z) be a solution to the system. If $x>0$ we have

$$x\ <\ f(x)\ =\ y\ <\ f(y)\ =\ z\ <\ f(z)\ =\ x\,,$$

a contradiction. Similarly, $x<0$ leads to a contradiction. Thus, the only possible solution is $x=y=z=0$, which is indeed a solution.

M190. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Given any three points in a unit square, show that a pair of them must be no further apart than $\sqrt{6}-\sqrt{2}$.

Solution by the proposer.

Let $DEFG$ be a unit square, labelled clockwise, and let A, B, and C be points in the square. Extending the sides of $\triangle ABC$ if necessary, we may assume that A, B, C are on the sides of $DEFG$. Sliding a side of $\triangle ABC$ if necessary, we may further assume that $A=D$. Since $1<\sqrt{6}-\sqrt{2}$, it suffices to consider that B is on EF and C is on FG. If $AB>\sqrt{6}-\sqrt{2}$, then, by the Pythagorean Theorem,

$$EB\ =\ \sqrt{AB^2-AE^2}\ >\ \sqrt{\left(8-4\sqrt{3}\right)-1}\ =\ \sqrt{7-4\sqrt{3}}\ =\ 2-\sqrt{3}\,.$$

Likewise, if $AC>\sqrt{6}-\sqrt{2}$, then $GC>2-\sqrt{3}$. Consequently,

$$BC\ =\ \sqrt{BF^2+CF^2}\ <\ \sqrt{2\left(\sqrt{3}-1\right)^2}\ =\ \sqrt{6}-\sqrt{2}\,.$$

M191. *Proposed by the Mayhem Staff.*

The surface areas of the six faces of a rectangular prism (box) are 1254, 1254, 770, 770, 1995, and 1995 cm^2. Determine the volume of the prism.

Solution by Geneviève Lalonde, Massey, ON.

Let the dimensions of the prism, in cm, be a, b, c, with $a\le b\le c$. Then $ab=770$, $ac=1254$, and $bc=1995$. Multiplying these three equations together, we get $(abc)^2=770\cdot 1254\cdot 1995=1926332100$. Thus, the volume of the prism, in cm^3, is $abc=\sqrt{1926332100}=43890$.

Also solved by Andrew Fischer and Frank Barlow, Humke's Raiders, Washington and Lee University, Lexington, VA.

M192. *Proposed by Victor Oxman, Western Galilee College, Israel.*

In triangles $A_1B_1C_1$ and $A_2B_2C_2$, we are given that $A_1C_1 = A_2C_2$, that the medians B_1M_1 and B_2M_2 are equal, and that the bisectors A_1D_1 and A_2D_2 are equal. Prove that the triangles are congruent.

Solution by the proposer, modified by the editor.

Consider an arbitrary triangle ABC. Let $a = BC$, $b = CA$, and $c = AB$. Let l be the length of the bisector AD, and let m be the length of the median BM. Then $m^2 = \frac{1}{2}(a^2 + c^2) - \frac{1}{4}b^2$, or equivalently,

$$a \;=\; \sqrt{2m^2 + \tfrac{1}{2}b^2 - c^2}\,, \qquad (1)$$

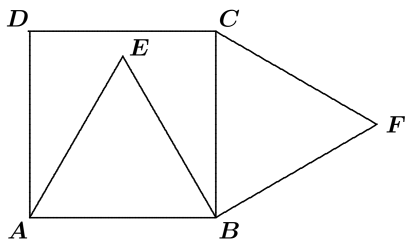
and $l^2 = bc\left(1 - \dfrac{a^2}{(b+c)^2}\right)$, or equivalently,

$$a \;=\; (b+c)\sqrt{1 - \frac{l^2}{bc}}\,. \qquad (2)$$

Now suppose that b, l, and m are fixed. Then each of the equations (1) and (2) defines a as a function of c. It is easy to check that the function in (1) is strictly decreasing (in the interval where it is defined), while the function in (2) is strictly increasing (in the interval where it is defined). Therefore, the system of equations consisting of (1) and (2) cannot have more than one solution. That is, there is at most one pair (a, c) satisfying the system. This implies the proposed result.

M193. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

On square $ABCD$, an equilateral triangle ABE is constructed internally and an equilateral triangle BCF is constructed externally. Prove that the points D, E, and F are collinear.



Solution by Titu Zvonaru, Comănești, Romania.

The triangle ADE is isosceles, and $\angle DAE = 90^\circ - 60^\circ = 30^\circ$. Similarly, the triangle EBF is isosceles, and $\angle EBF = 60^\circ + 30^\circ = 90^\circ$. Thus, we have

$$\begin{aligned} \angle DEF &= \angle DEA + \angle AEB + \angle BEF \\ &= \frac{180^\circ - 30^\circ}{2} + 60^\circ + \frac{180^\circ - 90^\circ}{2} \;=\; 180^\circ\,. \end{aligned}$$

Hence, the points D, E, and F are collinear.

Also solved by Luyun Zhong-Qiao, Columbia International College, Hamilton, ON.

M194. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Suppose $n - 1$ and $n + 1$ are twin primes where $n \in \mathbb{N}$ with $n \geq 3$. Show that $1, 2, 3, \ldots, n$ can be arranged in a row so that the sum of any two consecutive numbers is prime. (For example, when $n = 6$, one such arrangement is $6, 5, 2, 1, 4, 3$.)

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Consider the arrangement $n, 1, n-2, 3, n-4, 5, \ldots, 2, n-1$; that is, we create the sequence $\{a_k\}_{k=1}^{n}$, where

$$a_k = \begin{cases} n - (k-1) & \text{if } k \text{ is odd,} \\ k - 1 & \text{if } k \text{ is even.} \end{cases}$$

If k is odd $(1 \leq k < n)$, then $a_k + a_{k+1} = n - 1$ and $a_{k+1} + a_{k+2} = n + 1$; if k is even $(1 \leq k < n)$, then $a_k + a_{k+1} = n + 1$ and $a_{k+1} + a_{k+2} = n - 1$. Hence, the sum of any two consecutive numbers is prime, because $n - 1$ and $n + 1$ are primes.

Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

M195. *Proposed by J. Walter Lynch, Athens, GA, USA.*

A wire of unit length is divided into three pieces, which are used to construct a square, a circle, and an equilateral triangle such that each of them has the same area. Find the length of each of the three pieces of wire.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

Let s be the length of the edge of the square, r the radius of the circle, and b the length of the edge of the equilateral triangle. The perimeters and circumference satisfy the equation $4s + 2\pi r + 3b = 1$, and the areas, being equal, satisfy $s^2 = \pi r^2 = \frac{\sqrt{3}}{4}b^2$. Solving these last equations in terms of b, we get $s = \frac{1}{2}\sqrt[4]{3}b$ and $r = \frac{1}{2}\sqrt[4]{3}b/\sqrt{\pi}$. Substituting these values into the first equation gives $(2 \cdot \sqrt[4]{3} + \sqrt{\pi} \cdot \sqrt[4]{3} + 3)b = 1$. Thus,

$$b = \frac{1}{2 \cdot \sqrt[4]{3} + \sqrt{\pi} \cdot \sqrt[4]{3} + 3} \approx 0.125552\,.$$

Then $s = \frac{1}{2}\sqrt[4]{3}b \approx 0.082618$ and $r = s/\sqrt{\pi} \approx 0.046612$. It can be verified that these values do indeed satisfy the required equations.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.

M196. *Proposed by the Mayhem Staff.*

Committees are to be formed from a group of people. Show that the number of possible committees that can be formed with an odd number of members is exactly the same as the number of possible committees that can be formed with an even number of members.

Solution by Geneviève Lalonde, Massey, ON.

Note: We must assume that a committee of nobody and a committee of everybody are allowed. Otherwise, the result is true only when the number of people in the group is odd.

For convenience, we will refer to a committee as an *even committee* when it contains an even number of members, and an *odd committee* when it contains an odd number of members.

Suppose first that the number of people in the group is odd. Then, for each even committee, the number of people not in the committee is odd (and is therefore an odd committee). Similarly, for each odd committee, the number of people not in the committee is even (and is therefore an even committee). Thus, the even committees are in a one-to-one correspondence with the odd committees. It follow that the number of even committees is equal to the number of odd committees.

Next, suppose that the number of people in the group is even. Select one person, say John, and deal with him differently from the others. For each even committee C that does not contain John, there is a corresponding odd committee consisting of everybody who is not in C except John. For each even committee C that contains John, there is a corresponding odd committee consisting of John together with everybody who is not in C. Thus, the even committees are in one-to-one correspondence with the odd committees, and again we conclude that their numbers are equal.

M197. Corrected. *Proposed by Neven Jurič, Zagreb, Croatia.*

There are twelve ships situated on a 10×10 grid. The ships are denoted by the letters A through L, and each ship consists of three cells of the grid in either a horizontal or a vertical line, as shown in the diagram. Each ship contains a certain number of passengers. There are also some numbers in the last row and the last column of the diagram. These numbers represent the total number of passengers on all the ships intersected by that row or column. For example, the two ships B and L in the last (right-most) column together contain 9 passengers. How many passengers does each of the twelve ships contain, if there are no passengers on two of the ships and the remaining ten ships contain 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 passengers?

A	A	A							B	
					C	C	C		B	
	E	D	D	D					B	17
	E							J		
	E				I			J		15
					I			J		
F	F	F			I		K			19
		G					K		L	
		G	H	H	H		K		L	
		G							L	6
8		21			6				9	

[*Ed:* This problem, as originally printed, was not solvable because ship I was incorrectly positioned in the diagram. This error was the fault of the editors. It has been corrected in the above diagram.]

Solution by the proposer, expanded by the editor.

Let the letters A, B, $\ldots$, L represent the number of passengers on the respective ships. From the information in the last row and column of the diagram, we have

$$\begin{aligned} B+D+E &= 17\,, \\ E+I+J &= 15\,, \\ F+I+K &= 19\,, \\ G+L &= 6\,, \\ A+F &= 8\,, \\ A+D+F+G &= 21\,, \\ C+I+H &= 6\,, \\ B+L &= 9\,, \end{aligned}$$

which can be expressed as

$$\begin{aligned} L &= 9-B\,, && (1) \\ F &= 8-A\,, && (2) \\ G &= B-3\,, && (3) \\ D &= 16-B\,, && (4) \\ E &= 1\,, && (5) \\ C &= 6-I-H\,, && (6) \\ J &= 14-I\,, && (7) \\ K &= 11+A-I\,. && (8) \end{aligned}$$

Since $L \geq 0$, we obtain $B \leq 9$ from (1). Similarly, since $D \leq 10$, we obtain $B \geq 6$ from (4). If $B = 6$, then (1) and (3) imply that $L = G = 3$, which is impossible. If $B = 8$, then (4) implies that $D = 8$, which is also impossible. Therefore, $B = 7$ or $B = 9$. If $B = 7$, then $D = 9$; if $B = 9$, then $D = 7$.

From (7) we see that $I \geq 4$, and from (6) we get $I \leq 6$. Therefore, $I \in \{4,\ 5,\ 6\}$. If $I = 5$, then $J = 9$, which is impossible (because $B = 9$ or $D = 9$). Suppose that $I = 4$. Then $J = 10$ from (7), and $K = A + 7$ from (8). Since K cannot be 7, 9, or 10, we must have $A = 1$. But this is impossible, since $E = 1$. We conclude that $I = 6$.

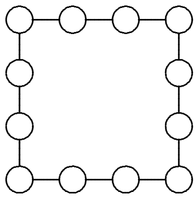
Since $I = 6$, we have $C = H = 0$ from (6), and $J = 8$ from (7). Also, $K = A + 5$ from (8). This is possible only if $A = 5$ and $K = 10$, since the other alternatives all use values that have already been assigned. Then $F = 3$ from (2).

Since $L \neq 0$, we must have $B = 7$ and $D = 9$, which gives us $L = 2$ by (1) and $G = 4$ by (3).

Thus, the only solution is

$$(A,B,C,D,E,F,G,H,I,J,K,L) \;=\; (5,7,0,9,1,3,4,0,6,8,10,2)\,.$$

M198. *Proposed by the Mayhem Staff.*



Each of the integers from 1 to 12 is to be placed in one of the circles in the figure so that the sum of the integers along each side of the figure is 25. Determine the sum of the four integers placed in the corners.

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

Place the values a, b, c, d, ..., k, and ℓ in the circles starting in the upper left corner and moving clockwise. Then we have the following equations:

$$\begin{aligned} a+b+c+d &= 25\,, & (1)\\ d+e+f+g &= 25\,, & (2)\\ g+h+i+j &= 25\,, & (3)\\ j+k+\ell+a &= 25\,, & (4)\\ a+b+c+d+\cdots+k+\ell &= 1+2+3+\cdots+12 \;=\; 78\,. & (5)\end{aligned}$$

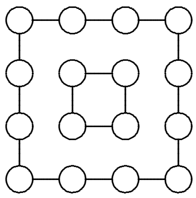
Summing equations (1) through (4) gives

$$2a+b+c+2d+e+f+2g+h+i+2j+k+\ell \;=\; 25\cdot 4 \;=\; 100\,.$$

Then, subtracting (5), we get $a+d+g+j=22$. This is the required sum.

Also solved by Makilini Balakrishan, Bell High School, Ottawa, ON; Andrea Ekholm, Bell High School, Ottawa, ON; Chelsey Gerrard, Bell High School, Ottawa, ON; Carolyn St-Amour, Bell High School, Ottawa, ON; and James Wallwork, Bell High School, Ottawa, ON.

M199. *Proposed by the Mayhem Staff.*



This is a modification of the previous problem. In this case, the requirement is to use all the integers from 1 to 16 once each so that the integers along each of the four outer edges of the large figure and the four integers that make up the inner figure have identical sums. What is the largest sum, if any, that can be obtained?

Solution by the editor.

Place the values a, b, c, d, ..., k, and ℓ in the circles that comprise the outer square, starting in the upper left corner and moving clockwise. Similarly, use m, n, o, and p for the circles that comprise the inner square. Let $S=m+n+o+p$. Then we also have

$$\begin{aligned} a+b+c+d &= S\,,\\ d+e+f+g &= S\,,\\ g+h+i+j &= S\,,\\ j+k+\ell+a &= S\,.\end{aligned}$$

Adding all five of the above equations, we get

$$\begin{aligned} 5S &= 2a+b+c+2d+e+f+2g+h+i+2j+k+\ell \\ &\qquad +m+n+o+p \\ &= (a+d+g+j)+(a+b+\cdots+p) \\ &= (a+d+g+j)+(1+2+\cdots+16) \\ &= (a+d+g+j)+136\,. \end{aligned}$$

Thus, the corner numbers, a, d, g, and j, must add to one less than a multiple of 5. To maximize S, we must maximize $a+d+g+j$. A little checking shows that $15+14+13+12=54$ works, giving $S=38$ as the maximum sum. A quick check shows that one maximal configuration is

$$\begin{aligned} &(a,b,c,d,e,f,g,h,i,j,k,\ell,m,n,o,p) \\ &\qquad = \quad (15,9,1,13,8,3,14,2,10,12,4,7,16,6,11,5)\,. \end{aligned}$$

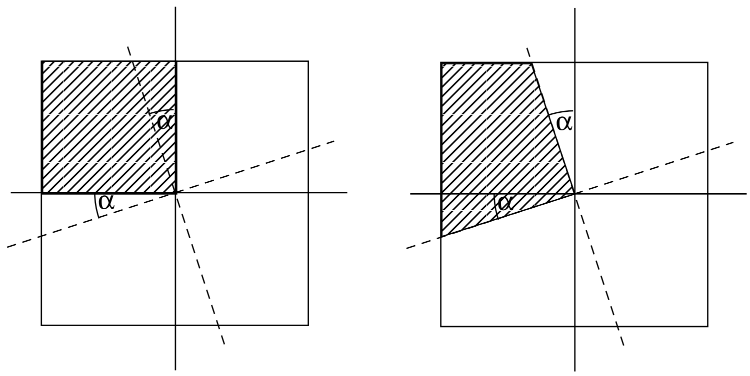
Also solved by Robert Bilinski, Collège Montmorency, Laval, QC. Bilinski points out that the English and French versions are not the same. We apologize for this oversight.

M200. *Proposed by the Mayhem Staff.*

Two perpendicular lines are drawn through the centre of a square with area 1 square unit, cutting the square into 4 pieces. What is the largest possible area for any of the pieces? Justify your answer.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

We give a proof without words, showing that the area of any of the pieces is always $\frac{1}{4}$ square unit.



Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.