SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3034. [2005 :175, 178] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let a, b, c, x, y, z be positive real numbers. Prove that

$$\begin{aligned}&(bc+ca+ab)(yz+zx+xy)\\ &\quad \geq\ bcyz+cazx+abxy+2\sqrt{abcxyz(a+b+c)(x+y+z)}\,,\end{aligned}$$

and determine when equality occurs.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Dividing the given inequality by $(ab+bc+ca)(xy+yz+zx)$ yields the equivalent inequality

$$AX+BY+CZ+2\sqrt{(AB+BC+CA)(XY+YZ+ZX)}\ \leq\ 1\,,$$

where

$$\begin{aligned}A &= \frac{bc}{ab+bc+ca}\,, & B &= \frac{ac}{ab+bc+ca}\,, & C &= \frac{ab}{ab+bc+ca}\,,\\ X &= \frac{yz}{xy+yz+zx}\,, & Y &= \frac{xz}{xy+yz+zx}\,, & Z &= \frac{xy}{xy+yz+zx}\,.\end{aligned}$$

By applying AM–GM Inequality and noting that $A+B+C=1$ and $X+Y+Z=1$, we obtain

$$\begin{aligned}&AX+BY+CZ+2\sqrt{(AB+BC+CA)(XY+YZ+ZX)}\\ &\qquad\leq\ AX+BY+CZ+AB+BC+CA+XY+YZ+ZX\\ &\qquad=\ AX+BY+CZ+\tfrac{1}{2}\big((A+B+C)^2-A^2-B^2-C^2\big)\\ &\qquad\qquad+\tfrac{1}{2}\big((X+Y+Z)^2-X^2-Y^2-Z^2\big)\\ &\qquad=\ 1-\tfrac{1}{2}\big((A-X)^2+(B-Y)^2+(C-Z)^2\big)\ \leq\ 1\,.\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; EDWARD DOOLITTLE, University of Regina, Regina, SK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Three correct solutions failed to indicate the conditions for equality.

Zhao indicated that this problem is essentially equivalent to an inequality that appeared in the 2001 Ukrainian Math Olympiad [2003 : 498; 2005 : 443], *which asks to prove the following*

$$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha a+\beta b+\gamma c)(ab+bc+ca)}\ \leq\ a+b+c\,,$$

given that a, b, c, α, β, and γ are positive real numbers such that $\alpha+\beta+\gamma=1$.

3035. [2005 : 175, 178] *Proposed by Ali Feiz Mohammadi, student, University of Toronto, Toronto, ON.*

Are there infinitely many prime numbers that cannot be written as the sum of a prime number and a power of 2?

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

The answer is *yes*, and it is known. In [1], P. Erdős introduced the concept of a *covering system of congruences* and used it to prove that any integer congruent to 3 modulo 62 and to 2036812 ($= 2^2 \cdot 509203$) modulo 5592405 ($= 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$) cannot be the sum of a prime number and a power of 2. By the Chinese Remainder Theorem, the system of congruences $x \equiv 3 \pmod{62}$ and $x \equiv 2036812 \pmod{5592405}$ has the solution set

$$S = \{x_0 + km \mid k \in \mathbb{Z} \text{ and } m = 62 \cdot 5592405 = 346729110\},$$

where x_0 is any given solution of the system of congruences. By Dirichlet's Theorem, there are infinitely many primes in S.

For further discussion, recent results, and references, see [2].

References

[1] P. Erdős, On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.* 2 (1950), 113–123.

[2] R.K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, 2004, pages 67–69.

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

In addition, one reader submitted a heuristic argument that a particular subset of those primes that cannot be written as $2^n + p$ constitute over 5 percent of all primes.

3036. [2005 : 175, 178] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let A, B, C be three distinct collinear fixed points. Let M be an arbitrary point not on the line ABC. The internal angle bisector of $\angle MAB$ intersects the line MB at a point X. The perpendicular at A to the line AX intersects the line MC at a point Y.

(a) Prove that the line XY passes through a fixed point D.

(b) Let Z be the projection of the point A onto the line XY. Prove that $\angle BZD = \angle CZD$.

I. *Solution by Titu Zvonaru, Cománeşti, Romania.*

(a) Suppose that XY meets the line ABC at D. Since AX is the internal bisector of $\angle MAB$ and AY is a bisector of $\angle MAC$ (external or internal according to the relative positions of A, B, and C), by the Bisector Theorem we have

$$\frac{YM}{YC} = \frac{AM}{AC} \quad \text{and} \quad \frac{XM}{XB} = \frac{AM}{AB}\,.$$

Applying the Theorem of Menelaus to the transversal YDX of $\triangle MCB$, we obtain

$$\frac{YM}{YC}\cdot\frac{DC}{DB}\cdot\frac{XB}{XM} = 1\,.$$

Hence,

$$\frac{DC}{DB} = \frac{YC}{YM}\cdot\frac{XM}{XB} = \frac{AC}{AM}\cdot\frac{AM}{AB} = \frac{AC}{AB}\,,$$

and the position of D is fixed with respect to A and B.

Editor's comment: The precise position of D on line AB relative to the segment AB depends, of course, on the position of C relative to segment AB. For a more thorough treatment, one should either employ directed distances and directed angles, or else treat three cases separately according to which of A, B, or C is between the other two. Zvonaru simply remarked that when A is the mid-point of the segment BC, then XY is parallel to BC, so that D is at infinity. Note further that when A is between C and B (so that ZD becomes the external bisector of $\angle BZC$), the correct condition to prove in part (b) is that $\angle BZA = \angle CZA$; alternatively, in the language of directed angles, prove that, for any position of C different from A and B on line AB, the angle from line BZ to DZ equals the angle from line DZ to CZ.

(b) Here is the argument for the case when B is between A and C. Denote $AB = b$ and $AC = c$. From $\frac{DC}{DB} = \frac{AC}{AB}$ in part (a), we find that

$$\frac{DC}{BC} = \frac{DC}{DB+DC} = \frac{c}{b+c}\,;$$

therefore,

$$DC = \frac{c(c-b)}{b+c}\,, \quad \text{and} \quad AD = c - \frac{c(c-b)}{b+c} = \frac{2bc}{b+c}\,.$$

Let $AZ = t$. By the Cosine Law, we obtain

$$\begin{aligned} ZB^2 &= t^2 + b^2 - 2tb\cos\angle ZAB \\ &= t^2 + b^2 - 2tb\frac{t(b+c)}{2bc} = \frac{b(bc-t^2)}{c}\,, \\ \text{and} \qquad ZC^2 &= t^2 + c^2 - 2tc\cos\angle ZAC \\ &= t^2 + c^2 - 2tc\frac{t(b+c)}{2bc} = \frac{c(bc-t^2)}{b}\,. \end{aligned}$$

It follows that

$$\frac{ZB^2}{ZC^2} = \frac{b^2}{c^2} = \frac{AB^2}{AC^2}$$

and, by the Bisector Theorem, AZ is the external bisector of $\angle BZC$; that is, ZD is the internal bisector of $\angle BZC$.

II. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

To avoid having to deal with parallel lines and non-existing intersection points, we will work in the projective plane.

(a) Let D be the intersection of lines XY and AB. We will prove that D is independent of M, which shows that it is the desired point. Let line MB meet line AY at X'. Since AX and AX' are the bisectors, internal and external, of $\angle A$ in $\triangle AMB$, it follows that X and X' are harmonic conjugates with respect to B and M. Consider the perspectivity with centre Y between the lines MB and AB. This sends the points B, M, X, X' to B, C, D, A, respectively. Thus, D and A are harmonic conjugates with respect to B and C. This property uniquely determines D in terms of A, B, and C; whence, D is independent of M. It follows that all lines XY pass through D.

(b) Let B' be the point on AC where it intersects the reflection of the line CZ in the mirror DZ. Then ZD is one bisector of $\angle CZB'$ and ZA is its other bisector (since $ZA \perp ZD$). Thus, A and D are harmonic conjugates with respect to B' and C. But we know from part (a) that A and D are harmonic conjugates with respect to B and C; hence, we must have $B = B'$, and the result follows.

Also solved by MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3037. [2005 : 175, 178] *Proposed by Ali Feiz Mohammadi, student, University of Toronto, Toronto, ON.*

There are 2005 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset K of senators such that for every senator in the senate, the number of enemies of that senator in the set K is an even number.

Solution by Joel Schlosberg, Bayside, NY, USA, modified by the editor.

Lemma. Let $\mathbb{F}$ be any field of characteristic 2, and let n be any odd positive integer. Suppose that $M \in M_n(\mathbb{F})$ is symmetric and that all its diagonal elements are zero. Then $\det M = 0$.

Proof: Let m_{ij} be the entry in row i and column j of M. Let S_n be the group of all permutations of $\{1, 2, \ldots, n\}$. Then

$$\det M \;=\; \sum_{\sigma\in S_n} (-1)^{\sigma} \prod_{i=1}^{n} m_{i\sigma(i)} \;=\; \sum_{\sigma\in S_n} \prod_{i=1}^{n} m_{i\sigma(i)}\,, \tag{1}$$

since $-1 = 1$ in $\mathbb{F}$.

If $\sigma \in S_n$ such that $\sigma^{-1} = \sigma$, then the disjoint cycles that make up σ have length 1 or 2. Since n is odd, there is at least one cycle containing just a single element k. For this k, we have $\sigma(k) = k$. Then $m_{k\sigma(k)} = m_{kk} = 0$ and hence, $\prod\limits_{i=1}^{n} m_{i\sigma(i)} = 0$.

Now let $\sigma \in S_n$ such that $\sigma^{-1} \neq \sigma$. Since M is symmetric,

$$\prod_{i=1}^{n} m_{i\sigma^{-1}(i)} \;=\; \prod_{i=1}^{n} m_{\sigma^{-1}(i)i} \;=\; \prod_{i=1}^{n} m_{\sigma^{-1}(\sigma(i))\sigma(i)} \;=\; \prod_{i=1}^{n} m_{i\sigma(i)}\,.$$

Therefore, since $\mathbb{F}$ has characteristic 2,

$$\prod_{i=1}^{n} m_{i\sigma^{-1}(i)} + \prod_{i=1}^{n} m_{i\sigma(i)} \;=\; 0\,.$$

Thus, all terms in (1) cancel out and $\det M = 0$. ∎

Returning to the original problem, we suppose that there are n senators $s_1, \ldots, s_n$, where n is any odd positive integer. Let $\mathbb{F} = \mathbb{Z}_2 = \{0, 1\}$, and let $M \in M_n(\mathbb{F})$ have entries m_{ij} defined by

$$m_{ij} \;=\; \begin{cases} 1 & \text{if } i \neq j \text{ and } s_i \text{ and } s_j \text{ are enemies,} \\ 0 & \text{otherwise.} \end{cases}$$

Then M is symmetric, and all its diagonal entries are 0. By the Lemma, $\det M = 0$.

Corresponding to any vector $v = (v_1, \ldots, v_n) \in \mathbb{Z}_2^n$, there is a unique set of senators, S_v, such that $s_i \in S_v$ if and only if $v_i = 1$. For each $k \in \{1, 2, \ldots, n\}$, the k^{th} entry of the vector Mv is the total number of enemies, modulo 2, that senator s_k has in the set S_v. Since $\det M = 0$, there is a non-zero vector $v \in \mathbb{Z}_2^n$ such that $Mv = 0$. For this vector v, let $K = S_v$. Since $Mv = 0$, each senator has 0 enemies, modulo 2, in K; that is, each senator has an even number of enemies in K.

Also solved by YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was also one incorrect solution submitted.

3038. [2005 : 175, 178] *Proposed by Virgil Nicula, Bucharest, Romania.*

Consider a triangle ABC in which $a = \max\{a, b, c\}$. Prove that the expressions $(a+b+c)\sqrt{2} - \left(\sqrt{a+b} + \sqrt{a-b}\right) \cdot \left(\sqrt{a+c} + \sqrt{a-c}\right)$ and $b^2 + c^2 - a^2$ have the same sign.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, modified by the editor.

Let $u = (a+b+c)\sqrt{2}$ and $v = \left(\sqrt{a+b}+\sqrt{a-b}\right)\left(\sqrt{a+c}+\sqrt{a-c}\right)$. Note that the sign of $u - v$ is the same as the sign of $u^2 - v^2$. We easily see that

$$u^2 - v^2 \;=\; 2(a+b+c)^2 - \left(2a + 2\sqrt{a^2-b^2}\right)\left(2a + 2\sqrt{a^2-c^2}\right).$$

We also note that

$$2(b^2+c^2-a^2) \;=\; 2(a+b+c)^2 - 4(a+c)(a+b)\,.$$

Fix $\sim$ as any one of the relations $<$, $=$, or $>$. Then the statements $b^2+c^2-a^2 \sim 0$, $c^2 \sim a^2-b^2$, and $c^2 \sim a^2-c^2$ are all equivalent to each other. Thus, we have

$$\begin{aligned} 2(a+b+c)^2 - \left(2a + 2\sqrt{a^2-b^2}\right)\left(2a + 2\sqrt{a^2-c^2}\right) \\ \sim\;\; 2(a+b+c)^2 - 4(a+c)(a+b)\,, \end{aligned}$$

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Cománeşti, Romania; and the proposer.

3039. [2005 : 237, 239] *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Let a, b be fixed non-zero real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f\left(x - \frac{b}{a}\right) + 2x \;\le\; \frac{a}{b}x^2 + 2\frac{b}{a} \;\le\; f\left(x + \frac{b}{a}\right) - 2x\,.$$

Combination of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Joel Schlosberg, Bayside, NY, USA.

If $y = x - \dfrac{b}{a}$, then the first inequality becomes

$$\begin{aligned} f(y) \;&\le\; \frac{a}{b}\left(y + \frac{b}{a}\right)^2 - 2\left(y + \frac{b}{a}\right) + 2\frac{b}{a} \\ &=\; \frac{a}{b}y^2 + 2y + \frac{b}{a} - 2y - 2\frac{b}{a} + 2\frac{b}{a} \\ &=\; \frac{a}{b}y^2 + \frac{b}{a}\,, \end{aligned}$$

while if $y = x + \frac{b}{a}$, then the second inequality becomes

$$\begin{aligned} f(y) &\geq \frac{a}{b}\left(y - \frac{b}{a}\right)^2 + 2\left(y - \frac{b}{a}\right) + 2\frac{b}{a} \\ &= \frac{a}{b}\,y^2 - 2y + \frac{b}{a} + 2y - 2\frac{b}{a} + 2\frac{b}{a} \\ &= \frac{a}{b}\,y^2 + \frac{b}{a}\,. \end{aligned}$$

Therefore, the only such function f is $f(x) = \frac{a}{b}\,x^2 + \frac{b}{a}$.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CURTIS COOPER, Central Missouri State University, Warrensburg, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

3040. [2005 : 237, 239] *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Prove that, for any three distinct natural numbers a, b, c greater than 1,

$$\left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right) \;\leq\; \frac{91}{8}\,.$$

Solution by Michel Bataille, Rouen, France.

Let $F(a,b,c) = \left(1+\frac{1}{a}\right)\left(2+\frac{1}{b}\right)\left(3+\frac{1}{c}\right)$. If $\min\{a,\,b,\,c\} \geq 3$, then

$$F(a,b,c) \;\leq\; \left(1+\frac{1}{3}\right)\left(2+\frac{1}{3}\right)\left(3+\frac{1}{3}\right) \;=\; \frac{280}{27} \;<\; \frac{91}{8}\,.$$

Otherwise, $\min\{a,\,b,\,c\} = 2$, and we have three cases.

(1) If $\min\{a,\,b,\,c\} = c = 2$, then $a \geq 3$ and $b \geq 3$, so that

$$F(a,b,c) \;=\; F(a,b,2) \;\leq\; \left(1+\frac{1}{3}\right)\left(2+\frac{1}{3}\right)\left(3+\frac{1}{2}\right) \;=\; \frac{98}{9} \;<\; \frac{91}{8}\,.$$

(2) If $\min\{a,\,b,\,c\} = b = 2$, then $a \geq 3$ and $c \geq 3$, so that

$$F(a,b,c) \;=\; F(a,2,c) \;\leq\; \left(1+\frac{1}{3}\right)\left(2+\frac{1}{2}\right)\left(3+\frac{1}{3}\right) \;=\; \frac{100}{9} \;<\; \frac{91}{8}\,.$$

(3) If $\min\{a,\,b,\,c\} = a = 2$, then either $b \geq 3$ and $c \geq 4$, or $b \geq 4$ and $c \geq 3$. Thus, we have either

$$F(a,b,c) \;=\; F(2,b,c) \;\leq\; \left(1+\frac{1}{2}\right)\left(2+\frac{1}{3}\right)\left(3+\frac{1}{4}\right) \;=\; \frac{91}{8}\,,$$

or

$$F(a,b,c) \;=\; F(2,b,c) \;\le\; \left(1+\tfrac{1}{2}\right)\left(2+\tfrac{1}{4}\right)\left(3+\tfrac{1}{3}\right) \;=\; \frac{90}{8} \;<\; \frac{91}{8}\,.$$

Therefore, $F(a,b,c) \le \frac{91}{8}$, with equality if and only if $(a,b,c) = (2,3,4)$.

Also solved by ARKADY ALT, San Jose, CA, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MIHÁLY BENCZE, Brasov, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESúS VILLAR RUBIO, Santander, Spain; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3041. [2004 : 237, 239] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Prove that

(a) $\sin x = 2^{n-1}\sum_{k=0}^{n-1}\sin\left(x+\frac{k\pi}{n}\right)$ for all $x\in\mathbb{R}$ and all integers $n\ge 1$;

(b) $n\cot nx = \sum_{k=0}^{n-1}\cot\left(x+\frac{k\pi}{n}\right)$ for $x\in\left(0,\frac{\pi}{n}\right)$.

[*Ed.:* As noted in the featured solution below, the formula in part (a) above is incorrect. This was the fault of the editors. We apologize to the proposer and to the readers.]

Solution by Michel Bataille, Rouen, France.

(a) It is easily seen that the formula as stated is incorrect. The intended formula was likely

$$\sin nx \;=\; 2^{n-1}\prod_{k=0}^{n-1}\sin\!\left(x+\frac{k\pi}{n}\right),$$

which is a well-known formula proved by Euler in his *Introductio in Analysis Infinitorum*, §240.

(b) The formula is obtained at once by logarithmic differentiation of the above formula for $\sin nx$. A direct proof can be found in Euler, *op. cit.*, §§249–250.

Also solved by ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

All of the solvers pointed out that the formula in (a) *was incorrect. Several then suggested the correct formula and gave either a proof or a reference for it. Alt also proved a formula for the sum on the right side of the published version of* (a)*:*

$$\sum_{k=0}^{n-1} \sin\Big(x + \frac{k\pi}{n}\Big) \;=\; \frac{\cos(x - \frac{\pi}{2n})}{\sin\frac{\pi}{2n}}\,.$$

This is a special case of a more general formula

$$\sum_{k=0}^{n-1} \sin(x + ky) \;=\; \sin\Big(x + \frac{n-1}{2}y\Big)\sin\frac{ny}{2}\csc\frac{y}{2}\,,$$

which is listed in §*1.341, p. 29, in I.S. Gradshteyn and I.M. Ryzhik,* Table of Integrals, Series, and Products. *The correct version of* (a) *is in this same reference,* §*1.392, p. 33. Zhou noted that the corrected formula in* (a) *and the formula in* (b) *are both in I.E.W. Hobson,* A Treatise on Plane and Advanced Trigonometry, *Dover, 2005, pp. 119–120. The proposer proved his* (correct) *version of* (a) *by defining a polynomial $P(z) = z^n - e^{i(2nx)}$, factoring this polynomial, and then calculating $P(1)$ using both the original form and the factored form of $P(z)$.*

3042. [2005 : 237, 240] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $x_1, x_2, \ldots, x_n$ be positive numbers such that $x_1x_2\cdots x_n = 1$. For $n \geq 3$ and $0 < \lambda \leq (2n-1)/(n-1)^2$, prove that

$$\frac{1}{\sqrt{1+\lambda x_1}} + \frac{1}{\sqrt{1+\lambda x_2}} + \cdots + \frac{1}{\sqrt{1+\lambda x_n}} \;\leq\; \frac{n}{\sqrt{1+\lambda}}\,.$$

Solution by the proposer, expanded by the editor.

Let $y_i = \lambda x_i$ for all $i = 1, 2, \ldots, n$. Then the given condition becomes $\prod\limits_{i=1}^{n} y_i = \lambda^n$ with $\lambda = \Big(\prod\limits_{i=1}^{n} y_i\Big)^{\frac{1}{n}} \leq \dfrac{2n-1}{(n-1)^2}$, and the inequality to be proved becomes

$$\sum_{i=1}^{n} \frac{1}{\sqrt{1+y_i}} \;\leq\; \frac{n}{\sqrt{1+\lambda}}\,. \tag{1}$$

We first show that (1) is true when $n = 2$ and $\lambda \leq 2$.

Let $y_1 = 2s^2$ and $y_2 = 2t^2$ where $s > 0$ and $t > 0$. Then the condition $y_1y_2 = \lambda^2$ becomes $4s^2t^2 = \lambda^2$, or $\lambda = 2st$, and (1) becomes

$$\frac{1}{\sqrt{1+2s^2}} + \frac{1}{\sqrt{1+2t^2}} \;\leq\; \frac{2}{\sqrt{1+2st}}\,. \tag{2}$$

Let $p = st$ and $q = \big(\frac{1}{2}(s+t)\big)^2$. Then $p \leq q$ and $s^2 + t^2 = 4q - 2p$. Since $\lambda \leq 2$, we also have $p \leq 1$. By squaring both sides, (2) can be rewritten as

$$\frac{1}{1+2s^2}+\frac{1}{1+2t^2}-\frac{2}{1+2p} \;\le\; \frac{2}{1+2p}-\frac{2}{\sqrt{(1+2s^2)(1+2t^2)}}\,. \qquad (3)$$

By straightforward but tedious computations and noting that

$$(1+2s^2)(1+2t^2) \;=\; 1+2(4q-2p)+4p^2 \;=\; 1+4(2q-p)+4p^2\,,$$

we have

$$\begin{aligned}
&\frac{1}{1+2s^2}+\frac{1}{1+2t^2}-\frac{2}{1+2p}\\
&\qquad = \frac{2+2(4q-2p)}{(1+2s^2)(1+2t^2)}-\frac{2}{1+2p}\\
&\qquad = \frac{2+4p+4(1+2p)(2q-p)-2(1+2(4q-2p)+4p^2)}{(1+2s^2)(1+2t^2)(1+2p)}\\
&\qquad = \frac{8(2p-1)(q-p)}{(1+2s^2)(1+2t^2)(1+2p)}\,,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2}{1+2p}-\frac{2}{\sqrt{(1+2s^2)(1+2t^2)}}\\
&\quad = \frac{2\big(\sqrt{(1+2s^2)(1+2t^2)}-(1+2p)\big)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}}\\
&\quad = \frac{-2\big((1+2p)^2-(1+2s^2)(1+2t^2)\big)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\left(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\right)}\\
&\quad = \frac{-2\big(1+4p+4p^2-(1+2(4q-2p)+4p^2)\big)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\left(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\right)}\\
&\quad = \frac{16(q-p)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\left(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\right)}\,.
\end{aligned}$$

Hence, (3) now becomes

$$\begin{aligned}
&\frac{8(2p-1)(q-p)}{(1+2s^2)(1+2t^2)(1+2p)}\\
&\quad\le \frac{16(q-p)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\left(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\right)}\,. \qquad (4)
\end{aligned}$$

If $p=q$, then (4) clearly holds. If $p\neq q$, then (4) can be rewritten as

$$\frac{2p-1}{\sqrt{(1+2s^2)(1+2t^2)}} \;\le\; \frac{2}{1+2p+\sqrt{(1+2s^2)(1+2t^2)}}\,,$$

or $4p^2 - 1 + (2p-1)\sqrt{(1+2s^2)(1+2t^2)} \le 2\sqrt{(1+2s^2)(1+2t^2)}$. This can be rewritten as $4p^2 - 1 \le (3-2p)\sqrt{8q+(1-2p)^2}$, which is true since

$$\begin{aligned} &(3-2p)\sqrt{8q+(1-2p)^2} - 4p^2 + 1 \\ &\qquad \ge \ (3-2p)\sqrt{8p+(1-2p)^2} - 4p^2 + 1 \\ &\qquad = \ (3-2p)(1+2p) - 4p^2 + 1 \\ &\qquad = \ 4(1+p-2p^2) \ = \ 4(1+2p)(1-p) \ \ge \ 0\,. \end{aligned}$$

Thus, (4) holds with equality if and only if $p = q$.

Hence, (2) is true with equality if and only if $s = t$.

Now, we proceed by induction and assume that (1) is valid for $n-1$ for some $n \ge 3$. Let $x = \sqrt[n-1]{y_1 y_2 \cdots y_{n-1}}$. Without loss of generality, we may assume that $y_1 \le y_2 \le \cdots y_n$. Then $x \le y_n$, which implies that $x^n = x^{n-1} \cdot x \le y_1 y_2 \cdots y_n = \lambda^n$. Thus, $x \le \lambda$. If $n = 3$, then $x \le \frac{5}{4} < 2$ and if $n > 3$, then $x \le \lambda < \dfrac{2n-1}{(n-1)^2} < \dfrac{2(n-1)-1}{\left((n-1)-1\right)^2}$, since the last inequality can be easily checked to be equivalent to $2n(n-2)+1 > 0$.

Hence, by the induction hypothesis, we have

$$\frac{1}{\sqrt{1+y_1}} + \frac{1}{\sqrt{1+y_2}} + \cdots + \frac{1}{\sqrt{1+y_{n-1}}} \ \le \ \frac{n-1}{\sqrt{1+x}}\,.$$

Thus, it remains to show that, for $x \le \lambda \le \dfrac{2n-1}{(n-1)^2}$, we have

$$\frac{1}{\sqrt{1+y_n}} + \frac{n-1}{\sqrt{1+x}} \ \le \ \frac{n}{\sqrt{1+\lambda}}\,. \tag{5}$$

Since $x^{n-1}y_n = \lambda^n$, (5) is equivalent to

$$\sqrt{\frac{x^{n-1}}{x^{n-1}+\lambda^n}} + \frac{n-1}{\sqrt{1+x}} \ \le \ \frac{n}{\sqrt{1+\lambda}}\,. \tag{6}$$

Consider the function $f : [0, \lambda] \to \mathbb{R}$ defined by

$$f(x) \ = \ \sqrt{\frac{x^{n-1}}{x^{n-1}+\lambda^n}} + \frac{n-1}{\sqrt{1+\lambda}}\,.$$

Then $f(\lambda) = \dfrac{n}{\sqrt{1+\lambda}}$ and thus (6) is equivalent to $f(x) \le f(\lambda)$.

Note also that $\lambda \le \dfrac{n-1}{(n-1)^2}$ is equivalent to $1 + \lambda \le \dfrac{n^2}{(n-1)^2}$, or $\dfrac{n}{\sqrt{1+\lambda}} \ge n-1 = f(0)$. Hence, $\lambda \le \dfrac{n-1}{(n-1)^2}$ is equivalent to $f(0) \le f(\lambda)$.

By direct computations, we find that

$$f'(x) \ = \ \frac{n-1}{2}\left(\frac{x^{(n-3)/2}\lambda^n}{(x^{n-1}+\lambda^n)^{3/2}} - \frac{1}{(x+1)^{3/2}}\right)\,,$$

which has the same sign as the function $g : [0, \lambda] \rightarrow \mathbb{R}$ defined by $g(x) = \lambda^{2n/3} x^{(n-3)/3}(x+1) - x^{n-1} - \lambda^n$.

Now,

$$g'(x) = \frac{\lambda^{2n/3}}{3}\left(nx^{(n-3)/3} + (n-3)x^{(n-6)/3}\right) - (n-1)x^{n-2},$$

which has the same sign as the function $h : [0, \lambda] \rightarrow \mathbb{R}$ defined by $h(x) = \lambda^{2n/3}(nx + n - 3) - 3(n-1)x^{2n/3}$. Note that

$$h'(x) = n\left(\lambda^{2n/3} - 2(n-1)x^{(2n-3)/3}\right),$$

which has the positive root $x_0 = \lambda\left(\frac{\lambda}{2n-2}\right)^{3/(2n-3)}$.

Since $2n - 2 > 2 > \lambda$, we have $0 < x_0 < \lambda$. Furthermore, $h'(x) > 0$ for $x \in [0, x_0)$ and $h'(x) < 0$ for $x \in (x_0, \lambda]$, which imply that $h(x)$ is strictly increasing on $[0, x_0]$ and strictly decreasing on $[x_0, \lambda]$. Since $h(0) = (n-3)\lambda^{2n/3} \geq 0$ and $h(\lambda) = n\lambda^{2n/3}(\lambda - 2) < 0$, we see that $h(x)$ has a single root x_1 in $(0, \lambda)$ and that $h(x) > 0$ for $x \in (0, x_1)$ and $h(x) < 0$ for $x \in (x_1, \lambda]$.

It follows that $g'(x_1) = 0$, $g'(x) > 0$ for $x \in (0, x_1)$, and $g'(x) < 0$ for $x \in (x_1, \lambda]$. There are two cases to be considered:

(i) If $g(0) < 0$, then, from $g(\lambda) = 0$, we deduce that there is some $x_2 \in (0, \lambda)$ such that $g(x_2) = 0$, $g(x) < 0$ for $x \in [0, x_2)$ and $g(x) > 0$ for $x \in (x_2, \lambda)$. Hence, $f'(x_2) = 0$, $f'(x) < 0$ for $x \in [0, x_2)$ and $f'(x) > 0$ for $x \in (x_2, \lambda)$. Consequently, $f(x)$ is strictly decreasing on $[0, x_2]$ and strictly increasing on $[x_2, \lambda]$. It follows that $f(x) \leq \max\{f(0)\}, f(\lambda)\} = f(\lambda)$.

(ii) If $g(0) \geq 0$, then, from $g(\lambda) = 0$, we deduce that $g(x) > 0$ for $x \in (0, \lambda)$. Hence, $f'(x) > 0$ for $x \in (0, \lambda)$ and it follows that $f(x)$ is strictly increasing on $[0, \lambda]$. Consequently, $f(x) \leq f(\lambda)$.

Our proof is now complete. In (1) equality holds if all the y_is are equal; that is, equality holds in the given inequality if all the x_is are equal.

Also solved by ARKADY ALT, San Jose, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

3043. [2005 : 238, 240] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

For any convex quadrilateral $ABCD$, prove that

$$\begin{aligned} & 1 - \cos(A+B)\cos(A+C)\cos(A+D) \\ & \quad \leq 2M \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{C+A}{2}\right), \end{aligned}$$

where $M = \max\{\sin A, \sin B, \sin C, \sin D\}$.

I. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Repeatedly applying product-to-sum identities, we have

$$\begin{aligned}
&\cos(A+B)\cos(A+C)\cos(A+D)\\
&\quad= \tfrac{1}{2}\Big(\cos(B-C)+\cos(2A+B+C)\Big)\cos(A+D)\\
&\quad= \tfrac{1}{2}\Big(\cos(B-C)+\cos(A-D)\Big)\cos(A+D)\\
&\quad= \tfrac{1}{2}\cos(B-C)\cos(A+D)+\tfrac{1}{2}\cos(A-D)\cos(A+D)\\
&\quad= \tfrac{1}{4}\cos(A-B+C+D)+\tfrac{1}{4}\cos(A+B-c+D)\\
&\qquad\qquad +\tfrac{1}{4}\cos 2A+\tfrac{1}{4}\cos 2D\\
&\quad= \tfrac{1}{4}(\cos 2A+\cos 2B+\cos 2C+\cos 2D)\\
&\quad= 1-\tfrac{1}{2}(\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D)\,.
\end{aligned}$$

Thus, the left side of the proposed inequality is simply

$$\tfrac{1}{2}(\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D)\,.$$

Using a similar method, we get

$$\begin{aligned}
&\sin\left(\tfrac{1}{2}(A+B)\right)\sin\left(\tfrac{1}{2}(B+C)\right)\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{2}\left[\cos\left(\tfrac{1}{2}(A-C)\right)-\cos\left(\tfrac{1}{2}(A+2B+C)\right)\right]\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{2}\left[\cos\left(\tfrac{1}{2}(A-C)\right)-\cos\left(\tfrac{1}{2}(B-D)\right)\right]\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{2}\cos\left(\tfrac{1}{2}(A-C)\right)\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\qquad\qquad +\tfrac{1}{2}\cos\left(\tfrac{1}{2}(B-D)\right)\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{4}\sin A+\tfrac{1}{4}\sin C+\tfrac{1}{4}\sin\left(\tfrac{1}{2}(A+B+C-D)\right)\\
&\qquad\qquad +\tfrac{1}{4}\sin\left(\tfrac{1}{2}(A-B+C+D)\right)\\
&\quad= \tfrac{1}{4}(\sin A+\sin B+\sin C+\sin D)\,.
\end{aligned}$$

Hence, the right side of the proposed inequality is simply

$$\tfrac{1}{4}M(\sin A+\sin B+\sin C+\sin D)\,.$$

Therefore, the proposed inequality is equivalent to

$$\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D \;\le\; M(\sin A+\sin B+\sin C+\sin D)\,.$$

Since the quadrilateral is convex, we have $0 < A$, B, C, $D < \pi$, which implies that $0 < \sin A$, $\sin B$, $\sin C$, $\sin D \le M$; whence,

$$\begin{aligned}
&\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D\\
&\qquad\le\; M\sin A+M\sin B+M\sin C+M\sin D\\
&\qquad=\; M(\sin A+\sin B+\sin C+\sin D)\,,
\end{aligned}$$

and we are done.

II. *Solution by Michel Bataille, Rouen, France.*

Let $x = \frac{1}{2}(B+C)$, $y = \frac{1}{2}(C+A)$, and $z = \frac{1}{2}(A+B)$. We will use the following known formulas:

$$\begin{aligned} 4\sin u\sin v\sin w &= \sin(u+v-w)+\sin(v+w-u) \\ &\qquad + \sin(w+u-v) - \sin(u+v+w)\,, \quad (1) \\ 4\cos u\cos v\cos w &= \cos(u+v-w)+\cos(v+w-u) \\ &\qquad + \cos(w+u-v)+\cos(u+v+w)\,. \quad (2) \end{aligned}$$

Applying (1) and the fact that $A+B+C+D = 2\pi$ yields

$$4\sin x\sin y\sin z = \sin A+\sin B+\sin C+\sin D\,;$$

applying (2) similarly yields

$$\begin{aligned} 4\cos 2x\cos 2y\cos 2z &= \cos 2A+\cos 2B+\cos 2C+\cos 2D \\ &= 4-2\big(\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D\big)\,. \end{aligned}$$

Now, the proposed inequality may be rewritten as

$$\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D \;\le\; M(\sin A+\sin B+\sin C+\sin D)\,,$$

and the argument proceeds as in solution I above.

Also solved by IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.