THE OLYMPIAD CORNER

No. 254

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To start your problem-solving challenges in this issue, we give Round 1 and Round 2 of the 2002/03 British Mathematical Olympiad. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for collecting them for our use.

BRITISH MATHEMATICAL OLYMPIAD 2002/3

Round 1

1. Given that $34! = 295\,232\,799\,cd9\,603\,140\,847\,618\,609\,643\,5ab\,000\,000$, determine the digits a, b, c, d.

2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.

Show that $BX = CX$ if and only if PQ is a diameter of S.

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$x^2yz + xy^2z + xyz^2 \;\le\; \tfrac{1}{3}\,.$$

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid. Determine the greatest possible value of k.

5. Find all solutions in positive integers a, b, c to the equation

$$a!b! \;=\; a! + b! + c!$$

Round 2

1. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples x, y, z of distinct positive integers satisying

(i) x, y, z are in arithmetic progression, and

(ii) $p(xyz) \le 3$.

2. Let ABC be a triangle, and let D be a point on AB such that $4AD = AB$. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of θ with DA, where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line ℓ at P, show that $PB = 2PD$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.

(a) Show that there is an arithmetic progression a, $a + d$, $a + 2d$, where $d > 0$, such that $f(a) < f(a + d) < f(a + 2d)$.

(b) Must there be an arithmetic progression a, $a+d$, $\dots$, $a+2003d$, where $d > 0$, such that $f(a) < f(a + d) < \cdots < f(a + 2003d)$?

[A permutation of $\mathbb{N}$ is a one-to-one function whose image is the whole of $\mathbb{N}$; that is, a function from $\mathbb{N}$ to $\mathbb{N}$ such that for all $m \in \mathbb{N}$ there is a unique $n \in \mathbb{N}$ such that $f(n) = m$.]

4. Let f be a function from the set of non-negative integers into itself such that, for all $n \geq 0$,

(i) $\left(f(2n + 1)\right)^2 - \left(f(2n)\right)^2 = 6f(n) + 1$, and

(ii) $f(2n) \geq f(n)$.

How many numbers less than 2003 are there in the image of f?

As a second set, we give selected problems of the Kazakh National Mathematical Olympiads 2002-2003. Thanks again go to Andy Liu for collecting them for the *Corner*.

KAZAKH NATIONAL MATHEMATICAL OLYMPIAD 2002–2003
Selected Problems

1. (*T. Akashev*) A quadrilateral $ABCD$ which is not a trapezoid is inscribed in a circle with centre O. Let M be the intersection point of the diagonals. Let K be an intersection point of the circumcircles of triangles BMC and DMA, and let L be an intersection point of the circumcircles of triangles AMB and CMD, where K, L, and M are distinct points. Prove that $OLMK$ is a rectangle.

2. (*S. Mukhanbetkaliev*) Angles B and C of triangle ABC are acute. Side KN of rectangle $KLMN$ belongs to segment BC, points L and M belong to segments AB and AC, respectively. Let O be the intersection point of the diagonals of $KLMN$. Let C_1 be the intersection point of lines BO and MN, and let B_1 be the intersection point of lines CO and LK. Prove that lines AO, BB_1, and CC_1 are concurrent.

3. (*U. Mukashev*) Find the maximal and minimal values of the sum $a+b+c$ if $a^2 + b^2 \leq c \leq 1$.

4. (*U. Mukashev*) Let two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ be such that $a_0 = b_0 = 0$ and for each positive integer n,

$$a_n = a_{n-1}^2 + 3 \quad \text{and} \quad b_n = b_{n-1}^2 + 2^n .$$

Compare the numbers a_{2003} and b_{2003}.

5. (*A. Kungozhin*) There are n grasshoppers in a row. Once a second at most one grasshopper can jump over exactly two neighbouring insects to the right or left side. For which values of n can the grasshoppers be rearranged in the reverse order?

As a final group of problems for your puzzling pleasure over the summer we give the 11th Form of the Ukrainian Mathematical Olympiad. Thanks again go to Andy Liu for collecting the contest for our use.

UKRAINIAN MATHEMATICAL OLYMPIAD
11th Form

1. Find all real k such that the following system of equations has a unique solution:

$$\begin{aligned} x^2 + y^2 &= 2k^2 , \\ kx - y &= 2k . \end{aligned}$$

2. Prove that for any triangle, if S denotes its area and r denotes the radius of its inscribed circle, then

$$\frac{S}{r^2} \geq 3\sqrt{3} .$$

3. Let $SABC$ be a triangular pyramid such that $SA + SB = CA + CB$, $SB + SC = AB + AC$, and $SC + SA = BC + BA$. Let O be the centre of its circumsphere, and let A_1, B_1, C_1 be the mid-points of the edges BC, CA, AB, respectively. Find the radius of the circumsphere of the triangular pyramid $OA_1B_1C_1$, in terms of the lengths $a = BC$, $b = CA$, and $c = AB$.

4. Let α be a real number such that five consecutive terms of the infinite sequence $\sin\alpha$, $\sin 2\alpha$, $\sin 3\alpha$, $\dots$, $\sin n\alpha$, $\dots$ are rational. Prove that *all* the terms of the sequence are rational.

5. Does there exist a number $q \in \mathbb{N}$ and a prime number $p \in \mathbb{N}$ such that

$$3^p + 7^p = 2 \cdot 5^q ?$$

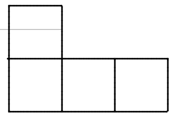
6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(xf(x) + f(y)\big) = x^2 + y$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

7. Let ω be the inscribed circle of the triangle ABC. Let K, M, and N be the points of tangency of ω with the sides AB, BC, and AC, respectively. The line containing the mid-points of the segments AK and AN intersects the line containing the mid-points of the segments CM and CN at the point P. Prove that the circumcircle of the triangle APC and the circle ω are tangent.

8. Given a positive integer n, let A_n be the number of different subdivisions (by the lattice lines) of the square $(6n) \times (6n)$ cell-like board into $6n^2$ rectangles of size 2×3 (they can be oriented arbitrarily), and let B_n be the number of different subdivisions (by the lattice lines) of the square $(12n) \times (12n)$ cell-like board into $36n^2$ figures of the type shown (they can be oriented arbitrarily).



Prove that

$$B_n \geq A_n \cdot 10^{6n^2} .$$

The first set of readers' solutions pertains to problems from the 8th Macedonian Mathematical Olympiad, which appeared in [2004 : 414–415].

1. Prove that, if $m \cdot s = 2000^{2001}$ where $m, s \in \mathbb{Z}$, then the equation $mx^2 - sy^2 = 3$ has no solution in $\mathbb{Z}$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

Let $m, s \in \mathbb{Z}$ such that $m \cdot s = 2001^{2001}$. Suppose, for the purpose of contradiction, that $mx^2 - sy^2 = 3$ for some integers x and y.

Note that $ms = 2^{8004}5^{6003}$. Since ms is even, m and s cannot both be odd. If s is even, then m must be odd, since $mx^2 = 3 + sy^2$ is odd. Thus, $m = 5^\alpha$ for some integer α with $0 \leq \alpha \leq 6003$, and $s = 2^{8004}5^{6003-\alpha}$. It follows that $5^\alpha x^2 = 3 + 2^{8004}6^{6003-\alpha}y^2$. Modulo 4, this yields $x^2 \equiv 3$, which is a contradiction, since a square is congruent to 0 or 1 modulo 4.

If m is even, then s is odd. Hence, $s = 5^\beta$ for some integer β with $0 \leq \beta \leq 6003$, and $m = 2^{8004}5^{6003-\beta}$. Thus, $3 + 5^\beta y^2 = 2^{8004}5^{6003-\beta}x^2$. This implies that β cannot lie strictly between 0 and 6003 (since 3 is not a multiple of 5). If $\beta = 0$, then $y^2 \equiv -3 \equiv 2 \pmod 5$; if $\beta = 6003$, then $x^2 \equiv 3 \pmod 5$ (since $2^{8004} = 4^{4002} \equiv (-1)^{4002} \equiv 1 \pmod 5$). We again have a contradiction, since a square is congruent to 0, 1 or 4, modulo 5.

2. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \geq 2$,

$$f\big(f(n-1)\big) = f(n+1) - f(n)\,?$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that there is no such function.

Assume, for a contradiction, that f is a function such that

$$f\big(f(n-1)\big) \;=\; f(n+1) - f(n)\,. \tag{1}$$

Then, for all $n \geq 2$, we have $f(n+1) - f(n) = f(f(n-1)) > 0$. Thus, f is increasing on $\{2,\,3,\,4,\,\dots\}$. Since $f(2) \geq 1$, it follows that $f(n) \geq n-1$ for all $n \geq 2$.

If $f(n) = n - 1$ for all $n \geq 2$, then, for $n \geq 4$,

$$\begin{aligned} f\big(f(n-1)\big) \;&=\; f(n-2) \;=\; n-3 \\ \text{and} \quad f(n+1) - f(n) \;&=\; n - (n-1) \;=\; 1\,. \end{aligned}$$

This contradicts (1) for $n \geq 5$.

Hence, there exists $n_0 \geq 2$ such that $f(n_0) \geq n_0$. As above, we deduce that $f(n) \geq n$ for all $n \geq n_0$.

Repeating the same reasoning twice (once for n and once for $n+1$), we prove that there exists $a \geq 2$ such that $f(n) \geq n+2$ for all $n \geq a$.

Now, let $b = f(a)$. Then $b - 2 \geq a$ and

$$\begin{aligned} f\big(f(a)\big) \;&=\; f(a+2) - f(a+1)\,, \\ f\big(f(a+1)\big) \;&=\; f(a+3) - f(a+2)\,, \\ &\;\;\vdots \\ f\big(f(b-2)\big) \;&=\; f(b) - f(b-1)\,. \end{aligned}$$

Summing, we obtain

$$\begin{aligned} &f\big(f(a)\big) + f\big(f(a+1)\big) + \cdots + f\big(f(b-2)\big) \\ &\quad=\; f(b) - f(a+1) \;=\; f\big(f(a)\big) - f(a+1)\,. \end{aligned}$$

Thus,

$$0 \;\leq\; f\big(f(a+1)\big) + \cdots + f\big(f(b-2)\big) \;=\; -f(a+1) \;<\; 0\,,$$

a contradiction.

4. Let M be a finite set and let $\Omega \subseteq \mathcal{P}(\mathcal{M})$ such that:

(i) If $|A \cap B| \geq 2$ for A, $B \in \Omega$, then $A = B$;

(ii) There are $A, B, C \in \Omega$ such that $A \neq B \neq C \neq A$ and $|A \cap B \cap C| = 1$;

(iii) For every $A \in \Omega$ and for every $a \in M \backslash A$, there is a unique $B \in \Omega$ such that $a \in B$ and $A \cap B = \emptyset$.

Prove that there are numbers p and s such that:

(a) For every $a \in M$ the number of sets which include the point a is p;

(b) $|A| = s$ for every $A \in \Omega$;

(c) $s + 1 \geq p$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) For every A, $B \in \Omega$, let us write $A \sim B$ to say that either $A = B$ or $A \cap B = \emptyset$. Clearly the relation $\sim$ is reflexive and symmetric. We will prove that it is transitive.

Assume that A, B, $C \in \Omega$ are such that $A \sim B$ and $B \sim C$. The only non-trivial case to study is where $A \cap B = \emptyset = B \cap C$. Suppose that $A \cap C \neq \emptyset$. Then there is some $x \in A \cap C$, and $x \notin B$. From (iii), there is only one $X \in \Omega$ such that $x \in X$ and $X \cap B = \emptyset$. Since both A and C satisfy these conditions, it follows that $A = C$; whence, $A \sim C$, as desired.

It follows that $\sim$ is an equivalence relation. Let $C_1, \ldots, C_p$ be the equivalence classes. For each i, let $n_i = |C_i|$ and $C_i(1), \ldots, C_i(n_i)$ be the elements of Ω which form the class C_i.

For each i, by definition of $\sim$, the sets $C_i(1), \ldots, C_i(n_i)$ are pairwise disjoint, and condition (iii) ensures that they form a partition of M. Thus, for each $x \in M$ and $i \in \{1, \ldots, p\}$, there is an unique element of C_i which contains x. It follows that for each $x \in M$ the number of elements of Ω which contain x is exactly the number p of classes of $\sim$.

Note that (ii) gives $p \geq 3$.

(b) **Lemma**. Let i, $i' \in \{1, \ldots, p\}$ with $i' \neq i$. For all $j \in \{1, \ldots, n_i\}$, we have $|C_i(j)| = n_{i'}$.

Proof: By construction of $\sim$ and from (i), for each $j \in \{1, \ldots, n_i\}$ and $j' \in \{1, \ldots, n_{i'}\}$, we have $|C_i(j) \cap C_{i'}(j')| = 1$. Since the $C_{i'}(j')$s form a partition of M, it follows that for a fixed j, each element of $C_i(j)$ is contained in exactly one of the $C_{i'}(j')$'s. Thus $|C_{i(j)}| = |C_{i'}| = n_{i'}$. ∎

From the lemma, since $p \geq 3$, it follows at once that $n_1 = \cdots = n_p$. Let s be this common value. The lemma ensures that $|A| = s$ for every $A \in \Omega$, and (b) is proved.

(c) Let $m = |M|$. Since the s sets which form C_1 form a partition of M, we deduce from (b) that $s^2 = m$. Let $a \in M$. From above, for each $i \geq 1$ there is a unique j such that $a \in C_i(j)$. With no loss of generality, we may assume that $a \in C_i(1)$ for all i.

Since $|C_i(1) \cap C_{i'}(1)| = 1$ as soon as $i \neq i'$, it follows that the sets $C_i(1) - \{a\}$, for $i = 1, \ldots, p$, are pairwise disjoint. Using the fact that each has cardinality $s - 1$, we deduce that $(s-1)p \leq m - 1 = s^2 - 1$, implying that $s + 1 \geq p$, and we are done.

Next we give solutions we have received to problems of the Latvian Mathematical Olympiad 2000/2001, Final Grade, 3rd Round given in [2004 : 415–416].

3. Is it possible to colour all grid points in the plane white and red so that no rectangle with vertices on grid points of one colour and sides parallel to the grid lines has area from the set $\{1, 2, 4, 8, \ldots, 2^n, \ldots\}$?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Yes, such a colouring does exist.

Let us colour red the points (x,y) such that $x+y \not\equiv 1 \pmod 3$, and all the other ones white. We will show that there is no rectangle with vertices on grid points of one colour and sides parallel to the grid lines whose area equals a power of 2.

Case 1. Let $\mathcal{W}$ be a rectangle with all its vertices white and sides parallel to the grid lines. Let (a,b) and (a,d) be two of its adjacent vertices. Then $a+b \equiv a+d \equiv 1 \pmod 3$, so that $b-d \equiv 0 \pmod 3$. It follows that the area of $\mathcal{W}$, which is a multiple of $b-d$, is divisible by 3. Thus, this area cannot be a power of 2.

Case 2. Let $\mathcal{R}$ be a rectangle with all its vertices red and sides parallel to the grid lines. Let (a,b), (a,d), (c,d), (c,b) be its vertices, with $c > a$ and $d > b$. Let $x = a+b$. Assume, for a contradiction, that the area of $\mathcal{W}$ is a power of 2. It follows that $c = a+2^p$ and $d = b+2^q$ for some non-negative integers p and q. Then, since the four vertices are red, we have

$$\begin{aligned} x &\not\equiv 1 \pmod 3 \ , \\ x+(-1)^p &\not\equiv 1 \pmod 3 \ , \\ x+(-1)^q &\not\equiv 1 \pmod 3 \ , \\ x+(-1)^p+(-1)^q &\not\equiv 1 \pmod 3 \ . \end{aligned}$$

Thus, modulo 3, we have $\{x,\, x+(-1)^p\} = \{0,\, 2\} = \{x,\, x+(-1)^q\}$. Then $x+(-1)^p \equiv x+(-1)^q \pmod 3$, and hence, p and q have the same parity. But this forces x, $x+(-1)^p$, $x+(-1)^p+(-1)^q$ to be distinct modulo 3. In particular, one of them is congruent to $1 \pmod 3$, a contradiction.

5. Prove that for each n there exists a finite graph without triangles such that in each colouring of the vertices with n colours there is an edge with equally coloured endpoints. (A known theorem.)

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let n be a positive integer, and let $k = 2^n$. Consider the graph $\mathcal{G}$ whose vertices are the points (a,b) in the plane such that $a,\, b \in \{1,\, 2,\, \dots,\, k\}$, and for which (a,b) and (c,d) are joined by an edge if and only if $a+b = c$ or $c+d = a$. For $i = 1,\, 2,\, \dots,\, k$, let Δ_i be the line with equation $x = i$.

Assume that (a,b), (c,d), (e,f) form a triangle. Without any loss of generality, we may assume that $a \le c \le e$. Then $a+b = c$ and $a+b = e$ and $c+d = e$, which forces $d = 0$, a contradiction. Thus, $\mathcal{G}$ contains no triangle.

Now, let $a,\, b \in \{1,\, 2,\, \dots,\, k\}$, with $a < b$. Then, $(a, b-a)$ is joined to (b,y) for all $y \in \{1,\, 2,\, \dots,\, k\}$. It follows that in each proper colouring of $\mathcal{G}$, each point belonging to Δ_b must have a colour different than the colour of $(a,b-a)$. Thus, in any proper colouring of $\mathcal{G}$, the set of colours determined by Δ_b must be distinct from the set of colours determined by Δ_a. Since a

and b are arbitrarily chosen, it follows that at least $k = 2^n$ pairwise distinct sets of colours must be available, and clearly none of them is the empty set. This forces $\chi(\mathcal{G}) > n$ (where $\chi(\mathcal{G})$ denotes the chromatic number of $\mathcal{G}$), so that in each colouring of $\mathcal{G}$ with only n colours, at least two adjacent vertices have the same colour.

Now we turn to the files of solutions from our readers to problems given in the December 2004 number of the *Corner*, beginning with the 13th Irish Mathematical Olympiad, which appeared at [2004 : 476–477].

1. Let S be the set of all numbers of the form $a(n) = n^2 + n + 1$, where n is a natural number. Prove that the product $a(n)a(n+1)$ is in S for all natural numbers n. Give, with proof, an example of a pair of elements s, $t \in S$ such that $st \notin S$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Robert Bilinski, Collège Montmorency, Laval, QC; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We first give Wang's comment.

The first part of this problem is the same as problem #2 of the 49th Mathematical Olympiad of Lithuania, which appeared previously in the *Corner* [2004 : 142; 2005 : 531]. The second part is easy. For example, $a(1) = 3$ and $a(3) = 13$, which means that $a(1)a(3) = 39$. Since the sequence $\{a(n)\}$ is clearly strictly increasing and $a(5) < 39 < a(6)$, it follows that $a(1)a(3) \notin S$.

Next we give Bilinski's solution.

For any natural numbers n and k,

$$a(n)a(n+k) = n^4+(2k+2)n^3+(k^2+3k+3)n^2+(k^2+3k+2)n+k^2+k+1\,.$$

This product is in S if and only if it has the form m^2+m+1 for some integer $m > 0$. Evidently, m must be a quadratic in n, say $m = An^2 + Bn + C$, for some integers A, B, C. Then

$$\begin{aligned} m^2 + m + 1 &= (An^2 + Bn + C)^2 + (An^2 + Bn + C) + 1 \\ &= A^2n^4 + 2ABn^3 + (B^2 + 2AC + A)n^2 + B(2C + 1)n + C^2 + C + 1\,. \end{aligned}$$

Now $a(n)a(n+k) = m^2 + m + 1$ if and only if the following equations are satisfied:

$$\begin{aligned} A^2 &= 1\,, \\ 2AB &= 2k + 2\,, \\ B^2 + 2AC + A &= k^2 + 3k + 3\,, \\ B(2C + 1) &= k^2 + 3k + 2\,, \\ C^2 + C + 1 &= k^2 + k + 1\,. \end{aligned}$$

These equations are satisfied if and only if

$$\begin{cases} A = 1 \\ B = k+1 \\ C = k = 1\,, \end{cases} \quad \text{or} \quad \begin{cases} A = -1 \\ B = -k-1 \\ C = k = -1\,. \end{cases}$$

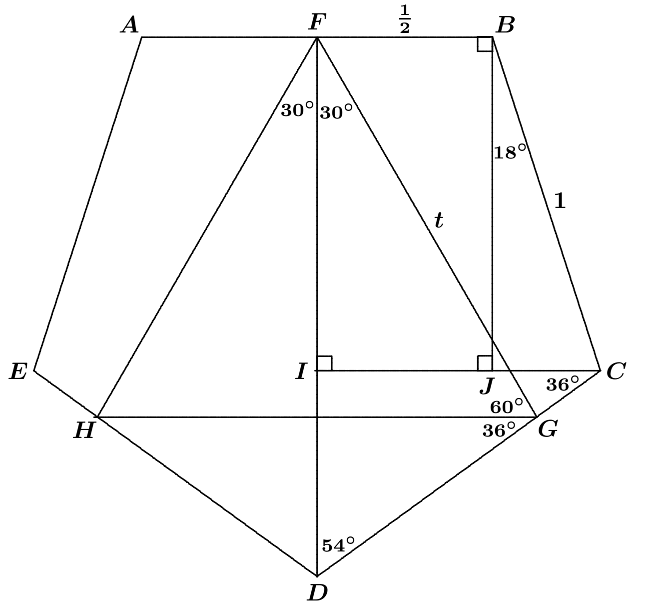
Hence, a product of the form $a(n)a(n+k)$, with $k > 0$, is in S if and only if $k = 1$. (Notice that this solves both parts of the problem.)

2. Let $ABCDE$ be a regular pentagon with its sides of length one. Let F be the mid-point of AB, and let G and H be points on the sides CD and DE, respectively, such that $\angle GFD = \angle HFD = 30^\circ$. Prove that the triangle GFH is equilateral. A square is inscribed in the triangle GFH with one side of the square along GH. Prove that FG has length

$$t = \frac{2\cos 18^\circ(\cos 36^\circ)^2}{\cos 6^\circ}\,,$$

and that the square has side length $\dfrac{t\sqrt{3}}{2+\sqrt{3}}$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



By symmetry, we have $FG = FH$; thus, $\triangle FGH$ is equilateral. Also, $\angle FDC = \frac{1}{2}\angle EDC = 54^\circ$. Drop a perpendicular CI from C to DF, and a perpendicular BJ from B to CI. We have

$$\begin{aligned} DF &= DI + JB = \sin 36^\circ + \cos 18^\circ = 2\cos 18^\circ \left(\sin 18^\circ + \tfrac{1}{2}\right) \\ &= 2\cos 18^\circ \cdot CI = 2\cos 18^\circ \cos 36^\circ\,. \end{aligned}$$

By the Law of Sines,

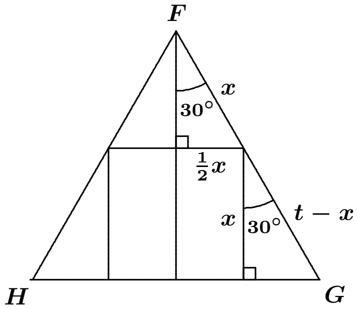
$$\frac{t}{\sin 54^\circ} = \frac{DF}{\sin 96^\circ};$$

whence,

$$t = \frac{\cos 36^\circ}{\cos 6^\circ} \cdot DF = \frac{2\cos 18^\circ(\cos 36^\circ)^2}{\cos 6^\circ}.$$

Let x be the length of a side of the square under consideration. It is clear from the diagram that $\frac{x}{\sqrt{3}} = \frac{t-x}{2}$, from which it follows easily that

$$x = \frac{t\sqrt{3}}{2+\sqrt{3}}.$$



3. Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.

Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's write-up.

Note that $f(x) = x(5x^{12} + 13x^4 + 9a)$ and $65 = 5 \times 13$. Let x be an integer.

If $x \equiv 0 \pmod{13}$, then $f(x) \equiv 0 \pmod{13}$. If $x \not\equiv 0 \pmod{13}$, then $5x^{12} + 13x^4 + 9a \equiv 5 + 9a \pmod{13}$, using Fermat's Little Theorem, and hence,

$$f(x) \equiv 0 \pmod{13} \quad \text{if and only if} \quad a \equiv -2 \pmod{13}. \qquad (1)$$

If $x \equiv 0 \pmod 5$, then $f(x) \equiv 0 \pmod 5$. If $x \not\equiv 0 \pmod 5$, then $5x^{12} + 13x^4 + 9a \equiv 3 + 9a \pmod 5$, using Fermat's Little Theorem again, and hence,

$$f(x) \equiv 0 \pmod 5 \quad \text{if and only if} \quad a \equiv -2 \pmod 5. \qquad (2)$$

From (1) and (2), we deduce that the least positive integer a such that $f(x) \equiv 0 \pmod{65}$ for all integers x, is defined by $a+2 = \mathrm{lcm}(5,13) = 65$. Thus, the desired integer is $a = 63$.

5. Consider all parabolas of the form $y = x^2 + 2px + q$ (for real p, q) which intersect the x– and y–axes in three distinct points. For such a pair p, q, let $C_{p,q}$ be the circle through the points of intersection of the parabola $y = x^2 + 2px + q$ with the axes. Prove that all the circles $C_{p,q}$ have a point in common.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; D.J. Smeenk, Zaltbommel, the Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give a comment by Michel Bataille, Rouen, France.

This problem was part of the 33rd Spanish Mathematical Olympiad 1997 (see [2001 : 93]). A solution is given in [2003 : 223].

6. Let $x \geq 0$, $y \geq 0$ be real numbers with $x + y = 2$. Prove that

$$x^2y^2(x^2+y^2) \;\leq\; 2\,.$$

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Vedula N. Murty, Dover, PA, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Murty's solution, modified by the editor.

By the AM–GM Inequality, we have $\sqrt{xy} \leq \frac{x+y}{2} = \frac{2}{2} = 1$; whence $xy \leq 1$. Then, using the AM–GM Inequality again, we get

$$x^2y^2 \;=\; \sqrt{x^4y^4} \;\leq\; \frac{x^4y^4+1}{2} \;\leq\; \frac{x^3y^3+1}{2}\,;$$

whence, $x^2y^2(2-xy) \leq 1$. We also have

$$x^2+y^2 \;=\; (x+y)^2 - 2xy \;=\; 4 - 2xy \;=\; 2(2-xy)\,.$$

Now $x^2y^2(x^2+y^2) = 2x^2y^2(2-xy) \leq 2$.

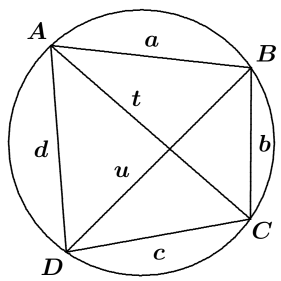
7. Let $ABCD$ be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c, d be the lengths of the sides of $ABCD$, and let Q be its area. Prove that

$$R^2 \;=\; \frac{(ab+cd)(ac+bd)(ad+bc)}{16Q^2}\,.$$

Deduce that $R \geq \frac{(abcd)^{\frac{3}{4}}}{Q\sqrt{2}}$, with equality if and only if $ABCD$ is a square.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Vedula N. Murty, Dover, PA, USA. We give Kandall's write-up.

Let $t = AC$ and $u = BD$. Let $[XYZ]$ denote the area of $\triangle XYZ$. By a well-known formula, we have $4R[ABC] = abt$ and $4R[ADC] = cdt$; hence, $4RQ = (ab+cd)t$. Analogously, $4RQ = (ad+bc)u$. Consequently,



$$16R^2Q^2 \;=\; (ab+cd)tu(ad+bc)\,.$$

By Ptolemy's Theorem, $tu = ac + bd$; thus, we easily obtain the desired expression for R^2.

By the AM–GM Inequality, each of $ab + cd$, $ac + bd$, $ad + bc$ is greater than or equal to $2(abcd)^{1/2}$. Therefore,

$$R^2 \ \geq \ \frac{(abcd)^{3/2}}{2Q^2}\,; \quad \text{that is,} \quad R \ \geq \ \frac{(abcd)^{3/4}}{Q\sqrt{2}}\,.$$

If equality holds, then $ab = cd$, $ac = bd$, and $ad = bc$, which implies that $a = b = c = d$. Thus, $ABCD$ is a rhombus and, being cyclic, a square. The converse is obvious.

8. For each positive integer n, determine, with proof, all positive integers m such that there exist positive integers $x_1 < x_2 < \cdots < x_n$ which satisfy $\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{n}{x_n} = m$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give Sinefakopoulos's presentation.

Let n be any fixed positive integer. We first suppose that there exist positive integers $x_1 < x_2 < \cdots < x_n$ such that $\sum\limits_{i=1}^{n} \frac{i}{x_i}$ is a positive integer m. Since $x_1, x_2, \ldots, x_n$ are positive integers and $x_1 < x_2 < \cdots < x_n$, we must have $x_i \geq i$ for all $i = 1, 2, \ldots, n$. Then

$$m \ = \ \sum_{i=1}^{n} \frac{i}{x_i} \ \leq \ \sum_{i=1}^{n} 1 \ = \ n\,.$$

Next we shall show that, for all integers m such that $1 \leq m \leq n$, such numbers x_i do exist. Indeed, for $m = n$, set $x_i = i$ and for $m = 1$, set $x_i = in$. It can be verified that $\sum\limits_{i=1}^{n} \frac{i}{x_i} = m$ in both cases. For $1 < m < n$, we write

$$\sum_{i=1}^{n} \frac{i}{x_i} \ = \ \underbrace{\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{m-1}{x_{m-1}}}_{m-1 \text{ terms}} + \underbrace{\frac{m}{x_m} + \cdots + \frac{n}{x_n}}_{n-m+1 \text{ terms}}$$

and note that, in order to get $\sum\limits_{i=1}^{n} \frac{i}{x_i} = m$, it suffices to make the first sum equal to $m - 1$ by setting $x_i = i$ for $1 \leq i \leq m - 1$, and the second sum equal to 1 by setting $x_i = i(n - m + 1)$ for $m \leq i \leq n$. It is easy to see that $x_1 < x_2 < \cdots < x_n$ in all cases, and the proof is complete.

9. Prove that in each set of ten consecutive integers there is one which is coprime with each of the other integers. For example, taking 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, the numbers 119 and 121 are each coprime with all the others. [Two integers a, b are coprime if their greatest common divisor is 1.]

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's solution.

Consider any set of ten consecutive integers. Among them are five odd integers. Among these five consecutive odd integers, at most two are divisible by 3, at most one is divisible by 5, and at most one is divisible by 7. Therefore, at least one of the ten integers, say n, is not divisible by 2, 3, 5 or 7.

Let k be any non-zero integer such that $-9 \leq k \leq 9$, and let $d = \gcd(n, n+k)$. Note that d is a divisor of k and n. It follows that d cannot have a prime divisor less than 10 (since d divides n), and d cannot have a prime divisor greater than 10 (since d divides k). Therefore, $d = 1$. Thus, n is coprime with all nine other integers.

10. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and $p(16) = 8$. Prove that $p(8) \leq 4$, and find, with proof, all such polynomials with $p(8) = 4$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the solution by Sinefakopoulos.

Applying the Cauchy-Schwarz Inequality,

$$(u_0v_0 + u_1v_1 + \cdots + u_nv_n)^2 \;\leq\; (u_0^2 + u_1^2 + \cdots + u_n^2)(v_0^2 + v_1^2 + \cdots + v_n^2)\,,$$

with $u_i = \sqrt{a_i}\,2^i$ and $v_i = \sqrt{a_i}\,4^i$ for $1 \leq i \leq n$, we get

$$p(8)^2 \;\leq\; p(4)p(16) \;=\; 2 \cdot 8 \;=\; 16\,.$$

Taking the square root gives $p(8) \leq 4$ (since $p(8) \geq 0$).

If equality holds, then $v_i = cu_i$ for some real c and for all i. Then, since $v_i = 2^iu_i$ for all i, all u_is but one must be equal to zero. This implies that $p(x) = a_ix^i$ for some i. Then it is easy to see that $i = 1$ and $a_i = \frac{1}{2}$, so that $p(x) = \frac{1}{2}x$. Conversely, if $p(x) = \frac{1}{2}x$, then equality holds.

Next we look at solutions to problems of the Third Hong-Kong Mathematical Olympiad which appeared at [2004 : 477–478].

2. Let $a_1 = 1$, $a_{n+1} = \dfrac{a_n}{n} + \dfrac{n}{a_n}$ for $n = 1, 2, 3, \ldots$. Find the greatest integer less than or equal to a_{2000}. Be sure to prove your claim.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Direct computation gives $a_2 = a_3 = 2$. Therefore, $\sqrt{3} < a_3 < 3/\sqrt{2}$. We prove by induction that $\sqrt{n} < a_n < n/\sqrt{n-1}$ for each integer $n \geq 3$.

Assume that this holds for some $n \geq 3$. Then we have

$$0 \ <\ \sqrt{n} \ <\ a_n \ <\ \frac{n}{\sqrt{n-1}} \ <\ n\,.$$

Note that $a_{n+1} = f(a_n)$, where $f(x) = \frac{x}{n} + \frac{n}{x}$. The function f is decreasing on $(0,n)$, since $f'(x) = \frac{x^2-n^2}{nx^2} < 0$ for $0 < x < n$. Therefore,

$$f\left(\frac{n}{\sqrt{n-1}}\right) \ <\ f(a_n) \ <\ f(\sqrt{n})\,.$$

But $f(\sqrt{n}) = \frac{1}{\sqrt{n}} + \sqrt{n} = \frac{n+1}{\sqrt{n}}$ and

$$f\left(\frac{n}{\sqrt{n-1}}\right) \ =\ \frac{1}{\sqrt{n-1}} + \sqrt{n-1} \ =\ \frac{n}{\sqrt{n-1}} \ >\ \sqrt{n+1}\,.$$

Thus, $\sqrt{n+1} < a_{n+1} < (n+1)/\sqrt{n}$, which ends the induction.

Now $44 < \sqrt{2000} < a_{2000} < \frac{2000}{\sqrt{1999}} < 45$. Therefore, $\lfloor a_{2000} \rfloor = 44$.

3. Find all prime numbers p and q such that $\frac{(7^p-2^p)(7^q-2^q)}{pq}$ is an integer.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Assume that (p,q) is a solution. Clearly, p, $q \notin \{2,\,7\}$.

Case 1. p divides $7^p - 2^p$.

Then $7^p - 2^p \equiv 7 - 2 = 5 \equiv 0 \pmod{p}$, by Fermat's Little Theorem. Thus $p = 5$. In that case, since $7^5 - 2^5 = 5^2 \times 11 \times 61$, we deduce that either $q \in \{5,\,11,\,61\}$ or q divides $7^q - 2^q$, the latter giving $q = 5$ as above. Thus, since p and q play symmetric parts, this leads to the solutions $(5,5)$, $(5,11)$, $(11,5)$, $(5,61)$, $(61,5)$.

Case 2. p does not divide $7^p - 2^p$ and q does not divide $7^q - 2^q$.

Then p divides $7^q - 2^q$, and q divides $7^p - 2^p$. We see that $p \neq q$. With no loss of generality, we may assume that $p > q$. Since p is prime, it follows that $\gcd(p, q-1) = 1$. Thus, from Bezout's Theorem, there exist two positive integers a and b such that $ap - b(q-1) = 1$.

Since q divides $7^p - 2^p$, we have $7^p \equiv 2^p \pmod{q}$. We also know that $7^{q-1} \equiv 2^{q-1} \pmod{q}$, by Fermat's Little Theorem. Thus,

$$7^{ap} \ \equiv\ 2^{ap} \pmod{q} \quad \text{and} \quad 7^{b(q-1)} \ \equiv\ 2^{b(q-1)} \pmod{q}\ .$$

Then $7 = 7^{ap-b(q-1)} \equiv 2^{ap-b(q-1)} = 2 \pmod{q}$. It follows that q divides $7 - 2 = 5$. Then $q = 5$. But then q divides $7^q - 2^q$, contradicting our assumption for Case 2.

Hence, the solutions are the pairs $(5,5)$, $(5,11)$, $(11,5)$, $(5,61)$, $(61,5)$.

4. In the coordinate plane, a *lattice point* is a point with integer coordinates. Find all positive integers $n \geq 3$ such that there exists an n–sided polygon with lattice points as vertices and all sides of equal length.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The desired integers are the even integers $n \geq 4$.

Assume that $n \geq 3$ is odd and that there exists an equilateral n–gon, say $A_1A_2 \ldots A_n$, with vertices on lattice points, and let $d > 0$ be its side length. With no loss of generality, we may assume that $A_1 = 0$ and that d is minimal.

For $i = 1, 2, \ldots, n$, let $[a_i, b_i] = \overrightarrow{A_iA_{i+1}}$ (where A_{n+1} is identified with A_1). Thus, both a_i and b_i are integers and $a_i^2 + b_i^2 = d^2$, from which we deduce that d^2 is an integer.

Moreover, since $\sum\limits_{i=1}^{n} \overrightarrow{A_iA_{i+1}} = \overrightarrow{0}$, we have $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} b_i = 0$. It follows that

$$0 = \left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 = \sum_{i=1}^{n}(a_i^2 + b_i^2) + 2\sum_{i \neq j}(a_ia_j + b_ib_j),$$

which leads to $2\sum\limits_{i \neq j}(a_ia_j + b_ib_j) = -nd^2$. Since n is odd, d^2 must be even.

First, suppose $d^2 \equiv 2 \pmod 4$. That is, $a_i^2 + b_i^2 \equiv 2 \pmod 4$, for $1 \leq i \leq n$, which ensures that a_i and b_i are odd. Thus, all the a_is and all the b_is are odd. It follows that the number $a_ia_j + b_ib_j$ is even for all i and j. Hence,

$$nd^2 = -2\sum_{i \neq j}(a_ia_j + b_ib_j) \equiv 0 \pmod 4 .$$

Since n is odd, this means that $d^2 \equiv 0 \pmod 4$, a contradiction.

Thus, $d^2 \equiv 0 \pmod 4$ and, as above, we prove that all the a_is and all the b_is are even. Since $A_1 = O$, it follows that both coordinates of A_i are even for $1 \leq i \leq n$. Therefore, using a homothety with centre O and ratio $1/2$, we deduce another equilateral n–gon with vertices on lattice points and with side length $d/2$, which contradicts the minimality of d.

Thus, in any case, we reach a contradiction, which proves that, if n is odd, there is no such equilateral polygon (note that we did not assume the polygon was convex).

Now consider the case where n is even. Let $n = 2m$ with $m \geq 2$. For rational numbers $a > 0$ and t, the point

$$M_t\left(a\frac{1-t^2}{1+t^2}, a\frac{2t}{1+t^2}\right)$$

has rational coordinates and belongs to the circle Γ_a with centre O and radius a. Choosing any m pairwise distinct rational values for $t \in (0, 1)$, we obtain m pairwise distinct points on Γ_a with positive y–coordinate. Let μ be the lowest common multiple of the denominators of all the coordinates of all the M_ts. Using an homothety with centre O and ratio μ, we deduce m pairwise distinct integer points on $\Gamma_{\mu a}$, each of them with positive y–coordinate. Then, for each chosen value of t, consider the point N_t symmetric to M_t with respect to O. We now have $2m = n$ pairwise distinct integer points which all are concylic; thus, all the vectors $\overrightarrow{OM_i}$ and $\overrightarrow{ON_i}$ have the same norm and no more than two of them have any given direction. Note that

$$\sum_{i=1}^{m} \left(\overrightarrow{OM}_i + \overrightarrow{ON}_i\right) \;=\; \overrightarrow{0}\,. \tag{1}$$

Let $P_1 = 0$ and $\overrightarrow{OP_{i+1}} = \sum\limits_{j=1}^{i} \overrightarrow{OM_j}$ for $1 \leq i \leq m$. Let

$$\overrightarrow{OP_{i+m+1}} \;=\; \left(\sum_{j=1}^{m} \overrightarrow{OM_j}\right) + \sum_{j=1}^{i} \overrightarrow{ON_j} \;=\; \overrightarrow{OP_{m+1}} + \sum_{j=1}^{i} \overrightarrow{ON_j}$$

for $1 \leq i \leq m$. Thus, from (1), we have $P_{2m+1} = 0$.

Moreover, for each $i = 1, 2, \ldots, n$, we have $\overrightarrow{P_iP_{i+1}} = \overrightarrow{OM_j}$ or $\overrightarrow{P_iP_{i+1}} = \overrightarrow{ON_j}$ for some j, which ensures that the polygon $\mathcal{P} = P_1P_2\ldots P_n$ is equilateral and, from the ordering, it is convex.

Then, if n is even, such a polygon does exist.

That completes the *Corner* for this issue. Over the next months please send me your nice solutions and generalizations.