SKOLIAD No. 94

Robert Bilinski

Please send your solutions to the problems in this edition by ***November 1, 2006***. A copy of **MATHEMATICAL MAYHEM Vol. 4** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our contest this month is the BC Colleges High School Mathematics Contest 2005, Senior Final Round, Part B. My thanks go to Clint Lee at Okanagan College.

BC Colleges High School Mathematics Contest 2005
Senior Final Round, Part B
Friday, May 6, 2005

1. The digits 1, 2, 3, 4, and 5 are each used once to compose a five-digit number $abcde$ such that the three-digit number abc is divisible by 4, bcd is divisible by 5, and cde is divisible by 3. Find the digit a.

2. An urn contains three white, six red, and four black balls.

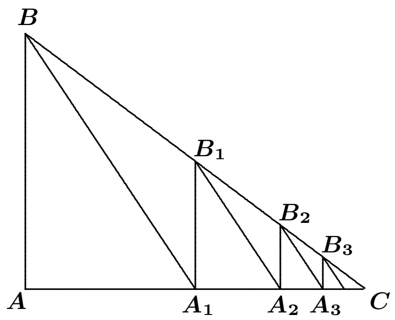
(a) If one ball is selected at random, what is the probability that the ball selected is red?

(b) If two balls are selected at random, what is the probability that they are both black?

(c) If two balls are selected at random, what is the probability that they are both black, given that they are the same colour?

3. In the diagram, ABC is a right-triangle with $\overline{AB} = 3$ and $\overline{AC} = 4$. Furthermore, each line segment A_iB_i is perpendicular to AC, A_1 bisects AC, and A_{i+1} bisects A_iC. Find the total length of the sequence of the diagonal segments:

$$\overline{BA_1} + \overline{B_1A_2} + \overline{B_2A_3} + \cdots$$



4. The equation

$$x^2 - 3x + q = 0$$

has two real roots, α and β. Knowing that $\alpha^3 + \beta^3 = 81$, find the value of q.

Hint: It is best not to use the quadratic formula.

5. A four-digit number which is a perfect square is created by writing Anne's age in years followed by Tom's age in years. Similarly, in 31 years, their ages in the same order will again form a four-digit perfect square. Determine the present ages of Anne and Tom.

Collèges de Colombie Britannique 2005
Concours Sénior de Mathématiques du Secondaire
Ronde Finale Partie B, Vendredi, 6 Mai 2005

1. Les chiffres 1, 2, 3, 4 et 5 sont tous utilisés une fois pour écrire un nombre à cinq chiffres $abcde$ tel que le nombre à trois chiffres abc est divisible par 4, bcd est divisible par 5, cde est divisible par 3. Quel est le chiffre a ?

2. Une urne contient trois boules blanches, six rouges et quatre noires.

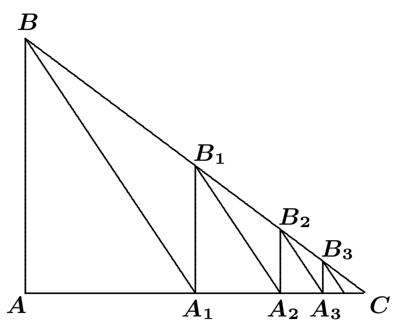
(a) Si une balle est choisie au hasard, quelle est la probabilité que la balle est rouge ?

(b) Si deux balles sont choisies au hasard, quelle est la probabilité qu'elles soient toutes les deux noires ?

(c) Si deux balles sont choisies au hasard, quelle est la probabilité qu'elles soient toutes les deux noires si elles sont de la même couleur ?

3. Dans le diagramme, ABC est un triangle rectangle avec $\overline{AB} = 3$ et $\overline{AC} = 4$. De plus, chaque segment A_iB_i est perpendiculaire à AC, A_1 coupe AC en deux, et A_{i+1} coupe A_iC en deux. Trouver la longueur totale de la sequence des segments diagonaux :

$$\overline{BA_1} + \overline{B_1A_2} + \overline{B_2A_3} + \cdots$$



4. L'équation

$$x^2 - 3x + q = 0$$

a deux racines réelles, α et β. Sachant que $\alpha^3 + \beta^3 = 81$, trouvez la valeur de q.

Indice : Il est mieux d'éviter la formule quadratique.

5. Un nombre à quatre chiffres qui est un carré parfait est créé en écrivant l'âge d'Anne en années suivi de l'âge à Tom en années. De la même manière, dans 31 ans, leurs âges dans le même ordre vont encore former un carré parfait à quatre chiffres. Determinez l'âge d'Anne et de Tom aujourd'hui.

Next we give the solutions to the 1999 New Zealand Junior Mathematics Competition [2005 : 417–420].

1999 New Zealand Junior Mathematics Competition
Sponsored by the University of Otago

1. Morris DuConfu multiplie les nombres à 2 chiffres en multipliant ensemble les chiffres des unités et des dizaines séparément puis en additionnant les résultats. On notera cette multiplication erronée par $(\times)$. Par exemple :

$$\begin{aligned} 36(\times)47 &= 42+12 = 54\,, \\ 23(\times)40 &= 0+8 = 8\,, \\ \text{et}\quad 65(\times)31 &= 5+18 = 23\,. \end{aligned}$$

Appelons cette opération le "Morris-produit".

(a) Que valent les Morris-produits $11(\times)18$, $91(\times)19$ et $35(\times)62$?

(b) Quel est le plus grand Morris-produit de 2 nombres à 2 chiffres?

(c) Trouver tous les nombres à 2 chiffres ab tels que $32(\times)ab = 32$.

(d) Quel est le plus grand produit réel de 2 nombres à 2 chiffres dont le Morris-produit est inférieur à 10?

(a) *Solution par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

On a

$$11(\times)18 = 8+1 = 9\,,\ 91(\times)19 = 9+9 = 18 \text{ et } 35(\times)62 = 10+18 = 28.$$

(b) *Solution identique par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC; et Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

Puisque le plus grand produit que l'on peut faire entre 2 chiffres est $9 \times 9 = 81$, le plus grand Morris-produit de 2 nombres à 2 chiffres est donc de $99(\times)99 = 81 + 81 = 162$.

(c) *Solution identique par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC; et Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

On a $32(\times)ab = 2b + 3a = 32$. Puisque $2b$ est pair et que 32 est pair, $3a$ doit être pair aussi, donc a doit être pair. Il ne reste qu'à essayer tous les chiffres pairs pour a. On voit assez vite que seulement $a = 6$ et $a = 8$ sont valables et donnent respectivement $b = 7$ et $b = 4$.

Il y a donc deux solutions au problème, 67 et 84.

(d) *Solution par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

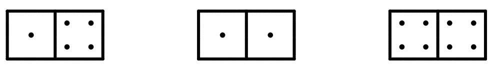
Prenons 2 nombres à 2 chiffres ab et cd et posons $ab > cd$ sans perdre de généralité. Leur Morris-produit est $ab(\times)cd = ac + bd < 10$, tandis que leur vrai produit est $100ac + 10ad + 10bc + bd$.

Pour avoir le plus grand produit possible tout en respectant la condition, il faut que $bd = 0$ et que $ac = 9$. Si $bd = 0$, alors $b = 0$ ou $d = 0$, alors $10ad = 0$ ou $10bc = 0$. Il est mieux de poser $b = 0$, car $a > c$ et il est donc mieux d'annuler c. Si $b = 0$, alors $d = 9$, car c'est sa plus grande valeur possible, donc la plus avantageuse. Puisque c est annulé par $b = 0$, il est mieux de lui donner la plus petite valeur possible pour augmenter celle de a puisque $ac = 9$. Nous avons donc $c = 1$, $a = 9$, $b = 0$ et $d = 9$.

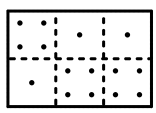
La réponse est donc $90(\times)19 = 0 + 9 = 9$ pour $90 \times 19 = 1710$.

Une solution erronée a été soumise pour (a). *La partie* (d) *a aussi été solutionnée par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.*

2. S'ennuyant à la plage, Barbara arrange des dominos sur la table. Avec les années, quelques dominos ont été perdus à tel point qu'il n'en reste que trois. Notamment :



(a) Comment arranger les dominos pour obtenir le rectangle ?

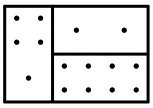


(b) Donner un exemple d'un rectangle similaire que l'on peut former exactement de 2 manières différentes avec ces trois dominos.

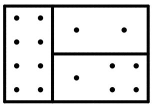
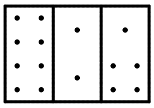
(c) Donner un exemple d'un rectangle similaire que l'on peut former exactement de 3 manières différentes avec ces trois dominos.

(d) Y a-t-il des rectangles que l'on peut former de 4 manières ?

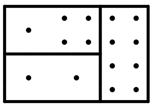
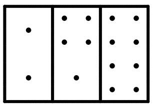
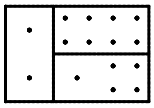
(a) *Solution identique par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC ; et Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*



(b) *Solution identique à la réflexion près par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC ; et Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

 et 

(c) *Solution identique à la réflexion près par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC ; et Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

 et  et 

(d) *Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.*

Il n'y a aucun rectangle que l'on puisse former de plus que trois manières parce qu'il y a seulement trois configurations possibles de dominos.

La partie (d) *a aussi été solutionnée par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

3. Le pays magique de EnZed est formé des îles Nord et Sud. Les habitants de l'île nord ne disent jamais la vérité, alors que ceux du sud le font tout le temps. Sur l'île du sud, ils produisent une potion magique brune appelée Spites. Un voyageur assoiffé entra un jour dans une taverne à la recherche de cette potion. Sur le comptoir il trouva 3 verres pleins et 5 personnes assis autour du bar. Son intuition lui disait que seulement un des verres contenait de la potion, alors que les autres contenaient une imitation peu savoureuse. Sans surprise, chacune des personnes autour du bar ne fit qu'un commentaire :

Andy : Le verre de gauche contient du Spites.

Brenda : Le verre de droite contient du Spites.

Carol : Andy et Brenda ne sont pas de la même île.

Deirdre : Soit Andy est de l'île nord ou Brenda est de l'île sud.

Ed : Soit je suis de l'île nord, ou Carol et Deirdre sont de la même île.

(a) En se rappelant que pour les EnZediens (et pour les mathématiciens partout) une expression du type "Soit X ou Y" est vrai si soit X ou Y ou les deux sont vrais, que peut-on conclure du commentaire à Ed ?

(b) Quel verre (gauche, centre or droite) le voyageur devrait-il prendre ?

(a) *Solution identique par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC; et Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

Ed ne peut pas venir du nord, car dire qu'il vient de l'île nord serait dire la vérité mais les gens du nord mentent toujours. Donc, il vient du sud et il dit la vérité, ce qui veut dire que Carol et Deirdre sont de la même île, car la première partie de sa déclaration est fausse.

(b) *Solution par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.*

Carol et Deirdre disent la vérité. Nous savons qu'ils sont de la même île, donc soit les deux mentent soit les deux disent la vérité. Si Carol ment, cela veut dire que Andy et Brenda sont de la même île. Or, si Deirdre ment, cela veut dire que Andy est du sud et que Brenda est du nord. Donc, les deux disent la vérité.

Comme on sait qu'ils ne sont pas de la même île, il faut que Andy soit du nord et que Brenda soit du sud, parce que l'inverse rendrait la déclaration de Deirdre fausse et que ce dernier dit la vérité.

Brenda dit donc la vérité et le voyageur doit prendre le verre de droite.

La partie (b) *a aussi été solutionnée par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.*

4. Une assiette circulaire est divisée en 20 secteurs égaux. Dix secteurs sont peints en bleu, et dix en jaune. Montrez que quelque part sur l'assiette il doit y avoir dix secteurs consécutifs, cinq étant bleus et cinq jaunes (quelque soit la manière que l'on a fait pour choisir les secteurs).

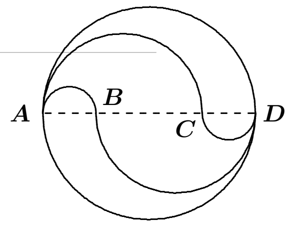
Solution par l'éditeur.

Supposons qu'il n'existe pas de telle suite. Choisissons un secteur comme étant s_1 et nommons-les en ordre jusqu'à s_{20}. Dans $(s_1, \ldots, s_{10})$, il n'y a pas 5 bleus et 5 jaunes. Il doit y en avoir plus de l'un que de l'autre. Par symétrie et sans perdre de généralité, supposons qu'il y a $n > 5$ bleus et $10 - n$ jaunes. Cela implique que dans $(s_{11}, \ldots, s_{20})$ c'est le contraire ($10 - n$ bleus et n jaunes).

Si on déplace maintenant l'intervalle que l'on étudie à $(s_2, \ldots, s_{11})$, ce que l'on fait en somme c'est remplacer s_1 par s_{11}. S'ils sont de la même couleur rien ne change à notre décompte. Par contre, s'ils sont de couleurs différentes, on passera soit à $n + 1$ bleus et $9 - n$ jaunes ou bien à $n - 1$ bleus et $11 - n$ jaunes. Mais en fait, ces 2 agencements de couleurs ont lieux en même temps car les secteurs que l'on n'étudie pas ont la répartition symétrique de celle que l'on étudie.

Compte tenu qu'à l'extérieur de notre main initiale, il y avait plus de jaunes que de bleus, on ne peut garder des mains "déséquilibrées" que pendant $\min\{n, 10 - n\} = 10 - n$ coups, ensuite, on devra piger le "surplus" de secteurs jaunes et retomber à 5 secteurs de chaque couleur.

5. Le roi Lear a l'intention de séparer sa fortune également parmi ses trois filles. Parmi ses possessions, on retrouve un large disque doré de 1m de diamètre. Pour des raisons esthétiques, il planifie que son forgeron le coupe en trois morceaux de même aire en utilisant des arcs semi-circulaires le long du diamètre AD comme sur le dessin (pas à l'échelle). Si AB et CD ont la même longueur, quelle devrait être cette longueur (exactement)?



(Le disque va être partagé selon les lignes pleines. La ligne en pointillés est seulement là pour indiquer le diamètre AD.)

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Soit A_1 l'aire de la section inférieure, A_2 celle de la section centrale et A_3 celle de la section supérieure. Donc, on a $A_1 = A_2 = A_3$. Soit A_t l'aire totale du disque ; donc $A_t = \pi$.

$$\begin{aligned} A_1 &= \tfrac{1}{2}\pi AB^2 + \left(\tfrac{1}{2}\pi AD^2 - \tfrac{1}{2}\pi BD^2\right) \qquad (\text{où } AD^2 = 1^2 = 1) \\ &= \tfrac{1}{2}\pi AB^2 + \tfrac{1}{2}\pi\left[1 - (1 - AB)^2\right] \\ &= \tfrac{1}{2}\pi AB^2 + \tfrac{1}{2}\pi\left[1 - (1 - 2AB + AB^2)\right] \\ &= \pi AB\,. \end{aligned}$$

Mais $A_1 + A_2 + A_3 = 3A_1 = A_t = \pi$. Donc, $3A_1 = 3\pi AB = \pi$, ou $AB = \frac{1}{3}$.

Aussi solutionné par Carl O'Connor, étudiant, Collège Montmorency, Laval, QC.

That brings us to the end of another issue. This month's winners of a past volume of Mayhem are Carl O'Connor and Jean-François Désilets. Congratulations!

Continue sending in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Veuillez nous transmettre vos solutions aux problèmes du présent numéro avant le **premier septembre 2006**. *Les solutions reçues après cette date ne seront prises en compte que s'il nous reste du temps avant la publication des solutions.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

M244. *Proposé par Mohammed Aassila, Strasbourg, France.*

Soit $ABCD$ un quadrilatère convexe, soit respectivement P, Q, R, S les points milieu de AB, BC, CD, DA. Supposons que quatre droites passant par P, Q, R, S se coupent en un point O. Dessiner quatre droites parallèles à ces dernières mais passant par les points milieu des côtés opposés. Montrer que ces quatre droites sont aussi concourantes.

M245. *Proposé par Ray Killgrove, Vista, CA, USA.*

Dans un triangle isocèle ABC avec $AB = AC$, les points D et E forment une trisection du troisième côté BC, c'est-à-dire $BD = DE = EC$. Pour de petits angles A, on dirait que les segments AD et AE forment une trisection de l'angle A. Montrer qu'au contraire, ce n'est jamais le cas.

M246. *Proposé par l'Équipe de Mayhem.*

Dix points sont donnés dans un plan de sorte qu'il n'y ait aucun sous-ensemble de trois points colinéaires. Quel est le nombre maximal de segments qu'il est possible de dessiner entre ces points de sorte qu'aucun triangle n'apparaisse dans la figure finale ?

M247. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit a, b et c trois nombres réels positifs avec $a + b + c = 1$. Sachant que $ab + bc + ca = \frac{1}{3}$, trouver les valeurs de :

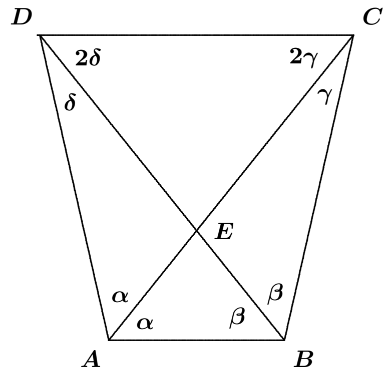
(a) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$, (b) $\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1}$.

M248. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Dans le quadrilatère convexe $ABCD$, les diagonales AC et BD forment, d'après la figure, une bisection et une trisection les angles opposés.

(a) Trouver l'angle (aigu) entre AC et BD.

(b) Montrer que $\frac{\pi}{7} < \alpha < \frac{3\pi}{7}$.



M249. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Déterminer les nombres réels a, b, c et d sachant que les racines de l'équation $x^2 + ax - b = 0$ sont a et c, et que celles de l'équation $x^2 + cx + d = 0$ sont b et d.

M250. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit $x_1, \ldots, x_n$ des nombres réels non négatifs satisfaisant $\sum_{i=1}^{n} x_i = n$. Soit $x_{n+1} = x_1$. Montrer que $\sum_{i=1}^{n} x_i x_{i+1} \leq n$ si $n \in \{1, 2, 3, 4\}$, mais non nécessairement si $n \geq 5$.

. . . .

M244. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $ABCD$ be a convex quadrilateral, and let P, Q, R, S be the mid-points of AB, BC, CD, DA, respectively. Suppose that four distinct lines each passing through one of P, Q, R, S concur at a point O. Draw lines parallel to these four lines but passing through the mid-points of the opposite sides. Prove that these four lines are also concurrent.

M245. *Proposed by Ray Killgrove, Vista, CA, USA.*

Given isosceles triangle ABC with $AB = AC$, let the points D and E trisect the third side BC; that is, $BD = DE = EC$. For small angles A, it appears as if $\angle A$ is trisected by the segments AD and AE. Prove that, to the contrary, $\angle BAC$ is never trisected by the segments AD and AE.

M246. *Proposed by the Mayhem Staff.*

Ten points are arranged in a plane so that no three are collinear. What is the maximum number of segments that can be drawn joining two of the points such that no three of these points are all joined to form a triangle?

M247. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let a, b, c be positive real numbers with $a + b + c = 1$. Given that $ab + bc + ca = \frac{1}{3}$, find the values of:

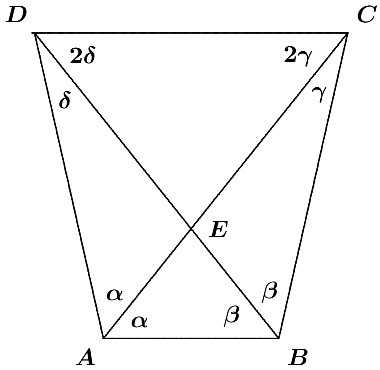
(a) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$, (b) $\frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1}$.

M248. *Proposed by K.R.S. Sastry, Bangalore, India.*

In the convex quadrilateral $ABCD$, the diagonals AC and BD bisect and trisect the opposite angles as shown.

(a) Find the (acute) angle between AC and BD.

(b) Show that $\frac{\pi}{7} < \alpha < \frac{3\pi}{7}$.



M249. *Proposed by K.R.S. Sastry, Bangalore, India.*

Determine the real numbers a, b, c, d given that the roots of the equation $x^2 + ax - b = 0$ are a and c, and the roots of the equation $x^2 + cx + d = 0$ are b and d.

M250. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers satisfying $\sum_{i=1}^{n} x_i = n$. Let $x_{n+1} = x_1$. Show that $\sum_{i=1}^{n} x_i x_{i+1} \le n$ if $n \in \{1, 2, 3, 4\}$, but not necessarily if $n \ge 5$.

Mayhem Solutions

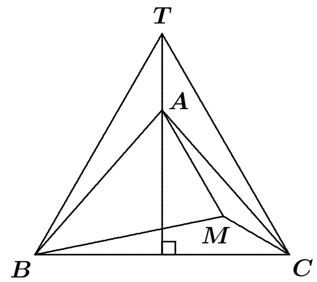
M188. *Proposed by Charalampos Stergiou, Chalkida, Greece.*

Consider triangle ABC in which $\angle B = \angle C = 35°$. In the interior of the triangle we take a point M such that $\angle MBC = 25°$ and $\angle MCB = 30°$. Prove, without trigonometry, that $\angle AMC = 150°$.

Solution by Titu Zvonaru, Cománeşti, Romania.

We will prove the following more general result: Given $\triangle ABC$ with $\angle B = \angle C = \alpha$, where $30° < \alpha < 60°$, let M be a point in the interior of the triangle such that $\angle MBC = 60° - \alpha$ and $\angle MCB = 30°$. Prove, without the aid of trigonometry, that $\angle AMC = 150°$. (For $\alpha = 35°$, we obtain the given problem.)

Let T be a point on the same side of BC as A such that $\triangle BCT$ is equilateral. Because $\triangle ABC$ and $\triangle BCT$ are both isosceles, AT is the perpendicular bisector of BC. Thus, $\angle BTA = 30°$. In triangles BAT and BMC, we have $\angle ABT = 60° - \alpha = \angle MBC$, $\angle ATB = 30° = \angle MCB$, and $BC = BT$. Hence, $\triangle BAT$ and $\triangle BMC$ are congruent.



It follows that $AB = AM$ and $\triangle ABM$ is isosceles. Then

$$\begin{aligned} \angle ABT &= \angle MBC = 60° - \alpha\,, \\ \angle ABM &= 60° - 2(60° - \alpha) = 2\alpha - 60°\,, \\ \angle BAM &= \frac{180° - \angle ABM}{2} = \frac{180° - (2\alpha - 60°)}{2} = 120° - \alpha\,, \\ \text{and}\quad \angle MAC &= \angle BAC - \angle BAM = (180° - 2\alpha) - (120° - \alpha) \\ &= 60° - \alpha\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \angle AMC &= 180° - \angle ACM - \angle MAC \\ &= 180° - (\alpha - 30°) - (60° - \alpha) = 150°\,. \end{aligned}$$

Also solved by Alper Cay, Uzman Private School, Kayseri, Turkey.

M189. *Proposed by Mihály Bencze, Brasov, Romania.*

Find all real solutions of the following system of equations:

$$\begin{aligned} x + \sqrt{x^2+1} &= 10^{y-x}\,, \\ y + \sqrt{y^2+1} &= 10^{z-y}\,, \\ z + \sqrt{z^2+1} &= 10^{x-z}\,. \end{aligned}$$

Solution by the proposer.

The system can be rewritten as

$$\begin{aligned} x+\log\left(x+\sqrt{x^2+1}\right) &= y\,, \\ y+\log\left(y+\sqrt{y^2+1}\right) &= z\,, \\ z+\log\left(z+\sqrt{z^2+1}\right) &= x\,; \end{aligned}$$

that is, $f(x)=y$, $f(y)=z$, $f(z)=x$, where $f(x)=x+\log\left(x+\sqrt{x^2+1}\right)$. If $x>0$, then $x+\sqrt{x^2+1}>1$, which implies that $f(x)>x$. Similarly, if $x<0$, then $f(x)<x$.

Let (x,y,z) be a solution to the system. If $x>0$ we have

$$x \;<\; f(x) \;=\; y \;<\; f(y) \;=\; z \;<\; f(z) \;=\; x\,,$$

a contradiction. Similarly, $x<0$ leads to a contradiction. Thus, the only possible solution is $x=y=z=0$, which is indeed a solution.

M190. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Given any three points in a unit square, show that a pair of them must be no further apart than $\sqrt{6}-\sqrt{2}$.

Solution by the proposer.

Let $DEFG$ be a unit square, labelled clockwise, and let A, B, and C be points in the square. Extending the sides of $\triangle ABC$ if necessary, we may assume that A, B, C are on the sides of $DEFG$. Sliding a side of $\triangle ABC$ if necessary, we may further assume that $A=D$. Since $1<\sqrt{6}-\sqrt{2}$, it suffices to consider that B is on EF and C is on FG. If $AB>\sqrt{6}-\sqrt{2}$, then, by the Pythagorean Theorem,

$$EB \;=\; \sqrt{AB^2-AE^2} \;>\; \sqrt{\left(8-4\sqrt{3}\right)-1} \;=\; \sqrt{7-4\sqrt{3}} \;=\; 2-\sqrt{3}\,.$$

Likewise, if $AC>\sqrt{6}-\sqrt{2}$, then $GC>2-\sqrt{3}$. Consequently,

$$BC \;=\; \sqrt{BF^2+CF^2} \;<\; \sqrt{2\left(\sqrt{3}-1\right)^2} \;=\; \sqrt{6}-\sqrt{2}\,.$$

M191. *Proposed by the Mayhem Staff.*

The surface areas of the six faces of a rectangular prism (box) are 1254, 1254, 770, 770, 1995, and 1995 cm^2. Determine the volume of the prism.

Solution by Geneviève Lalonde, Massey, ON.

Let the dimensions of the prism, in cm, be a, b, c, with $a\le b\le c$. Then $ab=770$, $ac=1254$, and $bc=1995$. Multiplying these three equations together, we get $(abc)^2=770\cdot1254\cdot1995=1926332100$. Thus, the volume of the prism, in cm^3, is $abc=\sqrt{1926332100}=43890$.

Also solved by Andrew Fischer and Frank Barlow, Humke's Raiders, Washington and Lee University, Lexington, VA.

M192. *Proposed by Victor Oxman, Western Galilee College, Israel.*

In triangles $A_1B_1C_1$ and $A_2B_2C_2$, we are given that $A_1C_1 = A_2C_2$, that the medians B_1M_1 and B_2M_2 are equal, and that the bisectors A_1D_1 and A_2D_2 are equal. Prove that the triangles are congruent.

Solution by the proposer, modified by the editor.

Consider an arbitrary triangle ABC. Let $a = BC$, $b = CA$, and $c = AB$. Let l be the length of the bisector AD, and let m be the length of the median BM. Then $m^2 = \frac{1}{2}(a^2 + c^2) - \frac{1}{4}b^2$, or equivalently,

$$a = \sqrt{2m^2 + \tfrac{1}{2}b^2 - c^2}\,, \tag{1}$$

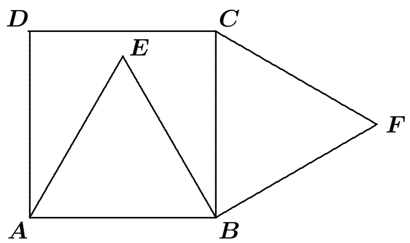
and $l^2 = bc\left(1 - \dfrac{a^2}{(b+c)^2}\right)$, or equivalently,

$$a = (b+c)\sqrt{1 - \frac{l^2}{bc}}\,. \tag{2}$$

Now suppose that b, l, and m are fixed. Then each of the equations (1) and (2) defines a as a function of c. It is easy to check that the function in (1) is strictly decreasing (in the interval where it is defined), while the function in (2) is strictly increasing (in the interval where it is defined). Therefore, the system of equations consisting of (1) and (2) cannot have more than one solution. That is, there is at most one pair (a, c) satisfying the system. This implies the proposed result.

M193. *Proposed by Robert Bilinski, Collège Montmorency, Laval, QC.*

On square $ABCD$, an equilateral triangle ABE is constructed internally and an equilateral triangle BCF is constructed externally. Prove that the points D, E, and F are collinear.



Solution by Titu Zvonaru, Comănești, Romania.

The triangle ADE is isosceles, and $\angle DAE = 90^\circ - 60^\circ = 30^\circ$. Similarly, the triangle EBF is isosceles, and $\angle EBF = 60^\circ + 30^\circ = 90^\circ$. Thus, we have

$$\begin{aligned}\angle DEF &= \angle DEA + \angle AEB + \angle BEF \\ &= \frac{180^\circ - 30^\circ}{2} + 60^\circ + \frac{180^\circ - 90^\circ}{2} = 180^\circ\,.\end{aligned}$$

Hence, the points D, E, and F are collinear.

Also solved by Luyun Zhong-Qiao, Columbia International College, Hamilton, ON.

M194. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Suppose $n-1$ and $n+1$ are twin primes where $n \in \mathbb{N}$ with $n \geq 3$. Show that $1, 2, 3, \ldots, n$ can be arranged in a row so that the sum of any two consecutive numbers is prime. (For example, when $n = 6$, one such arrangement is $6, 5, 2, 1, 4, 3$.)

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

Consider the arrangement $n, 1, n-2, 3, n-4, 5, \ldots, 2, n-1$; that is, we create the sequence $\{a_k\}_{k=1}^{n}$, where

$$a_k = \begin{cases} n-(k-1) & \text{if } k \text{ is odd,} \\ k-1 & \text{if } k \text{ is even.} \end{cases}$$

If k is odd $(1 \leq k < n)$, then $a_k + a_{k+1} = n-1$ and $a_{k+1} + a_{k+2} = n+1$; if k is even $(1 \leq k < n)$, then $a_k + a_{k+1} = n+1$ and $a_{k+1} + a_{k+2} = n-1$. Hence, the sum of any two consecutive numbers is prime, because $n-1$ and $n+1$ are primes.

Also solved by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

M195. *Proposed by J. Walter Lynch, Athens, GA, USA.*

A wire of unit length is divided into three pieces, which are used to construct a square, a circle, and an equilateral triangle such that each of them has the same area. Find the length of each of the three pieces of wire.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

Let s be the length of the edge of the square, r the radius of the circle, and b the length of the edge of the equilateral triangle. The perimeters and circumference satisfy the equation $4s + 2\pi r + 3b = 1$, and the areas, being equal, satisfy $s^2 = \pi r^2 = \frac{\sqrt{3}}{4}b^2$. Solving these last equations in terms of b, we get $s = \frac{1}{2}\sqrt[4]{3}b$ and $r = \frac{1}{2}\sqrt[4]{3}b/\sqrt{\pi}$. Substituting these values into the first equation gives $\left(2 \cdot \sqrt[4]{3} + \sqrt{\pi} \cdot \sqrt[4]{3} + 3\right)b = 1$. Thus,

$$b = \frac{1}{2 \cdot \sqrt[4]{3} + \sqrt{\pi} \cdot \sqrt[4]{3} + 3} \approx 0.125552\,.$$

Then $s = \frac{1}{2}\sqrt[4]{3}b \approx 0.082618$ and $r = s/\sqrt{\pi} \approx 0.046612$. It can be verified that these values do indeed satisfy the required equations.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.

M196. *Proposed by the Mayhem Staff.*

Committees are to be formed from a group of people. Show that the number of possible committees that can be formed with an odd number of members is exactly the same as the number of possible committees that can be formed with an even number of members.

Solution by Geneviève Lalonde, Massey, ON.

Note: We must assume that a committee of nobody and a committee of everybody are allowed. Otherwise, the result is true only when the number of people in the group is odd.

For convenience, we will refer to a committee as an *even committee* when it contains an even number of members, and an *odd committee* when it contains an odd number of members.

Suppose first that the number of people in the group is odd. Then, for each even committee, the number of people not in the committee is odd (and is therefore an odd committee). Similarly, for each odd committee, the number of people not in the committee is even (and is therefore an even committee). Thus, the even committees are in a one-to-one correspondence with the odd committees. It follow that the number of even committees is equal to the number of odd committees.

Next, suppose that the number of people in the group is even. Select one person, say John, and deal with him differently from the others. For each even committee C that does not contain John, there is a corresponding odd committee consisting of everybody who is not in C except John. For each even committee C that contains John, there is a corresponding odd committee consisting of John together with everybody who is not in C. Thus, the even committees are in one-to-one correspondence with the odd committees, and again we conclude that their numbers are equal.

M197. Corrected. *Proposed by Neven Jurič, Zagreb, Croatia.*

There are twelve ships situated on a 10×10 grid. The ships are denoted by the letters A through L, and each ship consists of three cells of the grid in either a horizontal or a vertical line, as shown in the diagram. Each ship contains a certain number of passengers. There are also some numbers in the last row and the last column of the diagram. These numbers represent the total number of passengers on all the ships intersected by that row or column. For example, the two ships B and L in the last (right-most) column together contain 9 passengers. How many passengers does each of the twelve ships contain, if there are no passengers on two of the ships and the remaining ten ships contain 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10 passengers?

A	A	A							B	
					C	C	C		B	
	E	D	D	D					B	17
	E							J		
	E				I			J		15
					I			J		
F	F	F			I		K			19
		G					K		L	
		G	H	H	H		K		L	
		G							L	6
8		21			6				9	

[*Ed:* This problem, as originally printed, was not solvable because ship I was incorrectly positioned in the diagram. This error was the fault of the editors. It has been corrected in the above diagram.]

Solution by the proposer, expanded by the editor.

Let the letters A, B, ..., L represent the number of passengers on the respective ships. From the information in the last row and column of the diagram, we have

$$\begin{aligned} B+D+E &= 17\,, \\ E+I+J &= 15\,, \\ F+I+K &= 19\,, \\ G+L &= 6\,, \\ A+F &= 8\,, \\ A+D+F+G &= 21\,, \\ C+I+H &= 6\,, \\ B+L &= 9\,, \end{aligned}$$

which can be expressed as

$$\begin{aligned} L &= 9-B\,, & (1) \\ F &= 8-A\,, & (2) \\ G &= B-3\,, & (3) \\ D &= 16-B\,, & (4) \\ E &= 1\,, & (5) \\ C &= 6-I-H\,, & (6) \\ J &= 14-I\,, & (7) \\ K &= 11+A-I\,. & (8) \end{aligned}$$

Since $L \geq 0$, we obtain $B \leq 9$ from (1). Similarly, since $D \leq 10$, we obtain $B \geq 6$ from (4). If $B = 6$, then (1) and (3) imply that $L = G = 3$, which is impossible. If $B = 8$, then (4) implies that $D = 8$, which is also impossible. Therefore, $B = 7$ or $B = 9$. If $B = 7$, then $D = 9$; if $B = 9$, then $D = 7$.

From (7) we see that $I \geq 4$, and from (6) we get $I \leq 6$. Therefore, $I \in \{4,\, 5,\, 6\}$. If $I = 5$, then $J = 9$, which is impossible (because $B = 9$ or $D = 9$). Suppose that $I = 4$. Then $J = 10$ from (7), and $K = A + 7$ from (8). Since K cannot be 7, 9, or 10, we must have $A = 1$. But this is impossible, since $E = 1$. We conclude that $I = 6$.

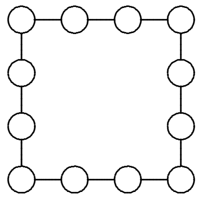
Since $I = 6$, we have $C = H = 0$ from (6), and $J = 8$ from (7). Also, $K = A + 5$ from (8). This is possible only if $A = 5$ and $K = 10$, since the other alternatives all use values that have already been assigned. Then $F = 3$ from (2).

Since $L \neq 0$, we must have $B = 7$ and $D = 9$, which gives us $L = 2$ by (1) and $G = 4$ by (3).

Thus, the only solution is

$$(A,B,C,D,E,F,G,H,I,J,K,L) \;=\; (5,7,0,9,1,3,4,0,6,8,10,2)\,.$$

M198. *Proposed by the Mayhem Staff.*



Each of the integers from 1 to 12 is to be placed in one of the circles in the figure so that the sum of the integers along each side of the figure is 25. Determine the sum of the four integers placed in the corners.

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

Place the values a, b, c, d, $\ldots$, k, and ℓ in the circles starting in the upper left corner and moving clockwise. Then we have the following equations:

$$\begin{aligned} a+b+c+d &= 25\,, & (1)\\ d+e+f+g &= 25\,, & (2)\\ g+h+i+j &= 25\,, & (3)\\ j+k+\ell+a &= 25\,, & (4)\\ a+b+c+d+\cdots+k+\ell &= 1+2+3+\cdots+12 \;=\; 78\,. & (5)\end{aligned}$$

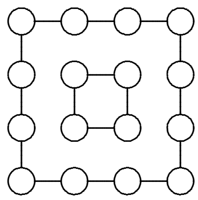
Summing equations (1) through (4) gives

$$2a+b+c+2d+e+f+2g+h+i+2j+k+\ell \;=\; 25\cdot 4 \;=\; 100\,.$$

Then, subtracting (5), we get $a+d+g+j=22$. This is the required sum.

Also solved by Makilini Balakrishan, Bell High School, Ottawa, ON; Andrea Ekholm, Bell High School, Ottawa, ON; Chelsey Gerrard, Bell High School, Ottawa, ON; Carolyn St-Amour, Bell High School, Ottawa, ON; and James Wallwork, Bell High School, Ottawa, ON.

M199. *Proposed by the Mayhem Staff.*



This is a modification of the previous problem. In this case, the requirement is to use all the integers from 1 to 16 once each so that the integers along each of the four outer edges of the large figure and the four integers that make up the inner figure have identical sums. What is the largest sum, if any, that can be obtained?

Solution by the editor.

Place the values a, b, c, d, $\ldots$, k, and ℓ in the circles that comprise the outer square, starting in the upper left corner and moving clockwise. Similarly, use m, n, o, and p for the circles that comprise the inner square. Let $S=m+n+o+p$. Then we also have

$$\begin{aligned} a+b+c+d &= S\,,\\ d+e+f+g &= S\,,\\ g+h+i+j &= S\,,\\ j+k+\ell+a &= S\,.\end{aligned}$$

Adding all five of the above equations, we get

$$\begin{aligned} 5S &= 2a+b+c+2d+e+f+2g+h+i+2j+k+\ell \\ &\qquad +m+n+o+p \\ &= (a+d+g+j)+(a+b+\cdots+p) \\ &= (a+d+g+j)+(1+2+\cdots+16) \\ &= (a+d+g+j)+136\,. \end{aligned}$$

Thus, the corner numbers, a, d, g, and j, must add to one less than a multiple of 5. To maximize S, we must maximize $a+d+g+j$. A little checking shows that $15+14+13+12=54$ works, giving $S=38$ as the maximum sum. A quick check shows that one maximal configuration is

$$\begin{aligned} &(a,b,c,d,e,f,g,h,i,j,k,\ell,m,n,o,p) \\ &\qquad = \quad (15,9,1,13,8,3,14,2,10,12,4,7,16,6,11,5)\,. \end{aligned}$$

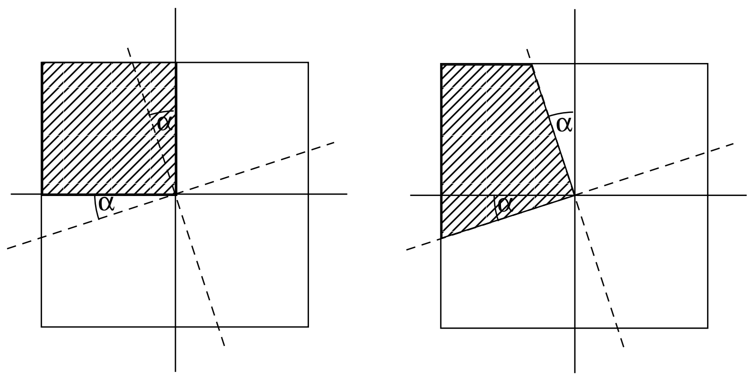
Also solved by Robert Bilinski, Collège Montmorency, Laval, QC. Bilinski points out that the English and French versions are not the same. We apologize for this oversight.

M200. *Proposed by the Mayhem Staff.*

Two perpendicular lines are drawn through the centre of a square with area 1 square unit, cutting the square into 4 pieces. What is the largest possible area for any of the pieces? Justify your answer.

Solution by Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentina.

We give a proof without words, showing that the area of any of the pieces is always $\frac{1}{4}$ square unit.



Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.

Problem of the Month

Ian VanderBurgh

It's time for a problem that is a bit more, shall we say, "complex".

Problem (1993 Fermat Contest)

If $i^2 = -1$ and $p^3 = 5 + \sqrt{2}i$ and $q^3 = 5 - \sqrt{2}i$, then there are three real values for $p + q$. One of these values is

(A) -5 (B) 10 (C) 2 (D) -2 (E) 5

Problems about complex numbers tend to appear very seldom on contests, at least in North America. Thus, there is some merit in doing a quick review of some concepts, which may or may not be useful.

If z is a complex number, then we can write $z = a + bi$ for real numbers a and b. The number a is called the *real part* of z (denoted $\mathfrak{Re}(z) = a$) and the number b is called the *imaginary part* of z (denoted $\mathfrak{Im}(z) = b$).

For $z = a + bi$, the *modulus* of z, denoted $|z|$, is the length of the vector in $\mathbb{R}^2$ induced by z; that is, $|z| = \sqrt{a^2 + b^2}$. Also, the *conjugate* of z is the complex number $\overline{z} = a - bi$.

Solution 1. For lack of any more clever idea, we start by letting $p = a + bi$ for some real numbers a and b. Then

$$\begin{aligned} p^3 &= (a+bi)^3 = a^3 + 3a^2bi + 3ab^2i^2 + b^3i^3 \\ &= a^3 + 3a^2bi - 3ab^2 - b^3i \qquad \text{(since } i^2 = -1) \\ &= (a^3 - 3ab^2) + i(3a^2b - b^3) \,. \end{aligned}$$

Since $p^3 = 5 + \sqrt{2}i$, then, by comparing real parts and imaginary parts, we obtain the system of equations

$$\begin{aligned} a^3 - 3ab^2 &= 5 \,, \\ 3a^2b - b^3 &= \sqrt{2} \,. \end{aligned}$$

This gets us into one of these proverbial "good news, bad news" situations: the good news is that we have a system of two equations in two unknowns; the bad news is, just look at the system!

Can we simplify this somehow? Look at the modulus of p and p^3. From the given information,

$$|p^3| = |5 + \sqrt{2}i| = \sqrt{5^2 + (\sqrt{2})^2} = \sqrt{27} = 3\sqrt{3} \,.$$

Racking our brains to remember any useful connection, we might remember that $|p^3| = |p|^3$ (in general, $|p^n| = |p|^n$). Thus, $|p|$, which is real and non-negative, is the real non-negative cube root of $|p^3| = 3\sqrt{3}$, which is $\sqrt{3}$. Hence, $|a + bi| = \sqrt{3}$; that is, $a^2 + b^2 = 3$, or $b^2 = 3 - a^2$.

How does this help? Substituting this into the first equation in the system above gives us an equation in a only, namely $a^3 - 3a(3 - a^2) = 5$, or $4a^3 - 9a - 5 = 0$. We can see that $a = -1$ is a root; thus, we get $(a+1)(4a^2 - 4a - 5) = 0$. The other two roots are going to be ugly, so we will focus on $a = -1$ for a minute. If $a = -1$, then $b^2 = 3 - a^2 = 2$, or $b = \pm\sqrt{2}$. If we check these back in the two equations in a and b above, we can see that $b = \sqrt{2}$ works. Thus, one value of p is $p = -1 + \sqrt{2}i$.

But we wanted a real value for $p+q$. In fact, since q^3 is the conjugate of p^3, then a possible value for q is the conjugate of p (since $(\overline{p})^3 = \overline{p}^3$—another long-lost fact about complex numbers). Thus, $q = -1 - \sqrt{2}i$ satisfies the required equation, and in this case $p + q = -2$, giving the answer (D).

Actually, we could have seen that $q = -1 - \sqrt{2}i$ gives the correct value for q^3 by noticing that, when we checked $b = -\sqrt{2}$ in the second equation, we got $-\sqrt{2}$ on the right side instead of $\sqrt{2}$. This would be exactly the equation we would have gotten were we focussing on q instead of p.

Now, we certainly have not found all possible values for $p+q$ or even all possible real values for $p+q$. We could have done this, though, by completely solving the cubic equation to get the remaining two possible values for a, finding the corresponding values of b, and finally sorting out the values of q. Possible, but not necessary here. (This is an advantage of doing multiple-choice questions—sometimes you can get out of doing the ugly work.)

Here is another solution, which looks a bit like something that a magician would pull out of a hat.

Solution 2 Since $p^3 = 5 + \sqrt{2}i$ and $q^3 = 5 - \sqrt{2}i$, then $p^3 + q^3 = 10$ (the imaginary parts cancel) and

$$p^3q^3 \;=\; (5 + \sqrt{2}i)(5 - \sqrt{2}i) \;=\; 5^2 + (\sqrt{2})^2 = 27\,.$$

(We might think to multiply these together, as they are conjugates, and multiplying conjugates often makes nice things happen.)

Since $p^3q^3 = 27$, then $(pq)^3 = 27$; whence, a possible value for pq is 3. (There are two other complex values of pq, but they are not so simple. With some luck, we may not have to worry about them. Let's see what we get with $pq = 3$.) We have a real value for $p^3 + q^3$ and we are seeking a real value for $p + q$. We might think to be sneaky and factor

$$p^3 + q^3 \;=\; (p+q)(p^2 - pq + q^2) \;=\; (p+q)\big((p+q)^2 - 3pq\big)\,.$$

We set $S = p + q$ and run with our $pq = 3$ value to see what happens. Substituting what we know, we get $10 = S\big(S^2 - 3(3)\big)$, or $S^3 - 9S - 10 = 0$.

We could use the Rational Roots Theorem (remember this?) to try to narrow down the list of possible rational roots, obtaining $S = -2$ as a root. We could then factor out $S + 2$ to get $(S+2)(S^2 - 2S - 5) = 0$. But wait! We have what we want already: a real value for $p + q = S$, which is -2.

Now that was pretty clever, which of course means that I didn't think of it! Hopefully, I won't develop a "complex" about this.

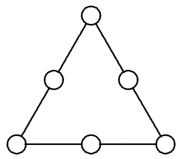
Pólya's Paragon

Magic Triangles: Beyond the Elementary Idea

John Grant McLoughlin

Puzzles commonly appear in mathematics texts as challenges. Usually the puzzles invite answers only, and the underlying mathematical principles are not discussed. The objective of this *Pólya's Paragon* is to unearth the mathematics beneath the surface of some seemingly elementary puzzles. This focus on the mathematical ideas makes the puzzles more advanced than they may initially appear. Specifically, the idea of magic triangles will be central to our discussion.

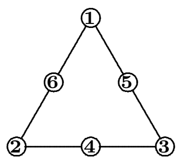
The basic magic triangle problem may be posed as follows: Consider the digits 1, 2, 3, 4, 5, and 6. Place each of the digits in one of the circles so that the sum of the digits along each side of the triangle is the same.



You are welcome to solve the problem; but the likely result is that you will find a suitable arrangement and stop. Technically it would be correct to say that you had successfully met the challenge. Mathematically there is much more than meets the eye. This particular problem has proven to be a rich teaching problem in my experiences with school age students and (prospective) teachers. Why? There is a sense of contentment among those who find a satisfactory solution. This sense is shaken by the realization that a neighbouring student has a different arrangement—in fact, a different sum. The arrangements are not merely reflections of one another but are different solutions to the same problem. Some people have never experienced such a moment in mathematics. They wonder "How many solutions are there?", or "How will I know when they have all been found?" Here we examine the problem in greater detail to answer these questions.

Observe that there are three identical sums each made up of three distinct digits selected from 1, 2, 3, 4, 5, and 6. Three of these digits will be placed at the vertices of the triangle and will thus appear in two such sums, whereas the remaining digits will appear in only one sum. Since each of 1, 2, 3, 4, 5, and 6 appears in at least one sum, the total of the three sums must be greater than 21. How much greater? By exactly the sum of the three digits placed at the vertices.

Suppose that the numbers 1, 2, and 3 appear at the vertices. The total of the three sums would become $21 + 6 = 27$, making the sum along each side equal to 9. We will refer to this common sum (that is, 9 in this case) as the magic sum. Is it possible to make a triangle under such conditions? Indeed, the arrangement falls out automatically, as shown to the right.

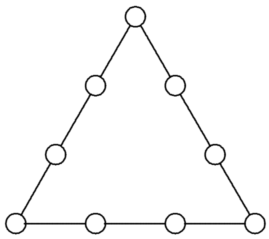


Now consider the other extreme possibility, where 4, 5, and 6 are placed at the vertices. Here the magic sum would be $(21 + 15)/3 = 12$. Again, a magic triangle falls out quickly with the placement of 3 between 4 and 5, 1 between 5 and 6, and 2 between 6 and 4.

We have found two solutions. Are there more? If so, they must have magic sums of 10 or 11. Consider the case of 10. Since the sum of the digits 1 through 6 is 21, the numbers at the vertices must total 9 (or $3 \times 10 - 21$). Two possible combinations of digits total 9: $(2, 3, 4)$ and $(1, 3, 5)$. The first combination will not produce a magic triangle because the 3 and 4 would need another 3 placed between them to obtain 10. The second combination does produce a magic triangle with the placement of 6 between 1 and 3, 2 between 3 and 5, and 4 between 5 and 1. We can apply similar reasoning to the case of 11 as a magic sum. Again we find that a solution exists.

In summary, there are four distinct magic sums possible, each of which corresponds to one solution. The elementary puzzle no longer appears like a five or ten minute challenge.

Now consider a larger triangle as shown at right. Place each of the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9 in one of the circles so that the sum of the digits along each side of the triangle is the same.



This problem may seem, at first glance, to be the same as the first problem, but the analysis is more complicated now. Observing that the sum of the digits 1 through 9 is 45, we can quickly verify that the smallest possible magic sum is $(45 + 1 + 2 + 3)/3$ and the largest is $(45 + 7 + 8 + 9)/3$. That is, the magic sums potentially range from 17 to 23 inclusive. Here is where the challenge is handed over to you, the reader about to be turned solver.

1. Show that magic triangles can be found with magic sums of 17 and 23.
2. Prove that a magic triangle with magic sum 22 does not exist.
3. Find all magic sums between 17 and 22 that produce magic triangles.

A more familiar member of the family of magical figures is the magic square in which the sum of each of the rows, columns, and main diagonals is the same. The best known example appears at right. Its magic sum is 15 and the entry in its middle square is 5. Prove that the middle entry in any 3×3 magic square must equal one-third of the magic sum.

8	1	6
3	5	7
4	9	2

I will close this Paragon with a problem given to me by Gerry Rising at University of Buffalo. It is based upon a square configuration with another twist. The entries in each row and column of the figure shown to the right are in arithmetic progression. Determine the value represented by $*$.

			*	
	74			
				186
		103		
0				

Enjoy the challenges and know that you are welcome to send along comments on any of the problems or ideas discussed in this feature.

THE OLYMPIAD CORNER

No. 254

R.E. Woodrow

To start your problem-solving challenges in this issue, we give Round 1 and Round 2 of the 2002/03 British Mathematical Olympiad. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for collecting them for our use.

BRITISH MATHEMATICAL OLYMPIAD 2002/3

Round 1

1. Given that $34! = 295\,232\,799\,cd9\,603\,140\,847\,618\,609\,643\,5ab\,000\,000$, determine the digits a, b, c, d.

2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.

Show that $BX = CX$ if and only if PQ is a diameter of S.

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$. Prove that

$$x^2yz + xy^2z + xyz^2 \;\leq\; \tfrac{1}{3}\,.$$

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid. Determine the greatest possible value of k.

5. Find all solutions in positive integers a, b, c to the equation

$$a!b! \;=\; a! + b! + c!$$

Round 2

1. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n. Determine all triples x, y, z of distinct positive integers satisying

(i) x, y, z are in arithmetic progression, and

(ii) $p(xyz) \leq 3$.

2. Let ABC be a triangle, and let D be a point on AB such that $4AD = AB$. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of θ with DA, where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line ℓ at P, show that $PB = 2PD$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.

(a) Show that there is an arithmetic progression a, $a + d$, $a + 2d$, where $d > 0$, such that $f(a) < f(a + d) < f(a + 2d)$.

(b) Must there be an arithmetic progression a, $a+d$, $\ldots$, $a+2003d$, where $d > 0$, such that $f(a) < f(a + d) < \cdots < f(a + 2003d)$?

[A permutation of $\mathbb{N}$ is a one-to-one function whose image is the whole of $\mathbb{N}$; that is, a function from $\mathbb{N}$ to $\mathbb{N}$ such that for all $m \in \mathbb{N}$ there is a unique $n \in \mathbb{N}$ such that $f(n) = m$.]

4. Let f be a function from the set of non-negative integers into itself such that, for all $n \geq 0$,

(i) $\big(f(2n + 1)\big)^2 - \big(f(2n)\big)^2 = 6f(n) + 1$, and

(ii) $f(2n) \geq f(n)$.

How many numbers less than 2003 are there in the image of f?

As a second set, we give selected problems of the Kazakh National Mathematical Olympiads 2002-2003. Thanks again go to Andy Liu for collecting them for the *Corner*.

KAZAKH NATIONAL MATHEMATICAL OLYMPIAD 2002–2003
Selected Problems

1. (*T. Akashev*) A quadrilateral $ABCD$ which is not a trapezoid is inscribed in a circle with centre O. Let M be the intersection point of the diagonals. Let K be an intersection point of the circumcircles of triangles BMC and DMA, and let L be an intersection point of the circumcircles of triangles AMB and CMD, where K, L, and M are distinct points. Prove that $OLMK$ is a rectangle.

2. (*S. Mukhanbetkaliev*) Angles B and C of triangle ABC are acute. Side KN of rectangle $KLMN$ belongs to segment BC, points L and M belong to segments AB and AC, respectively. Let O be the intersection point of the diagonals of $KLMN$. Let C_1 be the intersection point of lines BO and MN, and let B_1 be the intersection point of lines CO and LK. Prove that lines AO, BB_1, and CC_1 are concurrent.

3. (*U. Mukashev*) Find the maximal and minimal values of the sum $a+b+c$ if $a^2 + b^2 \leq c \leq 1$.

4. (*U. Mukashev*) Let two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ be such that $a_0 = b_0 = 0$ and for each positive integer n,

$$a_n = a_{n-1}^2 + 3 \quad \text{and} \quad b_n = b_{n-1}^2 + 2^n .$$

Compare the numbers a_{2003} and b_{2003}.

5. (*A. Kungozhin*) There are n grasshoppers in a row. Once a second at most one grasshopper can jump over exactly two neighbouring insects to the right or left side. For which values of n can the grasshoppers be rearranged in the reverse order?

As a final group of problems for your puzzling pleasure over the summer we give the 11th Form of the Ukrainian Mathematical Olympiad. Thanks again go to Andy Liu for collecting the contest for our use.

UKRAINIAN MATHEMATICAL OLYMPIAD
11th Form

1. Find all real k such that the following system of equations has a unique solution:

$$\begin{aligned} x^2 + y^2 &= 2k^2 , \\ kx - y &= 2k . \end{aligned}$$

2. Prove that for any triangle, if S denotes its area and r denotes the radius of its inscribed circle, then

$$\frac{S}{r^2} \geq 3\sqrt{3} .$$

3. Let $SABC$ be a triangular pyramid such that $SA + SB = CA + CB$, $SB + SC = AB + AC$, and $SC + SA = BC + BA$. Let O be the centre of its circumsphere, and let A_1, B_1, C_1 be the mid-points of the edges BC, CA, AB, respectively. Find the radius of the circumsphere of the triangular pyramid $OA_1B_1C_1$, in terms of the lengths $a = BC$, $b = CA$, and $c = AB$.

4. Let α be a real number such that five consecutive terms of the infinite sequence $\sin\alpha$, $\sin 2\alpha$, $\sin 3\alpha$, $\ldots$, $\sin n\alpha$, $\ldots$ are rational. Prove that *all* the terms of the sequence are rational.

5. Does there exist a number $q \in \mathbb{N}$ and a prime number $p \in \mathbb{N}$ such that

$$3^p + 7^p = 2 \cdot 5^q \,?$$

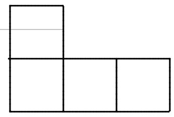
6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f\big(xf(x) + f(y)\big) = x^2 + y$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

7. Let ω be the inscribed circle of the triangle ABC. Let K, M, and N be the points of tangency of ω with the sides AB, BC, and AC, respectively. The line containing the mid-points of the segments AK and AN intersects the line containing the mid-points of the segments CM and CN at the point P. Prove that the circumcircle of the triangle APC and the circle ω are tangent.

8. Given a positive integer n, let A_n be the number of different subdivisions (by the lattice lines) of the square $(6n) \times (6n)$ cell-like board into $6n^2$ rectangles of size 2×3 (they can be oriented arbitrarily), and let B_n be the number of different subdivisions (by the lattice lines) of the square $(12n) \times (12n)$ cell-like board into $36n^2$ figures of the type shown (they can be oriented arbitrarily).



Prove that
$$B_n \geq A_n \cdot 10^{6n^2}.$$

The first set of readers' solutions pertains to problems from the 8th Macedonian Mathematical Olympiad, which appeared in [2004 : 414–415].

1. Prove that, if $m \cdot s = 2000^{2001}$ where $m, s \in \mathbb{Z}$, then the equation $mx^2 - sy^2 = 3$ has no solution in $\mathbb{Z}$.

Solved by Michel Bataille, Rouen, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give Bataille's write-up.

Let $m, s \in \mathbb{Z}$ such that $m \cdot s = 2001^{2001}$. Suppose, for the purpose of contradiction, that $mx^2 - sy^2 = 3$ for some integers x and y.

Note that $ms = 2^{8004}5^{6003}$. Since ms is even, m and s cannot both be odd. If s is even, then m must be odd, since $mx^2 = 3 + sy^2$ is odd. Thus, $m = 5^\alpha$ for some integer α with $0 \leq \alpha \leq 6003$, and $s = 2^{8004}5^{6003-\alpha}$. It follows that $5^\alpha x^2 = 3 + 2^{8004}6^{6003-\alpha}y^2$. Modulo 4, this yields $x^2 \equiv 3$, which is a contradiction, since a square is congruent to 0 or 1 modulo 4.

If m is even, then s is odd. Hence, $s = 5^\beta$ for some integer β with $0 \leq \beta \leq 6003$, and $m = 2^{8004}5^{6003-\beta}$. Thus, $3 + 5^\beta y^2 = 2^{8004}5^{6003-\beta}x^2$. This implies that β cannot lie strictly between 0 and 6003 (since 3 is not a multiple of 5). If $\beta = 0$, then $y^2 \equiv -3 \equiv 2 \pmod 5$; if $\beta = 6003$, then $x^2 \equiv 3 \pmod 5$ (since $2^{8004} = 4^{4002} \equiv (-1)^{4002} \equiv 1 \pmod 5$). We again have a contradiction, since a square is congruent to 0, 1 or 4, modulo 5.

2. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $n \geq 2$,
$$f\big(f(n-1)\big) = f(n+1) - f(n)\,?$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that there is no such function.

Assume, for a contradiction, that f is a function such that

$$f\big(f(n-1)\big) \;=\; f(n+1) - f(n)\,. \tag{1}$$

Then, for all $n \geq 2$, we have $f(n+1) - f(n) = f(f(n-1)) > 0$. Thus, f is increasing on $\{2,\,3,\,4,\,\dots\}$. Since $f(2) \geq 1$, it follows that $f(n) \geq n-1$ for all $n \geq 2$.

If $f(n) = n-1$ for all $n \geq 2$, then, for $n \geq 4$,

$$\begin{aligned} f\big(f(n-1)\big) \;&=\; f(n-2) \;=\; n-3 \\ \text{and}\quad f(n+1) - f(n) \;&=\; n-(n-1) \;=\; 1\,. \end{aligned}$$

This contradicts (1) for $n \geq 5$.

Hence, there exists $n_0 \geq 2$ such that $f(n_0) \geq n_0$. As above, we deduce that $f(n) \geq n$ for all $n \geq n_0$.

Repeating the same reasoning twice (once for n and once for $n+1$), we prove that there exists $a \geq 2$ such that $f(n) \geq n+2$ for all $n \geq a$.

Now, let $b = f(a)$. Then $b-2 \geq a$ and

$$\begin{aligned} f\big(f(a)\big) \;&=\; f(a+2) - f(a+1)\,, \\ f\big(f(a+1)\big) \;&=\; f(a+3) - f(a+2)\,, \\ &\;\;\vdots \\ f\big(f(b-2)\big) \;&=\; f(b) - f(b-1)\,. \end{aligned}$$

Summing, we obtain

$$\begin{aligned} &f\big(f(a)\big) + f\big(f(a+1)\big) + \cdots + f\big(f(b-2)\big) \\ &\quad=\; f(b) - f(a+1) \;=\; f\big(f(a)\big) - f(a+1)\,. \end{aligned}$$

Thus,

$$0 \;\leq\; f\big(f(a+1)\big) + \cdots + f\big(f(b-2)\big) \;=\; -f(a+1) \;<\; 0\,,$$

a contradiction.

4. Let M be a finite set and let $\Omega \subseteq \mathcal{P}(\mathcal{M})$ such that:

(i) If $|A \cap B| \geq 2$ for $A,\,B \in \Omega$, then $A = B$;

(ii) There are $A, B, C \in \Omega$ such that $A \neq B \neq C \neq A$ and $|A \cap B \cap C| = 1$;

(iii) For every $A \in \Omega$ and for every $a \in M \backslash A$, there is a unique $B \in \Omega$ such that $a \in B$ and $A \cap B = \emptyset$.

Prove that there are numbers p and s such that:

(a) For every $a \in M$ the number of sets which include the point a is p;

(b) $|A| = s$ for every $A \in \Omega$;

(c) $s+1 \geq p$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) For every A, $B \in \Omega$, let us write $A \sim B$ to say that either $A = B$ or $A \cap B = \emptyset$. Clearly the relation $\sim$ is reflexive and symmetric. We will prove that it is transitive.

Assume that A, B, $C \in \Omega$ are such that $A \sim B$ and $B \sim C$. The only non-trivial case to study is where $A \cap B = \emptyset = B \cap C$. Suppose that $A \cap C \neq \emptyset$. Then there is some $x \in A \cap C$, and $x \notin B$. From (iii), there is only one $X \in \Omega$ such that $x \in X$ and $X \cap B = \emptyset$. Since both A and C satisfy these conditions, it follows that $A = C$; whence, $A \sim C$, as desired.

It follows that $\sim$ is an equivalence relation. Let $C_1, \ldots, C_p$ be the equivalence classes. For each i, let $n_i = |C_i|$ and $C_i(1), \ldots, C_i(n_i)$ be the elements of Ω which form the class C_i.

For each i, by definition of $\sim$, the sets $C_i(1), \ldots, C_i(n_i)$ are pairwise disjoint, and condition (iii) ensures that they form a partition of M. Thus, for each $x \in M$ and $i \in \{1, \ldots, p\}$, there is an unique element of C_i which contains x. It follows that for each $x \in M$ the number of elements of Ω which contain x is exactly the number p of classes of $\sim$.

Note that (ii) gives $p \geq 3$.

(b) **Lemma**. Let i, $i' \in \{1, \ldots, p\}$ with $i' \neq i$. For all $j \in \{1, \ldots, n_i\}$, we have $|C_i(j)| = n_{i'}$.

Proof: By construction of $\sim$ and from (i), for each $j \in \{1, \ldots, n_i\}$ and $j' \in \{1, \ldots, n_{i'}\}$, we have $|C_i(j) \cap C_{i'}(j')| = 1$. Since the $C_{i'}(j')$s form a partition of M, it follows that for a fixed j, each element of $C_i(j)$ is contained in exactly one of the $C_{i'}(j')$'s. Thus $|C_{i(j)}| = |C_{i'}| = n_{i'}$. ∎

From the lemma, since $p \geq 3$, it follows at once that $n_1 = \cdots = n_p$. Let s be this common value. The lemma ensures that $|A| = s$ for every $A \in \Omega$, and (b) is proved.

(c) Let $m = |M|$. Since the s sets which form C_1 form a partition of M, we deduce from (b) that $s^2 = m$. Let $a \in M$. From above, for each $i \geq 1$ there is a unique j such that $a \in C_i(j)$. With no loss of generality, we may assume that $a \in C_i(1)$ for all i.

Since $|C_i(1) \cap C_{i'}(1)| = 1$ as soon as $i \neq i'$, it follows that the sets $C_i(1) - \{a\}$, for $i = 1, \ldots, p$, are pairwise disjoint. Using the fact that each has cardinality $s - 1$, we deduce that $(s-1)p \leq m - 1 = s^2 - 1$, implying that $s + 1 \geq p$, and we are done.

Next we give solutions we have received to problems of the Latvian Mathematical Olympiad 2000/2001, Final Grade, 3^{rd} Round given in [2004 : 415–416].

3. Is it possible to colour all grid points in the plane white and red so that no rectangle with vertices on grid points of one colour and sides parallel to the grid lines has area from the set $\{1, 2, 4, 8, \ldots, 2^n, \ldots\}$?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Yes, such a colouring does exist.

Let us colour red the points (x, y) such that $x+y \not\equiv 1 \pmod 3$, and all the other ones white. We will show that there is no rectangle with vertices on grid points of one colour and sides parallel to the grid lines whose area equals a power of 2.

Case 1. Let $\mathcal{W}$ be a rectangle with all its vertices white and sides parallel to the grid lines. Let (a, b) and (a, d) be two of its adjacent vertices. Then $a + b \equiv a + d \equiv 1 \pmod 3$, so that $b - d \equiv 0 \pmod 3$. It follows that the area of $\mathcal{W}$, which is a multiple of $b - d$, is divisible by 3. Thus, this area cannot be a power of 2.

Case 2. Let $\mathcal{R}$ be a rectangle with all its vertices red and sides parallel to the grid lines. Let (a, b), (a, d), (c, d), (c, b) be its vertices, with $c > a$ and $d > b$. Let $x = a + b$. Assume, for a contradiction, that the area of $\mathcal{W}$ is a power of 2. It follows that $c = a + 2^p$ and $d = b + 2^q$ for some non-negative integers p and q. Then, since the four vertices are red, we have

$$\begin{aligned} x &\not\equiv 1 \pmod 3 , \\ x + (-1)^p &\not\equiv 1 \pmod 3 , \\ x + (-1)^q &\not\equiv 1 \pmod 3 , \\ x + (-1)^p + (-1)^q &\not\equiv 1 \pmod 3 . \end{aligned}$$

Thus, modulo 3, we have $\{x,\, x + (-1)^p\} = \{0,\, 2\} = \{x,\, x + (-1)^q\}$. Then $x + (-1)^p \equiv x + (-1)^q \pmod 3$, and hence, p and q have the same parity. But this forces x, $x + (-1)^p$, $x + (-1)^p + (-1)^q$ to be distinct modulo 3. In particular, one of them is congruent to $1 \pmod 3$, a contradiction.

5. Prove that for each n there exists a finite graph without triangles such that in each colouring of the vertices with n colours there is an edge with equally coloured endpoints. (A known theorem.)

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let n be a positive integer, and let $k = 2^n$. Consider the graph $\mathcal{G}$ whose vertices are the points (a, b) in the plane such that $a,\, b \in \{1,\, 2,\, \dots,\, k\}$, and for which (a, b) and (c, d) are joined by an edge if and only if $a + b = c$ or $c + d = a$. For $i = 1,\, 2,\, \dots,\, k$, let Δ_i be the line with equation $x = i$.

Assume that (a, b), (c, d), (e, f) form a triangle. Without any loss of generality, we may assume that $a \le c \le e$. Then $a + b = c$ and $a + b = e$ and $c + d = e$, which forces $d = 0$, a contradiction. Thus, $\mathcal{G}$ contains no triangle.

Now, let $a,\, b \in \{1,\, 2,\, \dots,\, k\}$, with $a < b$. Then, $(a, b - a)$ is joined to (b, y) for all $y \in \{1,\, 2,\, \dots,\, k\}$. It follows that in each proper colouring of $\mathcal{G}$, each point belonging to Δ_b must have a colour different than the colour of $(a, b - a)$. Thus, in any proper colouring of $\mathcal{G}$, the set of colours determined by Δ_b must be distinct from the set of colours determined by Δ_a. Since a

and b are arbitrarily chosen, it follows that at least $k = 2^n$ pairwise distinct sets of colours must be available, and clearly none of them is the empty set. This forces $\chi(\mathcal{G}) > n$ (where $\chi(\mathcal{G})$ denotes the chromatic number of $\mathcal{G}$), so that in each colouring of $\mathcal{G}$ with only n colours, at least two adjacent vertices have the same colour.

Now we turn to the files of solutions from our readers to problems given in the December 2004 number of the *Corner*, beginning with the 13th Irish Mathematical Olympiad, which appeared at [2004 : 476–477].

1. Let S be the set of all numbers of the form $a(n) = n^2+n+1$, where n is a natural number. Prove that the product $a(n)a(n+1)$ is in S for all natural numbers n. Give, with proof, an example of a pair of elements s, $t \in S$ such that $st \notin S$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Robert Bilinski, Collège Montmorency, Laval, QC; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We first give Wang's comment.

The first part of this problem is the same as problem #2 of the 49th Mathematical Olympiad of Lithuania, which appeared previously in the *Corner* [2004 : 142; 2005 : 531]. The second part is easy. For example, $a(1) = 3$ and $a(3) = 13$, which means that $a(1)a(3) = 39$. Since the sequence $\{a(n)\}$ is clearly strictly increasing and $a(5) < 39 < a(6)$, it follows that $a(1)a(3) \notin S$.

Next we give Bilinski's solution.

For any natural numbers n and k,

$$a(n)a(n+k) = n^4+(2k+2)n^3+(k^2+3k+3)n^2+(k^2+3k+2)n+k^2+k+1\,.$$

This product is in S if and only if it has the form m^2+m+1 for some integer $m > 0$. Evidently, m must be a quadratic in n, say $m = An^2+Bn+C$, for some integers A, B, C. Then

$$\begin{aligned} m^2+m+1 &= (An^2+Bn+C)^2+(An^2+Bn+C)+1 \\ &= A^2n^4+2ABn^3+(B^2+2AC+A)n^2+B(2C+1)n+C^2+C+1\,. \end{aligned}$$

Now $a(n)a(n+k) = m^2+m+1$ if and only if the following equations are satisfied:

$$\begin{aligned} A^2 &= 1\,, \\ 2AB &= 2k+2\,, \\ B^2+2AC+A &= k^2+3k+3\,, \\ B(2C+1) &= k^2+3k+2\,, \\ C^2+C+1 &= k^2+k+1\,. \end{aligned}$$

These equations are satisfied if and only if

$$\begin{cases} A = 1 \\ B = k+1 \\ C = k = 1\,, \end{cases} \qquad \text{or} \qquad \begin{cases} A = -1 \\ B = -k-1 \\ C = k = -1\,. \end{cases}$$

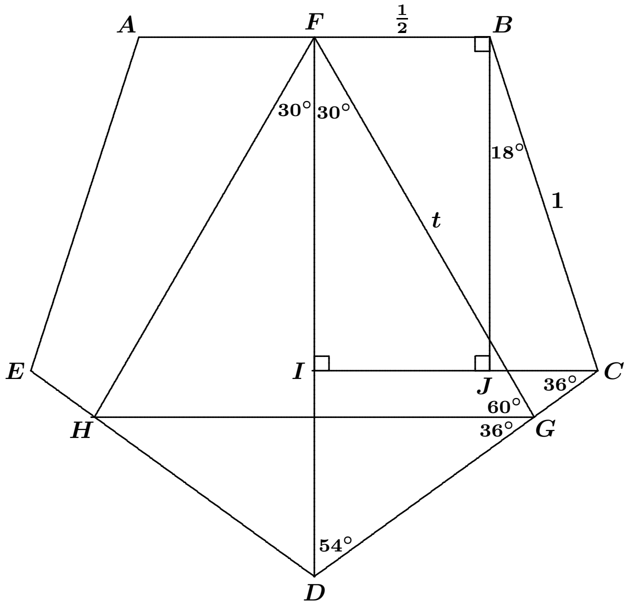
Hence, a product of the form $a(n)a(n+k)$, with $k > 0$, is in S if and only if $k = 1$. (Notice that this solves both parts of the problem.)

2. Let $ABCDE$ be a regular pentagon with its sides of length one. Let F be the mid-point of AB, and let G and H be points on the sides CD and DE, respectively, such that $\angle GFD = \angle HFD = 30^\circ$. Prove that the triangle GFH is equilateral. A square is inscribed in the triangle GFH with one side of the square along GH. Prove that FG has length

$$t = \frac{2\cos 18^\circ(\cos 36^\circ)^2}{\cos 6^\circ}\,,$$

and that the square has side length $\dfrac{t\sqrt{3}}{2+\sqrt{3}}$.

Solution by Geoffrey A. Kandall, Hamden, CT, USA.



By symmetry, we have $FG = FH$; thus, $\triangle FGH$ is equilateral. Also, $\angle FDC = \frac{1}{2}\angle EDC = 54^\circ$. Drop a perpendicular CI from C to DF, and a perpendicular BJ from B to CI. We have

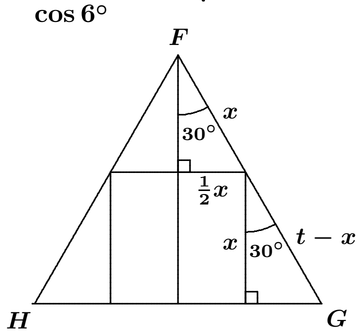
$$\begin{aligned} DF &= DI + JB = \sin 36^\circ + \cos 18^\circ = 2\cos 18^\circ \left(\sin 18^\circ + \tfrac{1}{2}\right) \\ &= 2\cos 18^\circ \cdot CI = 2\cos 18^\circ \cos 36^\circ\,. \end{aligned}$$

By the Law of Sines,

$$\frac{t}{\sin 54^\circ} = \frac{DF}{\sin 96^\circ};$$

whence,

$$t = \frac{\cos 36^\circ}{\cos 6^\circ}\cdot DF = \frac{2\cos 18^\circ(\cos 36^\circ)^2}{\cos 6^\circ}.$$



Let x be the length of a side of the square under consideration. It is clear from the diagram that $\frac{x}{\sqrt{3}} = \frac{t-x}{2}$, from which it follows easily that

$$x = \frac{t\sqrt{3}}{2+\sqrt{3}}.$$

3. Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.

Solved by Houda Anoun, Bordeaux, France; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's write-up.

Note that $f(x) = x(5x^{12} + 13x^4 + 9a)$ and $65 = 5 \times 13$. Let x be an integer.

If $x \equiv 0 \pmod{13}$, then $f(x) \equiv 0 \pmod{13}$. If $x \not\equiv 0 \pmod{13}$, then $5x^{12} + 13x^4 + 9a \equiv 5 + 9a \pmod{13}$, using Fermat's Little Theorem, and hence,

$$f(x) \equiv 0 \pmod{13} \quad \text{if and only if} \quad a \equiv -2 \pmod{13}. \qquad (1)$$

If $x \equiv 0 \pmod 5$, then $f(x) \equiv 0 \pmod 5$. If $x \not\equiv 0 \pmod 5$, then $5x^{12} + 13x^4 + 9a \equiv 3 + 9a \pmod 5$, using Fermat's Little Theorem again, and hence,

$$f(x) \equiv 0 \pmod 5 \quad \text{if and only if} \quad a \equiv -2 \pmod 5. \qquad (2)$$

From (1) and (2), we deduce that the least positive integer a such that $f(x) \equiv 0 \pmod{65}$ for all integers x, is defined by $a+2 = \operatorname{lcm}(5,13) = 65$. Thus, the desired integer is $a = 63$.

5. Consider all parabolas of the form $y = x^2 + 2px + q$ (for real p, q) which intersect the x– and y–axes in three distinct points. For such a pair p, q, let $C_{p,q}$ be the circle through the points of intersection of the parabola $y = x^2 + 2px + q$ with the axes. Prove that all the circles $C_{p,q}$ have a point in common.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; D.J. Smeenk, Zaltbommel, the Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give a comment by Michel Bataille, Rouen, France.

This problem was part of the 33rd Spanish Mathematical Olympiad 1997 (see [2001 : 93]). A solution is given in [2003 : 223].

6. Let $x \geq 0$, $y \geq 0$ be real numbers with $x + y = 2$. Prove that

$$x^2y^2(x^2+y^2) \;\leq\; 2\,.$$

Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; Vedula N. Murty, Dover, PA, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Murty's solution, modified by the editor.

By the AM–GM Inequality, we have $\sqrt{xy} \leq \frac{x+y}{2} = \frac{2}{2} = 1$; whence $xy \leq 1$. Then, using the AM–GM Inequality again, we get

$$x^2y^2 \;=\; \sqrt{x^4y^4} \;\leq\; \frac{x^4y^4+1}{2} \;\leq\; \frac{x^3y^3+1}{2}\,;$$

whence, $x^2y^2(2-xy) \leq 1$. We also have

$$x^2+y^2 \;=\; (x+y)^2 - 2xy \;=\; 4-2xy \;=\; 2(2-xy)\,.$$

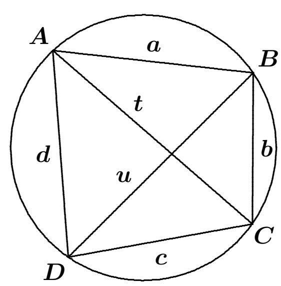
Now $x^2y^2(x^2+y^2) = 2x^2y^2(2-xy) \leq 2$.

7. Let $ABCD$ be a cyclic quadrilateral and R the radius of the circumcircle. Let a, b, c, d be the lengths of the sides of $ABCD$, and let Q be its area. Prove that

$$R^2 \;=\; \frac{(ab+cd)(ac+bd)(ad+bc)}{16Q^2}\,.$$

Deduce that $R \geq \frac{(abcd)^{\frac{3}{4}}}{Q\sqrt{2}}$, with equality if and only if $ABCD$ is a square.

Solved by Geoffrey A. Kandall, Hamden, CT, USA; and Vedula N. Murty, Dover, PA, USA. We give Kandall's write-up.



Let $t = AC$ and $u = BD$. Let $[XYZ]$ denote the area of $\triangle XYZ$. By a well-known formula, we have $4R[ABC] = abt$ and $4R[ADC] = cdt$; hence, $4RQ = (ab+cd)t$. Analogously, $4RQ = (ad+bc)u$. Consequently,

$$16R^2Q^2 \;=\; (ab+cd)tu(ad+bc)\,.$$

By Ptolemy's Theorem, $tu = ac+bd$; thus, we easily obtain the desired expression for R^2.

By the AM–GM Inequality, each of $ab+cd$, $ac+bd$, $ad+bc$ is greater than or equal to $2(abcd)^{1/2}$. Therefore,

$$R^2 \;\geq\; \frac{(abcd)^{3/2}}{2Q^2}\,; \quad \text{that is,} \quad R \;\geq\; \frac{(abcd)^{3/4}}{Q\sqrt{2}}\,.$$

If equality holds, then $ab = cd$, $ac = bd$, and $ad = bc$, which implies that $a = b = c = d$. Thus, $ABCD$ is a rhombus and, being cyclic, a square. The converse is obvious.

8. For each positive integer n, determine, with proof, all positive integers m such that there exist positive integers $x_1 < x_2 < \cdots < x_n$ which satisfy $\frac{1}{x_1} + \frac{2}{x_2} + \frac{3}{x_3} + \cdots + \frac{n}{x_n} = m$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give Sinefakopoulos's presentation.

Let n be any fixed positive integer. We first suppose that there exist positive integers $x_1 < x_2 < \cdots < x_n$ such that $\sum\limits_{i=1}^{n} \frac{i}{x_i}$ is a positive integer m. Since x_1, x_2, $\ldots$, x_n are positive integers and $x_1 < x_2 < \cdots < x_n$, we must have $x_i \geq i$ for all $i = 1$, 2, $\ldots$, n. Then

$$m \;=\; \sum_{i=1}^{n} \frac{i}{x_i} \;\leq\; \sum_{i=1}^{n} 1 \;=\; n\,.$$

Next we shall show that, for all integers m such that $1 \leq m \leq n$, such numbers x_i do exist. Indeed, for $m = n$, set $x_i = i$ and for $m = 1$, set $x_i = in$. It can be verified that $\sum\limits_{i=1}^{n} \frac{i}{x_i} = m$ in both cases. For $1 < m < n$, we write

$$\sum_{i=1}^{n} \frac{i}{x_i} \;=\; \underbrace{\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{m-1}{x_{m-1}}}_{m-1 \text{ terms}} + \underbrace{\frac{m}{x_m} + \cdots + \frac{n}{x_n}}_{n-m+1 \text{ terms}}$$

and note that, in order to get $\sum\limits_{i=1}^{n} \frac{i}{x_i} = m$, it suffices to make the first sum equal to $m-1$ by setting $x_i = i$ for $1 \leq i \leq m-1$, and the second sum equal to 1 by setting $x_i = i(n-m+1)$ for $m \leq i \leq n$. It is easy to see that $x_1 < x_2 < \cdots < x_n$ in all cases, and the proof is complete.

9. Prove that in each set of ten consecutive integers there is one which is coprime with each of the other integers. For example, taking 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, the numbers 119 and 121 are each coprime with all the others. [Two integers a, b are coprime if their greatest common divisor is 1.]

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's solution.

Consider any set of ten consecutive integers. Among them are five odd integers. Among these five consecutive odd integers, at most two are divisible by 3, at most one is divisible by 5, and at most one is divisible by 7. Therefore, at least one of the ten integers, say n, is not divisible by 2, 3, 5 or 7.

Let k be any non-zero integer such that $-9 \leq k \leq 9$, and let $d = \gcd(n, n+k)$. Note that d is a divisor of k and n. It follows that d cannot have a prime divisor less than 10 (since d divides n), and d cannot have a prime divisor greater than 10 (since d divides k). Therefore, $d = 1$. Thus, n is coprime with all nine other integers.

10. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with non-negative real coefficients. Suppose that $p(4) = 2$ and $p(16) = 8$. Prove that $p(8) \leq 4$, and find, with proof, all such polynomials with $p(8) = 4$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Geoffrey A. Kandall, Hamden, CT, USA; and Achilleas Sinefakopoulos, student, Cornell University, Ithaca, NY, USA. We give the solution by Sinefakopoulos.

Applying the Cauchy-Schwarz Inequality,

$$(u_0v_0 + u_1v_1 + \cdots + u_nv_n)^2 \;\leq\; (u_0^2 + u_1^2 + \cdots + u_n^2)(v_0^2 + v_1^2 + \cdots + v_n^2)\,,$$

with $u_i = \sqrt{a_i}\,2^i$ and $v_i = \sqrt{a_i}\,4^i$ for $1 \leq i \leq n$, we get

$$p(8)^2 \;\leq\; p(4)p(16) \;=\; 2 \cdot 8 \;=\; 16\,.$$

Taking the square root gives $p(8) \leq 4$ (since $p(8) \geq 0$).

If equality holds, then $v_i = cu_i$ for some real c and for all i. Then, since $v_i = 2^i u_i$ for all i, all u_is but one must be equal to zero. This implies that $p(x) = a_i x^i$ for some i. Then it is easy to see that $i = 1$ and $a_i = \frac{1}{2}$, so that $p(x) = \frac{1}{2}x$. Conversely, if $p(x) = \frac{1}{2}x$, then equality holds.

Next we look at solutions to problems of the Third Hong-Kong Mathematical Olympiad which appeared at [2004 : 477–478].

2. Let $a_1 = 1$, $a_{n+1} = \dfrac{a_n}{n} + \dfrac{n}{a_n}$ for $n = 1, 2, 3, \ldots$. Find the greatest integer less than or equal to a_{2000}. Be sure to prove your claim.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Direct computation gives $a_2 = a_3 = 2$. Therefore, $\sqrt{3} < a_3 < 3/\sqrt{2}$. We prove by induction that $\sqrt{n} < a_n < n/\sqrt{n-1}$ for each integer $n \geq 3$.

Assume that this holds for some $n \geq 3$. Then we have

$$0 < \sqrt{n} < a_n < \frac{n}{\sqrt{n-1}} < n\,.$$

Note that $a_{n+1} = f(a_n)$, where $f(x) = \frac{x}{n} + \frac{n}{x}$. The function f is decreasing on $(0, n)$, since $f'(x) = \frac{x^2 - n^2}{nx^2} < 0$ for $0 < x < n$. Therefore,

$$f\left(\frac{n}{\sqrt{n-1}}\right) < f(a_n) < f(\sqrt{n})\,.$$

But $f(\sqrt{n}) = \frac{1}{\sqrt{n}} + \sqrt{n} = \frac{n+1}{\sqrt{n}}$ and

$$f\left(\frac{n}{\sqrt{n-1}}\right) = \frac{1}{\sqrt{n-1}} + \sqrt{n-1} = \frac{n}{\sqrt{n-1}} > \sqrt{n+1}\,.$$

Thus, $\sqrt{n+1} < a_{n+1} < (n+1)/\sqrt{n}$, which ends the induction.

Now $44 < \sqrt{2000} < a_{2000} < \frac{2000}{\sqrt{1999}} < 45$. Therefore, $\lfloor a_{2000} \rfloor = 44$.

3. Find all prime numbers p and q such that $\frac{(7^p - 2^p)(7^q - 2^q)}{pq}$ is an integer.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Assume that (p, q) is a solution. Clearly, p, $q \notin \{2, 7\}$.

Case 1. p divides $7^p - 2^p$.

Then $7^p - 2^p \equiv 7 - 2 = 5 \equiv 0 \pmod{p}$, by Fermat's Little Theorem. Thus $p = 5$. In that case, since $7^5 - 2^5 = 5^2 \times 11 \times 61$, we deduce that either $q \in \{5, 11, 61\}$ or q divides $7^q - 2^q$, the latter giving $q = 5$ as above. Thus, since p and q play symmetric parts, this leads to the solutions $(5,5)$, $(5,11)$, $(11,5)$, $(5,61)$, $(61,5)$.

Case 2. p does not divide $7^p - 2^p$ and q does not divide $7^q - 2^q$.

Then p divides $7^q - 2^q$, and q divides $7^p - 2^p$. We see that $p \neq q$. With no loss of generality, we may assume that $p > q$. Since p is prime, it follows that $\gcd(p, q-1) = 1$. Thus, from Bezout's Theorem, there exist two positive integers a and b such that $ap - b(q-1) = 1$.

Since q divides $7^p - 2^p$, we have $7^p \equiv 2^p \pmod{q}$. We also know that $7^{q-1} \equiv 2^{q-1} \pmod{q}$, by Fermat's Little Theorem. Thus,

$$7^{ap} \equiv 2^{ap} \pmod{q} \quad \text{and} \quad 7^{b(q-1)} \equiv 2^{b(q-1)} \pmod{q}\,.$$

Then $7 = 7^{ap-b(q-1)} \equiv 2^{ap-b(q-1)} = 2 \pmod{q}$. It follows that q divides $7 - 2 = 5$. Then $q = 5$. But then q divides $7^q - 2^q$, contradicting our assumption for Case 2.

Hence, the solutions are the pairs $(5,5)$, $(5,11)$, $(11,5)$, $(5,61)$, $(61,5)$.

4. In the coordinate plane, a *lattice point* is a point with integer coordinates. Find all positive integers $n \geq 3$ such that there exists an n–sided polygon with lattice points as vertices and all sides of equal length.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The desired integers are the even integers $n \geq 4$.

Assume that $n \geq 3$ is odd and that there exists an equilateral n–gon, say $A_1 A_2 \ldots A_n$, with vertices on lattice points, and let $d > 0$ be its side length. With no loss of generality, we may assume that $A_1 = 0$ and that d is minimal.

For $i = 1, 2, \ldots, n$, let $[a_i, b_i] = \overrightarrow{A_i A_{i+1}}$ (where A_{n+1} is identified with A_1). Thus, both a_i and b_i are integers and $a_i^2 + b_i^2 = d^2$, from which we deduce that d^2 is an integer.

Moreover, since $\sum_{i=1}^{n} \overrightarrow{A_i A_{i+1}} = \overrightarrow{0}$, we have $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 0$. It follows that

$$0 = \left(\sum_{i=1}^{n} a_i\right)^2 + \left(\sum_{i=1}^{n} b_i\right)^2 = \sum_{i=1}^{n}(a_i^2 + b_i^2) + 2\sum_{i \neq j}(a_i a_j + b_i b_j),$$

which leads to $2\sum_{i \neq j}(a_i a_j + b_i b_j) = -nd^2$. Since n is odd, d^2 must be even.

First, suppose $d^2 \equiv 2 \pmod 4$. That is, $a_i^2 + b_i^2 \equiv 2 \pmod 4$, for $1 \leq i \leq n$, which ensures that a_i and b_i are odd. Thus, all the a_is and all the b_is are odd. It follows that the number $a_i a_j + b_i b_j$ is even for all i and j. Hence,

$$nd^2 = -2\sum_{i \neq j}(a_i a_j + b_i b_j) \equiv 0 \pmod 4 .$$

Since n is odd, this means that $d^2 \equiv 0 \pmod 4$, a contradiction.

Thus, $d^2 \equiv 0 \pmod 4$ and, as above, we prove that all the a_is and all the b_is are even. Since $A_1 = O$, it follows that both coordinates of A_i are even for $1 \leq i \leq n$. Therefore, using a homothety with centre O and ratio $1/2$, we deduce another equilateral n–gon with vertices on lattice points and with side length $d/2$, which contradicts the minimality of d.

Thus, in any case, we reach a contradiction, which proves that, if n is odd, there is no such equilateral polygon (note that we did not assume the polygon was convex).

Now consider the case where n is even. Let $n = 2m$ with $m \geq 2$. For rational numbers $a > 0$ and t, the point

$$M_t\left(a\frac{1-t^2}{1+t^2}, a\frac{2t}{1+t^2}\right)$$

has rational coordinates and belongs to the circle Γ_a with centre O and radius a. Choosing any m pairwise distinct rational values for $t \in (0, 1)$, we obtain m pairwise distinct points on Γ_a with positive y–coordinate. Let μ be the lowest common multiple of the denominators of all the coordinates of all the M_ts. Using an homothety with centre O and ratio μ, we deduce m pairwise distinct integer points on $\Gamma_{\mu a}$, each of them with positive y–coordinate. Then, for each chosen value of t, consider the point N_t symmetric to M_t with respect to O. We now have $2m = n$ pairwise distinct integer points which all are concylic; thus, all the vectors $\overrightarrow{OM_i}$ and $\overrightarrow{ON_i}$ have the same norm and no more than two of them have any given direction. Note that

$$\sum_{i=1}^{m} \left(\overrightarrow{OM}_i + \overrightarrow{ON}_i\right) \;=\; \overrightarrow{0}\,. \qquad (1)$$

Let $P_1 = 0$ and $\overrightarrow{OP_{i+1}} = \sum_{j=1}^{i} \overrightarrow{OM_j}$ for $1 \le i \le m$. Let

$$\overrightarrow{OP_{i+m+1}} \;=\; \left(\sum_{j=1}^{m} \overrightarrow{OM_j}\right) + \sum_{j=1}^{i} \overrightarrow{ON_j} \;=\; \overrightarrow{OP_{m+1}} + \sum_{j=1}^{i} \overrightarrow{ON_j}$$

for $1 \le i \le m$. Thus, from (1), we have $P_{2m+1} = 0$.

Moreover, for each $i = 1, 2, \ldots, n$, we have $\overrightarrow{P_iP_{i+1}} = \overrightarrow{OM_j}$ or $\overrightarrow{P_iP_{i+1}} = \overrightarrow{ON_j}$ for some j, which ensures that the polygon $\mathcal{P} = P_1P_2\ldots P_n$ is equilateral and, from the ordering, it is convex.

Then, if n is even, such a polygon does exist.

That completes the *Corner* for this issue. Over the next months please send me your nice solutions and generalizations.

BOOK REVIEW

John Grant McLoughlin

From Calculus to Computers:
Using the Last 200 Years of Mathematics History in the Classroom
By Amy Shell-Gellasch & Dick Jardine (Eds.), published by the Mathematical Association of America, 2005 (MAA Notes #68).
ISBN 0-88385-178-4, paperbound, 255+xii pages, US$48.95.
Reviewed by ***John Grant McLoughlin****, University of New Brunswick, Fredericton, NB.*

A modified title, "Using the Last 200 Years of Mathematics History in the Classroom: From Calculus to Computers", would better reflect the emphasis of this book. The collection of twenty-two articles focuses attention on 19^{th} and 20^{th} century mathematics. The majority of articles stem from a contributed papers' session at the 2001 MAA MathFest in Madison, Wisconsin, and a follow-up session a year later in Burlington, Vermont. The articles, most ranging from six to twelve pages in length, are divided into four broad sections (number of articles in brackets): I. Algebra, Number Theory, Calculus, and Dynamical Systems (5); II. Geometry (3); III. Discrete Mathematics, Computer Science, Numerical Methods, Logic, and Statistics (7); and, IV. History of Mathematics and Pedagogy (7).

The articles generally shed insight into the historical development relevant to a particular topic in mathematics, such as elliptic curves, predator-prey models (an example of mathematical modeling), or logic via Turing machines. Ideas for classroom projects, extensive references, and personal experiences with undergraduate teaching of the topics appear as the subtext throughout the collection. The volume as a whole offers a rich blend of mathematics, history, and classroom teaching ideas in a manner that will be of interest to mathematicians of many stripes. Whether your interest is Galois theory, cryptography, computer programming, or modern geometry, there is an article of specific interest to you. More importantly, any mathematician will be drawn to several articles in the book while likely finding others to be relatively uninteresting in contrast. This book would make an excellent addition to a departmental reading room or a library collection.

Mindful that this is being prepared for a problem-solving journal, the reviewer will devote the remainder of the review to aspects that may be particularly pertinent to ***CRUX with MAYHEM*** readers. The book does not appear like a problem-solving book and is clearly not intended as such. However, descriptions of courses and projects combine with the discussions of historical contexts scattered throughout the book to provide fertile ground for posing interesting problems and for revisiting problems that were historically significant. A little browsing reveals problems nestled into the structure of the articles. The most vivid example may be "Euler on Cevians" by Eisso J. Atzema and Homer White. The authors introduce the article as "a historical examination of a little-known contribution of Euler to classical Euclidean geometry, combined with a free-floating elaboration on some of Euler's results". The proofs and discussions of results offer one evident source of problems. The authors then conclude the article with an appendix consisting of various challenging mathematical "exercises" intended to deepen students' mathematical understanding of these topics. Indeed, these are problems!

The final chapter of the book is entitled "Teaching History of Mathematics Through Problems". The author, John R. Prather, presents students with a list of problems that are historically motivated as a means of engaging students in the learning of mathematical history. For example, a set of construction problems using a straightedge and compass is presented with no indication of the relative level of difficulty associated with the constructions. Indeed the trisection of a given angle will prove to be quite a challenge. Consider it to be an open problem with solutions welcomed at any time. Prather adds that it is the number theory and discrete math problems that actually seem to work best in class as they lend themselves to building conjectures, examples, and results. Readers are invited to contact the author at prather@ohiou.edu if they wish to receive a current copy of the complete problem set. Prior to placing this information here, I requested and received a copy of the problem set, as well as assurance that correspondence from readers of this review would be welcome.

In summary, this publication is not one that problem solvers will necessarily wish to add to their collections. However, it offers a wealth of ideas that can contribute to various facets of mathematical teaching and learning, particularly at the undergraduate level. Those with an interest in the history of mathematics and/or the teaching of undergraduate mathematics will enjoy the book and will possibly find the problems contained within to be a bonus. The exchange of problems and communication with John Prather has offered such a benefit to my own experience as a reviewer of the book. Likewise, I am confident that those who obtain the book for their department, institution, or personal library will find some component of it to be particularly pleasing, while gaining an appreciation for the entire collection as a worthy contribution to mathematics (education). The Mathematical Association of America should be applauded for its commitment to publish this collection.

Cycles of Residues Generated by Divisibility Tests

James T. Bruening and Deanna Kindhart

Introduction

Divisibility tests have long fascinated mathematicians and mathematics students. Tests for divisibility by 2, 3, 5, 9, and 11, for example, are very familiar and are based on modular arithmetic. The tests for 2 and 5 work because $10 \equiv 0 \pmod{2}$ and $10 \equiv 0 \pmod{5}$; tests for 3 and 9 work because $10 \equiv 1 \pmod{3}$ and $10 \equiv 1 \pmod{9}$ and the test for 11 works because $10 \equiv -1 \pmod{11}$. The history of mathematics has recorded many efforts to devise tests for divisibility by other positive integers, especially primes [1], [2], [3], [5], [7]. This paper will study cycles of residues generated by repeating tests for divisibility by a prime, and we will show relationships between the cycles and aspects of group theory from abstract algebra.

Divisibility Tests

To check for divisibility of the positive integer N by the prime 7, we write N as $10t + u$, where t and u are integers with $0 \leq u < 10$. We have $N = 10t + u \equiv 3t - 6u \equiv 3(t - 2u) \pmod{7}$. Since 3 is relatively prime to 7, we have $N \equiv 0 \pmod{7}$ if and only if $t - 2u \equiv t + 5u \equiv 0 \pmod{7}$. We will say that $t + 5u$ is a divisibility test for 7.

In general, for a prime p, we will say that $t + ku$, $1 \leq k \leq p - 1$, is a *divisibility test* for p if, for each positive integer N,

$$N \;=\; 10t + u \;\equiv\; 0 \pmod{p} \quad \text{if and only if} \quad t + ku \;\equiv\; 0 \pmod{p}\ .$$

See [1], [5]. We first prove the following.

Lemma 1 Let p be a prime that does not divide 10 and let k_p denote the smallest positive solution of $10x \equiv 1 \pmod{p})$. Then k_p is the only integer for which $t + k_p u$ is a divisibility test for p.

Proof: Since $\gcd(10, p) = 1$, the congruence $10x \equiv 1 \pmod{p}$ has exactly one incongruent solution. Hence, $10k_p \equiv 1 \pmod{p}$. Thus, for each positive integer $N = 10t + u$, one has

$$\begin{aligned} p \mid N \quad &\Longleftrightarrow \quad 10t + u \;\equiv\; 0 \pmod{p} \\ &\Longleftrightarrow \quad 10t + 10k_p u \;\equiv\; 0 \pmod{p} \\ &\Longleftrightarrow \quad t + k_p u \;\equiv\; 0 \pmod{p}\ . \end{aligned}$$

■

Examples of Cycles

Example 1: Let $p = 7$. As noted earlier, $t + 5u$ is a divisibility test for 7. If $N = 104$, then $t = 10$ and $u = 4$, and we have $t + 5u = 30$. Note that $30 \equiv 2 \pmod 7$. Since $t + 5u = 30$ in the first step, we next let $N = 30$. For this value of N, we have $t = 3$ and $u = 0$, and $t + 5u = 3$. Continuing in this fashion, one can check that the cycle of residues that one generates is

N	104	30	3	15	26	32	13
$t+5u$	30	3	15	26	32	13	16
Res	2	3	1	5	4	6	2

This is the cyclic group $\langle 5 \rangle \pmod 7$. (We chose 5 as the generator, since $k_7 = 5$.) Later we prove that this is true in general. Note that the remainder when 104 is divided by 7 is 6, the last number in the cycle before it repeats. We will also prove this in general. See Theorem 1.

Example 2: A divisibility test for the prime 13 is of the form $t + 4u$ (from Lemma 1), since $10 \cdot 4 \equiv 1 \pmod{13}$. For $N = 107$, we generate the following residues:

N	107	38	35	23	14	17	29
$t+4u$	38	35	23	14	17	29	38
Res	12	9	10	1	4	3	12

Again, the residues form a cycle, also of length 6. But this time the cycle is the cyclic group $\langle 4 \rangle \pmod{13}$, since $k_{13} = 4$ and $4^6 \equiv 1 \pmod{13}$. The remainder when 107 is divided by 13 is 3, as seen by the residue that appears just before the cycle repeats.

Example 3: Consider the cycle of residues when 93 is divided by 13.

N	93	21	6	24	18	33	15
$t+4u$	21	6	24	18	33	15	21
Res	8	6	11	5	7	2	8

This cycle of residues also contains six elements, but there are no elements in common with the cycle generated when 107 is divided by 13, because the last cycle is the left coset $8 \cdot \langle 4 \rangle \pmod{13}$. Thus, when the two cycles are combined, they give exactly the multiplicative group $\mathbb{Z}_{13}^*$, or the set of positive integers less than the divisor 13. We will say that the prime 13 has two cycles of length 6.

Cycles Generated by Repeating the Divisibility Test

We want to investigate the cycle of residues formed by repeating the $t + k_p u$ divisibility test for a given number N and a given prime $p > 5$. Let $N = N_1 = 10t_1 + u_1$, $0 \leq u_1 \leq 9$, and assume that $N \equiv \gamma \pmod p$, $1 \leq \gamma \leq p - 1$. Thus, γ is the remainder when N is divided by p. Find γ_1 such that $t_1 + k_p u_1 \equiv \gamma_1 \pmod p$, $1 \leq \gamma_1 \leq p - 1$. To generate the cycle, let $N_2 = t_1 + k_p u_1 = 10t_2 + u_2$, $0 \leq u_2 \leq 9$, and then calculate

$\gamma_2 \equiv t_2 + k_p u_2 \pmod{p}$, $1 \leq \gamma_2 \leq p-1$. Continuing this process, we can form the sequence $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$, where $1 \leq \gamma_i \leq p-1$ for all i. We have implied that the cycles of residues are either the cyclic subgroup $\langle k_p \rangle \pmod{p}$ or a coset of this subgroup. We will show that this is true in general.

Lemma 2 Let p be a prime that does not divide 10. Then the sequence of residues $\{\gamma_1, \gamma_2, \gamma_3, \ldots\}$ satisfies $\gamma_{i+1} \equiv k_p \gamma_i \pmod{p}$, for $i = 1, 2, \ldots$.

Proof: We know that $t_1 + k_p u_1 \equiv \gamma_1 \pmod{p}$. Hence, $10k_p \equiv 1 \pmod{p}$ implies that

$$N_1 \;=\; 10t_1 + u_1 \;\equiv\; \gamma \;\equiv\; 10t_1 + 10k_p u_1 \;\equiv\; 10k_p\gamma \pmod{p}\ .$$

Since 10 and p are relatively prime, we have $t_1 + k_p u_1 \equiv k_p \gamma \pmod{p}$. Therefore, $\gamma_1 \equiv k_p\gamma \pmod{p}$. Repeating the test for $N_2 = t_1 + k_p u_1$ gives $\gamma_2 \equiv k_p \gamma_1 \pmod{p}$. It follows by induction that for all i, one has $\gamma_{i+1} \equiv k_p \gamma_i \pmod{p}$. ∎

It follows at once that if a cycle of residues contains an element $\gamma_j = 1$, then this cycle forms the cyclic subgroup $\langle k_p \rangle = \{k_p, k_p^2, k_p^3, \ldots, k_p^{L_p} = 1\}$ of the multiplicative group $\mathbb{Z}_p^*$, where L_p is the number of elements in a cycle. That is, $\gamma_i \neq \gamma_j$, $1 \leq i < j \leq L_p$ and $\gamma_{L_p+1} = \gamma_1$. From this we can establish the following result.

Theorem 1 Let p be a prime that does not divide 10. Then γ_{L_p}, the last element of the cycle formed by applying the divisibility test $t + k_p u$ to N, is the remainder when N is divided by p.

Proof: As usual, let $N = 10t + u$, $0 \leq u \leq 9$. Then $k_p^{L_p} \equiv 1 \pmod{p}$, $10k_p \equiv 1 \pmod{p}$, $t + k_p u \equiv \gamma_1 \pmod{p}$, and $\gamma_{i+1} \equiv k_p \gamma_i \pmod{p}$. Thus,

$$\begin{aligned} 10t + u \;&\equiv\; 10k_p^{L_p} t + k_p^{L_p} u \;\equiv\; 10k_p k_p^{L_p-1} t + k_p^{L_p} u \\ &\equiv\; k_p^{L_p-1} t + k_p^{L_p} u \;\equiv\; k_p^{L_p-1}(t + k_p u) \\ &\equiv\; k_p^{L_p-1}\gamma_1 \;\equiv\; \gamma_{L_p} \pmod{p}\ . \end{aligned}$$

Since $1 \leq \gamma_{L_p} < p$, we see that γ_{L_p} is the remainder when N is divided by p. ∎

The last computation in the proof of Theorem 1 shows that we do not need to calculate the whole cycle $\{\gamma_1, \gamma_2, \ldots, \gamma_{L_p}\}$ to find the remainder. The fact that $10k_p \equiv 1 \pmod{p}$ implies that 10 is the inverse of k_p in the multiplicative group $\mathbb{Z}_p^*$. Hence, from $\gamma_1 \equiv \gamma_{L_p+1} \equiv k_p \gamma_{L_p} \pmod{p}$, we conclude that $\gamma_{L_p} \equiv k_p^{-1}\gamma_1 \equiv 10\gamma_1 \pmod{p}$. For example, if $N = 125$ and $p = 17$, then $k_{17} = 12$. Performing the $t + k_{17}u$ divisibility test on $N = 125$ gives $12 + 12(5) = 72$. Then $72 \equiv 4 \pmod{17}$. Thus, $\gamma_1 = 4$, and $\gamma_{L_p} \equiv 10 \cdot 4 \equiv 6 \pmod{17}$. The remainder when 125 is divided by 17 is 6.

We now relate this study of cycles of residues to established theorems and results from group theory. Using Theorem 1 and the fact that

$\langle 10 \rangle = \langle k_p^{-1} \rangle = \langle k_p \rangle$, we have $\{\gamma_1, \gamma_2, \ldots, \gamma_{L_p}\} = r \cdot \langle 10 \rangle$, where r denotes the remainder on division of N by p. Hence, the cycle is a group if and only if $r \in \langle 10 \rangle$. That is, if and only if $10^j \equiv r \pmod p$ for some non-negative integer j. The examples given above illustrate these facts.

Note that the length L_p of the cycle is also the order of k_p in $\mathbb{Z}_p^*$, since $k_p^{L_p} \equiv 1 \pmod p$ and $k_p^j \not\equiv 1 \pmod p$ for $1 \le j < L_p$. Thus, either k_p or 10 generates the cyclic subgroup $\mathbb{Z}_p^*$ if and only if $L_p = p - 1$, see [4, pp. 74-75].

Furthermore, from Lagrange's Theorem [4, p. 137], we know that the order of $\langle k_p \rangle$ is a divisor of $p - 1$, the order of the group $\mathbb{Z}_p^*$. Hence, there are exactly $m_p = (p-1)/L_p$ different cycles, and all of these cycles can be obtained by applying the $t + k_p u$ divisibility test to appropriate elements of any reduced system of positive residues $\bmod p$.

Consider also the period of the decimal expansion of $1/p$ which has been shown to be the order of 10 $\pmod p$ [6]. We have discussed earlier that 10 has order L_p modulo p; whence, the period of the decimal expansion of $1/p$ will be L_p also. For example, $1/7 = .\overline{142857}$, and the period of the decimal expansion of $p = 7$ is 6, the same as the length of the cycle illustrated in Example 1. For $p = 13$, one has $1/13 = .\overline{076923}$. Thus, the period of the decimal expansion and the length of the cycles illustrated in the second and third examples are all 6.

Cycles of Residues for Different Forms of N

Generalizing the definition of a divisibility test for p given for the case that N is written as $N = 10t + u$, $0 \le u < 10$, we say that $t + k_p^s b$ is the divisibility test for p when N is written as $N = 10^s t + b$, $0 \le b < 10^s$, where $10k_p \equiv 1 \pmod p$ (which implies that $10^s k_p^s \equiv 1 \pmod p$.) As before,

$$\begin{aligned} p \mid N &\iff N = 10^s t + b \equiv 0 \pmod p \\ &\iff 10^s t + 10^s k_p^s b \equiv 0 \pmod p \\ &\iff t + k_p^s u \equiv 0 \pmod p . \end{aligned}$$

Therefore, $t + k_p^s u$ is a divisibility test for the prime p when N is written in the form $N = 10^s t + b$.

Let us again look at the example with $N = 107$ and $p = 13$. If N is written in the form $N = 100t + b$, then $4^2 = 16 \equiv 3 \pmod{13}$ implies that $t + 3b$ is now the divisibility test for the prime 13. We then get

N	107	22	66	198
$t + 3u$	22	66	198	295
Res	9	1	3	9

The length of this cycle is now only three, and this cycle is a proper subset of the earlier cycle $\{12, 9, 10, 1, 4, 3\}$. This is not surprising, since $k_{13} = 4$, and $4^6 \equiv 1 \pmod{13}$ implies that $4^6 = (4^2)^3 \equiv 3^3 \equiv 1 \pmod{13}$. Therefore, k_{13}^2 has order 3.

If $L_{p,s}$ is the length of the cycle of residues generated by repeating the divisibility test $t+k_p^s u$ for some number $N = 10^s t+b$, $0 \leq b < 10^s$, divided by the prime p, then each cycle of residues is a coset of $\langle k_p^s \rangle \pmod{p}$, and the number of distinct cycles is $m_{p,s} = (p-1)/L_{p,s}$. It is well known that k_p^s has order $L_p/\gcd(L_p, s)$; whence, $L_{p,s} = L_p/\gcd(L_p, s)$, and $m_{p,s} = (p-1) \cdot \gcd L_{p,s}/L_p = m_p \cdot \gcd(L_p, s)$.

To illustrate the formulas for $L_{p,s}$ and $m_{p,s}$, consider the prime $p = 17$ and the divisibility test $t+12u$. Now 12 is a primitive root modulo 17. Thus, $L_{17} = 16$ and $m_{17} = 1$. For $s = 2$, one has $L_{17,2} = L_{17}/\gcd(L_{17}, 2) = 8$, and $m_{17,2} = m_{17} \cdot \gcd(L_{17}, 2) = 2$. On the other hand, for $s = 3$, we have $L_{17,2} = L_{17}/\gcd(L_{17}, 3) = 16$, and $m_{17,3} = m_{17} \cdot \gcd(L_{17}, 3) = 1$. These values can be verified by direct calculation.

We have worked in bases of the form $b = 10^s$, with $s \geq 1$ and p a prime that does not divide 10. One can readily verify that all of the results in this paper are also true in an arbitrary base b that is not a multiple of p.

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PROBLEMS

Toutes solutions aux problèmes dans ce numéro doivent nous parvenir au plus tard le **1er décembre 2006**. *Une étoile* ($\star$) *après le numéro indique que le problème a été soumis sans solution.*

Chaque problème sera publié dans les deux langues officielles du Canada (anglais et français). Dans les numéros 1, 3, 5 et 7, l'anglais précédera le français, et dans les numéros 2, 4, 6 et 8, le français précédera l'anglais. Dans la section des solutions, le problème sera publié dans la langue de la principale solution présentée.

La rédaction souhaite remercier Jean-Marc Terrier et Martin Goldstein, de l'Université de Montréal, d'avoir traduit les problèmes.

3139. *Proposé par Michel Bataille, Rouen, France.*

Soit ε l'ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Deux parallèles, tangentes à ε, coupent une troisième tangente en $M_1(x_1,y_1)$ et $M_2(x_2,y_2)$. Montrer que la valeur de

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right)$$

est indépendante des tangentes choisies.

3140. *Proposé par Michel Bataille, Rouen, France.*

Soit a_1, a_2, $\dots$, a_n n nombres réels positifs distincts, où $n\geq 2$. Pour $i=1$, 2, $\dots$, n, soit $p_i=\prod\limits_{j\neq i}(a_j-a_i)$. Montrer que $\prod\limits_{i=1}^{n}a_i^{\frac{1}{p_i}}<1$.

3141. *Proposé par José Luis Díaz-Barrero, Université Polytechnique de Catalogne, Barcelone, Espagne.*

Soit a, b et c les côtés d'un triangle scalène ABC. Montrer que

$$\sum_{\text{cyclique}}\frac{(a+1)bc}{(\sqrt{a}-\sqrt{b})(\sqrt{a}-\sqrt{c})}\;<\;\frac{a^4+b^4+c^4}{abc}\,.$$

3142. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Si $x_k>0$ pour $k=1$, 2, $\dots$, n, montrer que

(a) $\cos\left(\dfrac{n}{\sum\limits_{k=1}^{n}x_k}\right)-\sin\left(\dfrac{n}{\sum\limits_{k=1}^{n}x_k}\right)\;\geq\;\dfrac{1}{n}\sum\limits_{k=1}^{n}\left(\cos\dfrac{1}{x_k}-\sin\dfrac{1}{x_k}\right)$;

(b) $\dfrac{\sum\limits_{k=1}^{n}\sin\dfrac{1}{x_k}}{\sum\limits_{k=1}^{n}\cos\dfrac{1}{x_k}}\;\geq\;\tan\left(\dfrac{n}{\sum\limits_{k=1}^{n}x_k}\right)$.

3143. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit $a_n = 1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}$, où $n \geq 1$. Montrer que

$$\sum_{k=1}^{n} \frac{\sqrt[k]{k}}{a_k^2} < \frac{2n+1+(\ln n)^2}{n+1+\frac{1}{2}(\ln n)^2}.$$

3144. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit A, $B \in M_n(\mathbb{C})$, et soit $\omega = e^{2\pi/n}$. Montrer que

$$\sum_{k=1}^{n} \det(A + \omega^{k-1}B) + \sum_{k=1}^{n} \det(B + \omega^{k-1}A) = 2n(\det A + \det B).$$

3145★. *Proposé par Yuming Chen, Université Wilfrid Laurier, Waterloo, ON.*

Soit $f(x) = x - c^2 \tanh(x)$, où $c > 1$ est une constante arbitraire. Il n'est pas difficile de montrer que f est décroissante sur l'intervalle $[-x_0, x_0]$, où $x_0 = \ln\bigl(c + \sqrt{c^2 - 1}\bigr)$ est la racine positive de l'équation $\cosh x = c$. Pour tout $x \in (-x_0, x_0)$, la droite horizontale passant par $\bigl(x, f(x)\bigr)$ coupe le graphe de f en deux autres points d'abscisse $x_1(x)$ et $x_2(x)$. On définit une fonction $g : (-x_0, x_0) \to \mathbb{R}$ par :

$$g(x) = x + c^2 \tanh\bigl(x_1(x)\bigr) + c^2 \tanh\bigl(x_2(x)\bigr).$$

Montrer si oui ou non, $g(x) > 0$ pour tout $x \in (0, x_0)$.

3146. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit $p > 1$, et soit a, b, c, $d \in \bigl[1/\sqrt{p}, \sqrt{p}\bigr]$. Montrer que

(a) $\displaystyle \frac{p}{1+p} + \frac{2}{1+\sqrt{p}} \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}}$;

(b) $\displaystyle \frac{p}{1+p} + \frac{3}{1+\sqrt[3]{p}} \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \leq \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}$.

3147. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie, et Gabriel Dospinescu, Paris, France.*

Soit $n \geq 3$, et soit x_1, x_2, ..., x_n des nombres réels positifs tels que $x_1 x_2 \cdots x_n = 1$. Pour $n = 3$ et $n = 4$, montrer que

$$\frac{1}{x_1^2 + x_1 x_2} + \frac{1}{x_2^2 + x_2 x_3} + \cdots + \frac{1}{x_n^2 + x_n x_1} \geq \frac{n}{2}.$$

3148. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit $0 < m < 1$, et soit a, b, $c \in \left[\sqrt{m}, 1/\sqrt{m}\right]$. Montrer que

$$\frac{a^3+b^3+c^3+3(1+m)abc}{ab(a+b)+bc(b+c)+ca(c+a)} \;\geq\; 1+\frac{m}{2}\,.$$

3149. *Proposé par Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, Chine.*

Soit x, y et z trois entiers positifs. Montrer que

$$\frac{1}{x+y+z+1}-\frac{1}{(x+1)(y+1)(z+1)} \;\leq\; \frac{1}{8}\,.$$

3150. *Proposé par Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, Chine.*

Soit a, b et c les trois côtés d'un triangle, et soit h_a, h_b et h_c les hauteurs abaissées sur les côtés respectifs a, b et c. Montrer que

$$\frac{h_a^2}{b^2+c^2}\cdot\frac{h_b^2}{c^2+a^2}\cdot\frac{h_c^2}{a^2+b^2} \;\leq\; \left(\frac{3}{8}\right)^2\,.$$

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3139. *Proposed by Michel Bataille, Rouen, France.*

Let ε be the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Two parallel tangents to ε intersect a third tangent at $M_1(x_1,y_1)$ and $M_2(x_2,y_2)$. Show that the value of

$$\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1\right)$$

is independent of the chosen tangents.

3140. *Proposed by Michel Bataille, Rouen, France.*

Let a_1, a_2, $\dots$, a_n be n distinct positive real numbers, where $n \geq 2$. For $i = 1$, 2, $\dots$, n, let $p_i = \prod\limits_{j\neq i}(a_j - a_i)$. Show that $\prod\limits_{i=1}^{n} a_i^{\frac{1}{p_i}} < 1$.

3141. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let a, b, and c be the sides of a scalene triangle ABC. Prove that

$$\sum_{\text{cyclic}}\frac{(a+1)bc}{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}-\sqrt{c}\right)} \;<\; \frac{a^4+b^4+c^4}{abc}\,.$$

3142. *Proposed by Mihály Bencze, Brasov, Romania.*

If $x_k > 0$ for $k = 1, 2, \ldots, n$, prove that

(a) $$\cos\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right) - \sin\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right) \geq \frac{1}{n}\sum_{k=1}^{n}\left(\cos\frac{1}{x_k} - \sin\frac{1}{x_k}\right);$$

(b) $$\frac{\sum\limits_{k=1}^{n} \sin\frac{1}{x_k}}{\sum\limits_{k=1}^{n} \cos\frac{1}{x_k}} \geq \tan\left(\frac{n}{\sum\limits_{k=1}^{n} x_k}\right).$$

3143. *Proposed by Mihály Bencze, Brasov, Romania.*

For $n \geq 1$ let $a_n = 1 + \sqrt{2} + \sqrt[3]{3} + \cdots + \sqrt[n]{n}$. Prove that

$$\sum_{k=1}^{n} \frac{\sqrt[k]{k}}{a_k^2} < \frac{2n + 1 + (\ln n)^2}{n + 1 + \frac{1}{2}(\ln n)^2}.$$

3144. *Proposed by Mihály Bencze, Brasov, Romania.*

Let A, $B \in M_n(\mathbb{C})$, and let $\omega = e^{2\pi/n}$. Prove that

$$\sum_{k=1}^{n} \det(A + \omega^{k-1}B) + \sum_{k=1}^{n} \det(B + \omega^{k-1}A) = 2n(\det A + \det B).$$

3145★. *Proposed by Yuming Chen, Wilfrid Laurier University, Waterloo, ON.*

Let $f(x) = x - c^2 \tanh x$, where $c > 1$ is an arbitrary constant. It is not hard to show that $f(x)$ is decreasing on the interval $[-x_0, x_0]$, where $x_0 = \ln\bigl(c + \sqrt{c^2 - 1}\bigr)$ is the positive root of the equation $\cosh x = c$. For each $x \in (-x_0, x_0)$, the horizontal line passing through $\bigl(x, f(x)\bigr)$ intersects the graph of f at two other points with abscissas $x_1(x)$ and $x_2(x)$. Define a function $g : (-x_0, x_0) \to \mathbb{R}$ as follows:

$$g(x) = x + c^2 \tanh\bigl(x_1(x)\bigr) + c^2 \tanh\bigl(x_2(x)\bigr).$$

Prove or disprove that $g(x) > 0$ for all $x \in (0, x_0)$.

3146. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $p > 1$, and let $a, b, c, d \in \left[1/\sqrt{p}, \sqrt{p}\right]$. Prove that

(a) $$\frac{p}{1+p} + \frac{2}{1+\sqrt{p}} \le \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \le \frac{1}{1+p} + \frac{2\sqrt{p}}{1+\sqrt{p}};$$

(b) $$\frac{p}{1+p} + \frac{3}{1+\sqrt[3]{p}} \le \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+d} + \frac{d}{d+a} \le \frac{1}{1+p} + \frac{3\sqrt[3]{p}}{1+\sqrt[3]{p}}.$$

3147. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania; and Gabriel Dospinescu, Paris, France.*

Let $n \ge 3$, and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1x_2\cdots x_n = 1$. For $n = 3$ and $n = 4$, prove that

$$\frac{1}{x_1^2 + x_1x_2} + \frac{1}{x_2^2 + x_2x_3} + \cdots + \frac{1}{x_n^2 + x_nx_1} \ge \frac{n}{2}.$$

3148. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $0 < m < 1$, and let $a, b, c \in \left[\sqrt{m}, 1/\sqrt{m}\right]$. Prove that

$$\frac{a^3 + b^3 + c^3 + 3(1+m)abc}{ab(a+b) + bc(b+c) + ca(c+a)} \ge 1 + \frac{m}{2}.$$

3149. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let x, y, z be positive integers. Prove that

$$\frac{1}{x+y+z+1} - \frac{1}{(x+1)(y+1)(z+1)} \le \frac{1}{8}.$$

3150. *Proposed by Zhang Yun, High School attached to Xi′ An Jiao Tong University, Xi′ An City, Shan Xi, China.*

Let a, b, c be the three sides of a triangle, and let h_a, h_b, h_c be the altitudes to the sides a, b, c, respectively. Prove that

$$\frac{h_a^2}{b^2+c^2} \cdot \frac{h_b^2}{c^2+a^2} \cdot \frac{h_c^2}{a^2+b^2} \le \left(\frac{3}{8}\right)^2.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

3034. [2005 :175, 178] *Proposed by Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.*

Let a, b, c, x, y, z be positive real numbers. Prove that

$$\begin{aligned}&(bc+ca+ab)(yz+zx+xy)\\ &\quad\geq\ bcyz+cazx+abxy+2\sqrt{abcxyz(a+b+c)(x+y+z)}\,,\end{aligned}$$

and determine when equality occurs.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Dividing the given inequality by $(ab+bc+ca)(xy+yz+zx)$ yields the equivalent inequality

$$AX+BY+CZ+2\sqrt{(AB+BC+CA)(XY+YZ+ZX)}\ \leq\ 1\,,$$

where

$$\begin{aligned}A&=\frac{bc}{ab+bc+ca}\,,\quad &B&=\frac{ac}{ab+bc+ca}\,,\quad &C&=\frac{ab}{ab+bc+ca}\,,\\ X&=\frac{yz}{xy+yz+zx}\,,\quad &Y&=\frac{xz}{xy+yz+zx}\,,\quad &Z&=\frac{xy}{xy+yz+zx}\,.\end{aligned}$$

By applying AM–GM Inequality and noting that $A+B+C=1$ and $X+Y+Z=1$, we obtain

$$\begin{aligned}&AX+BY+CZ+2\sqrt{(AB+BC+CA)(XY+YZ+ZX)}\\ &\qquad\leq\ AX+BY+CZ+AB+BC+CA+XY+YZ+ZX\\ &\qquad=\ AX+BY+CZ+\tfrac{1}{2}\big((A+B+C)^2-A^2-B^2-C^2\big)\\ &\qquad\qquad+\tfrac{1}{2}\big((X+Y+Z)^2-X^2-Y^2-Z^2\big)\\ &\qquad=\ 1-\tfrac{1}{2}\big((A-X)^2+(B-Y)^2+(C-Z)^2\big)\ \leq\ 1\,.\end{aligned}$$

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; EDWARD DOOLITTLE, University of Regina, Regina, SK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. Three correct solutions failed to indicate the conditions for equality.

Zhao indicated that this problem is essentially equivalent to an inequality that appeared in the 2001 Ukrainian Math Olympiad [2003 : 498; 2005 : 443], *which asks to prove the following*

$$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha a+\beta b+\gamma c)(ab+bc+ca)}\ \leq\ a+b+c\,,$$

given that a, b, c, α, β, and γ are positive real numbers such that $\alpha+\beta+\gamma=1$.

3035. [2005 : 175, 178] *Proposed by Ali Feiz Mohammadi, student, University of Toronto, Toronto, ON.*

Are there infinitely many prime numbers that cannot be written as the sum of a prime number and a power of 2?

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

The answer is *yes*, and it is known. In [1], P. Erdős introduced the concept of a *covering system of congruences* and used it to prove that any integer congruent to 3 modulo 62 and to 2036812 ($= 2^2 \cdot 509203$) modulo 5592405 ($= 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241$) cannot be the sum of a prime number and a power of 2. By the Chinese Remainder Theorem, the system of congruences $x \equiv 3 \pmod{62}$ and $x \equiv 2036812 \pmod{5592405}$ has the solution set

$$S \;=\; \{x_0 + km \mid k \in \mathbb{Z} \text{ and } m = 62 \cdot 5592405 = 346729110\}\,,$$

where x_0 is any given solution of the system of congruences. By Dirichlet's Theorem, there are infinitely many primes in S.

For further discussion, recent results, and references, see [2].

References

[1] P. Erdős, On integers of the form $2^k + p$ and some related problems, *Summa Brasil. Math.* 2 (1950), 113–123.

[2] R.K. Guy, *Unsolved Problems in Number Theory*, 3rd ed., Springer, 2004, pages 67–69.

Also solved by JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

In addition, one reader submitted a heuristic argument that a particular subset of those primes that cannot be written as $2^n + p$ constitute over 5 percent of all primes.

3036. [2005 : 175, 178] *Proposed by Virgil Nicula, Bucharest, Romania.*

Let A, B, C be three distinct collinear fixed points. Let M be an arbitrary point not on the line ABC. The internal angle bisector of $\angle MAB$ intersects the line MB at a point X. The perpendicular at A to the line AX intersects the line MC at a point Y.

(a) Prove that the line XY passes through a fixed point D.

(b) Let Z be the projection of the point A onto the line XY. Prove that $\angle BZD = \angle CZD$.

I. *Solution by Titu Zvonaru, Comǎneşti, Romania.*

(a) Suppose that XY meets the line ABC at D. Since AX is the internal bisector of $\angle MAB$ and AY is a bisector of $\angle MAC$ (external or internal according to the relative positions of A, B, and C), by the Bisector Theorem we have

$$\frac{YM}{YC} = \frac{AM}{AC} \quad \text{and} \quad \frac{XM}{XB} = \frac{AM}{AB}\,.$$

Applying the Theorem of Menelaus to the transversal YDX of $\triangle MCB$, we obtain

$$\frac{YM}{YC}\cdot\frac{DC}{DB}\cdot\frac{XB}{XM} = 1\,.$$

Hence,

$$\frac{DC}{DB} = \frac{YC}{YM}\cdot\frac{XM}{XB} = \frac{AC}{AM}\cdot\frac{AM}{AB} = \frac{AC}{AB}\,,$$

and the position of D is fixed with respect to A and B.

Editor's comment: The precise position of D on line AB relative to the segment AB depends, of course, on the position of C relative to segment AB. For a more thorough treatment, one should either employ directed distances and directed angles, or else treat three cases separately according to which of A, B, or C is between the other two. Zvonaru simply remarked that when A is the mid-point of the segment BC, then XY is parallel to BC, so that D is at infinity. Note further that when A is between C and B (so that ZD becomes the external bisector of $\angle BZC$), the correct condition to prove in part (b) is that $\angle BZA = \angle CZA$; alternatively, in the language of directed angles, prove that, for any position of C different from A and B on line AB, the angle from line BZ to DZ equals the angle from line DZ to CZ.

(b) Here is the argument for the case when B is between A and C. Denote $AB = b$ and $AC = c$. From $\frac{DC}{DB} = \frac{AC}{AB}$ in part (a), we find that

$$\frac{DC}{BC} = \frac{DC}{DB+DC} = \frac{c}{b+c}\,;$$

therefore,

$$DC = \frac{c(c-b)}{b+c}\,, \quad \text{and} \quad AD = c - \frac{c(c-b)}{b+c} = \frac{2bc}{b+c}\,.$$

Let $AZ = t$. By the Cosine Law, we obtain

$$\begin{aligned} ZB^2 &= t^2+b^2-2tb\cos\angle ZAB \\ &= t^2+b^2-2tb\frac{t(b+c)}{2bc} = \frac{b(bc-t^2)}{c}\,, \\ \text{and} \quad ZC^2 &= t^2+c^2-2tc\cos\angle ZAC \\ &= t^2+c^2-2tc\frac{t(b+c)}{2bc} = \frac{c(bc-t^2)}{b}\,. \end{aligned}$$

It follows that

$$\frac{ZB^2}{ZC^2} = \frac{b^2}{c^2} = \frac{AB^2}{AC^2}$$

and, by the Bisector Theorem, AZ is the external bisector of $\angle BZC$; that is, ZD is the internal bisector of $\angle BZC$.

II. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

To avoid having to deal with parallel lines and non-existing intersection points, we will work in the projective plane.

(a) Let D be the intersection of lines XY and AB. We will prove that D is independent of M, which shows that it is the desired point. Let line MB meet line AY at X'. Since AX and AX' are the bisectors, internal and external, of $\angle A$ in $\triangle AMB$, it follows that X and X' are harmonic conjugates with respect to B and M. Consider the perspectivity with centre Y between the lines MB and AB. This sends the points B, M, X, X' to B, C, D, A, respectively. Thus, D and A are harmonic conjugates with respect to B and C. This property uniquely determines D in terms of A, B, and C; whence, D is independent of M. It follows that all lines XY pass through D.

(b) Let B' be the point on AC where it intersects the reflection of the line CZ in the mirror DZ. Then ZD is one bisector of $\angle CZB'$ and ZA is its other bisector (since $ZA \perp ZD$). Thus, A and D are harmonic conjugates with respect to B' and C. But we know from part (a) that A and D are harmonic conjugates with respect to B and C; hence, we must have $B = B'$, and the result follows.

Also solved by MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

3037. [2005 : 175, 178] *Proposed by Ali Feiz Mohammadi, student, University of Toronto, Toronto, ON.*

There are 2005 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset K of senators such that for every senator in the senate, the number of enemies of that senator in the set K is an even number.

Solution by Joel Schlosberg, Bayside, NY, USA, modified by the editor.

Lemma. Let $\mathbb{F}$ be any field of characteristic 2, and let n be any odd positive integer. Suppose that $M \in M_n(\mathbb{F})$ is symmetric and that all its diagonal elements are zero. Then $\det M = 0$.

Proof: Let m_{ij} be the entry in row i and column j of M. Let S_n be the group of all permutations of $\{1, 2, \ldots, n\}$. Then

$$\det M = \sum_{\sigma \in S_n} (-1)^\sigma \prod_{i=1}^{n} m_{i\sigma(i)} = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i\sigma(i)}, \quad (1)$$

since $-1 = 1$ in $\mathbb{F}$.

If $\sigma \in S_n$ such that $\sigma^{-1} = \sigma$, then the disjoint cycles that make up σ have length 1 or 2. Since n is odd, there is at least one cycle containing just a single element k. For this k, we have $\sigma(k) = k$. Then $m_{k\sigma(k)} = m_{kk} = 0$ and hence, $\prod_{i=1}^{n} m_{i\sigma(i)} = 0$.

Now let $\sigma \in S_n$ such that $\sigma^{-1} \neq \sigma$. Since M is symmetric,

$$\prod_{i=1}^{n} m_{i\sigma^{-1}(i)} = \prod_{i=1}^{n} m_{\sigma^{-1}(i)i} = \prod_{i=1}^{n} m_{\sigma^{-1}(\sigma(i))\sigma(i)} = \prod_{i=1}^{n} m_{i\sigma(i)}.$$

Therefore, since $\mathbb{F}$ has characteristic 2,

$$\prod_{i=1}^{n} m_{i\sigma^{-1}(i)} + \prod_{i=1}^{n} m_{i\sigma(i)} = 0.$$

Thus, all terms in (1) cancel out and $\det M = 0$. ■

Returning to the original problem, we suppose that there are n senators $s_1, \ldots, s_n$, where n is any odd positive integer. Let $\mathbb{F} = \mathbb{Z}_2 = \{0, 1\}$, and let $M \in M_n(\mathbb{F})$ have entries m_{ij} defined by

$$m_{ij} = \begin{cases} 1 & \text{if } i \neq j \text{ and } s_i \text{ and } s_j \text{ are enemies,} \\ 0 & \text{otherwise.} \end{cases}$$

Then M is symmetric, and all its diagonal entries are 0. By the Lemma, $\det M = 0$.

Corresponding to any vector $v = (v_1, \ldots, v_n) \in \mathbb{Z}_2^n$, there is a unique set of senators, S_v, such that $s_i \in S_v$ if and only if $v_i = 1$. For each $k \in \{1, 2, \ldots, n\}$, the k^{th} entry of the vector Mv is the total number of enemies, modulo 2, that senator s_k has in the set S_v. Since $\det M = 0$, there is a non-zero vector $v \in \mathbb{Z}_2^n$ such that $Mv = 0$. For this vector v, let $K = S_v$. Since $Mv = 0$, each senator has 0 enemies, modulo 2, in K; that is, each senator has an even number of enemies in K.

Also solved by YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was also one incorrect solution submitted.

3038. [2005 : 175, 178] *Proposed by Virgil Nicula, Bucharest, Romania.*

Consider a triangle ABC in which $a = \max\{a, b, c\}$. Prove that the expressions $(a+b+c)\sqrt{2} - \left(\sqrt{a+b} + \sqrt{a-b}\right) \cdot \left(\sqrt{a+c} + \sqrt{a-c}\right)$ and $b^2 + c^2 - a^2$ have the same sign.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, modified by the editor.

Let $u = (a+b+c)\sqrt{2}$ and $v = \left(\sqrt{a+b}+\sqrt{a-b}\right)\left(\sqrt{a+c}+\sqrt{a-c}\right)$. Note that the sign of $u - v$ is the same as the sign of $u^2 - v^2$. We easily see that

$$u^2 - v^2 \;=\; 2(a+b+c)^2 - \left(2a + 2\sqrt{a^2-b^2}\right)\left(2a + 2\sqrt{a^2-c^2}\right).$$

We also note that

$$2(b^2 + c^2 - a^2) \;=\; 2(a+b+c)^2 - 4(a+c)(a+b).$$

Fix $\sim$ as any one of the relations $<$, $=$, or $>$. Then the statements $b^2 + c^2 - a^2 \sim 0$, $c^2 \sim a^2 - b^2$, and $c^2 \sim a^2 - c^2$ are all equivalent to each other. Thus, we have

$$\begin{aligned} 2(a+b+c)^2 - \left(2a + 2\sqrt{a^2-b^2}\right)\left(2a + 2\sqrt{a^2-c^2}\right) & \\ \sim\;\; 2(a+b+c)^2 - 4(a+c)(a+b)\,, & \end{aligned}$$

which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PANOS E. TSAOUSSOGLOU, Athens, Greece; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Comănești, Romania; and the proposer.

3039. [2005 : 237, 239] *Proposed by Dorin Mărghidanu, Colegiul Național "A.I. Cuza", Corabia, Romania.*

Let a, b be fixed non-zero real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all $x \in \mathbb{R}$,

$$f\left(x - \frac{b}{a}\right) + 2x \;\le\; \frac{a}{b}x^2 + 2\frac{b}{a} \;\le\; f\left(x + \frac{b}{a}\right) - 2x\,.$$

Combination of solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria; and Joel Schlosberg, Bayside, NY, USA.

If $y = x - \dfrac{b}{a}$, then the first inequality becomes

$$\begin{aligned} f(y) &\le \frac{a}{b}\left(y + \frac{b}{a}\right)^2 - 2\left(y + \frac{b}{a}\right) + 2\frac{b}{a} \\ &= \frac{a}{b}y^2 + 2y + \frac{b}{a} - 2y - 2\frac{b}{a} + 2\frac{b}{a} \\ &= \frac{a}{b}y^2 + \frac{b}{a}\,, \end{aligned}$$

while if $y = x + \frac{b}{a}$, then the second inequality becomes

$$\begin{aligned} f(y) &\geq \frac{a}{b}\left(y - \frac{b}{a}\right)^2 + 2\left(y - \frac{b}{a}\right) + 2\frac{b}{a} \\ &= \frac{a}{b}\,y^2 - 2y + \frac{b}{a} + 2y - 2\frac{b}{a} + 2\frac{b}{a} \\ &= \frac{a}{b}\,y^2 + \frac{b}{a}\,. \end{aligned}$$

Therefore, the only such function f is $f(x) = \frac{a}{b}\,x^2 + \frac{b}{a}$.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; CURTIS COOPER, Central Missouri State University, Warrensburg, MO, USA; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3040. [2005 : 237, 239] *Proposed by Dorin Mărghidanu, Colegiul Naţional "A.I. Cuza", Corabia, Romania.*

Prove that, for any three distinct natural numbers a, b, c greater than 1,

$$\left(1 + \frac{1}{a}\right)\left(2 + \frac{1}{b}\right)\left(3 + \frac{1}{c}\right) \;\leq\; \frac{91}{8}\,.$$

Solution by Michel Bataille, Rouen, France.

Let $F(a,b,c) = \left(1 + \frac{1}{a}\right)\left(2 + \frac{1}{b}\right)\left(3 + \frac{1}{c}\right)$. If $\min\{a,\,b,\,c\} \geq 3$, then

$$F(a,b,c) \;\leq\; \left(1 + \frac{1}{3}\right)\left(2 + \frac{1}{3}\right)\left(3 + \frac{1}{3}\right) \;=\; \frac{280}{27} \;<\; \frac{91}{8}\,.$$

Otherwise, $\min\{a,\,b,\,c\} = 2$, and we have three cases.

(1) If $\min\{a,\,b,\,c\} = c = 2$, then $a \geq 3$ and $b \geq 3$, so that

$$F(a,b,c) \;=\; F(a,b,2) \;\leq\; \left(1 + \frac{1}{3}\right)\left(2 + \frac{1}{3}\right)\left(3 + \frac{1}{2}\right) \;=\; \frac{98}{9} \;<\; \frac{91}{8}\,.$$

(2) If $\min\{a,\,b,\,c\} = b = 2$, then $a \geq 3$ and $c \geq 3$, so that

$$F(a,b,c) \;=\; F(a,2,c) \;\leq\; \left(1 + \frac{1}{3}\right)\left(2 + \frac{1}{2}\right)\left(3 + \frac{1}{3}\right) \;=\; \frac{100}{9} \;<\; \frac{91}{8}\,.$$

(3) If $\min\{a,\,b,\,c\} = a = 2$, then either $b \geq 3$ and $c \geq 4$, or $b \geq 4$ and $c \geq 3$. Thus, we have either

$$F(a,b,c) \;=\; F(2,b,c) \;\leq\; \left(1 + \frac{1}{2}\right)\left(2 + \frac{1}{3}\right)\left(3 + \frac{1}{4}\right) \;=\; \frac{91}{8}\,,$$

or

$$F(a,b,c) \;=\; F(2,b,c) \;\le\; \left(1+\tfrac{1}{2}\right)\left(2+\tfrac{1}{4}\right)\left(3+\tfrac{1}{3}\right) \;=\; \frac{90}{8} \;<\; \frac{91}{8}\,.$$

Therefore, $F(a,b,c) \le \dfrac{91}{8}$, with equality if and only if $(a,b,c) = (2,3,4)$.

Also solved by ARKADY ALT, San Jose, CA, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MIHÁLY BENCZE, Brasov, Romania; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PANOS E. TSAOUSSOGLOU, Athens, Greece; Mª JESúS VILLAR RUBIO, Santander, Spain; STAN WAGON, Macalester College, St. Paul, MN, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

3041. [2004 : 237, 239] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

Prove that

(a) $\sin x = 2^{n-1}\displaystyle\sum_{k=0}^{n-1} \sin\left(x+\frac{k\pi}{n}\right)$ for all $x \in \mathbb{R}$ and all integers $n \ge 1$;

(b) $n\cot nx = \displaystyle\sum_{k=0}^{n-1} \cot\left(x+\frac{k\pi}{n}\right)$ for $x \in \left(0,\frac{\pi}{n}\right)$.

[*Ed.:* As noted in the featured solution below, the formula in part (a) above is incorrect. This was the fault of the editors. We apologize to the proposer and to the readers.]

Solution by Michel Bataille, Rouen, France.

(a) It is easily seen that the formula as stated is incorrect. The intended formula was likely

$$\sin nx \;=\; 2^{n-1}\prod_{k=0}^{n-1}\sin\!\left(x+\frac{k\pi}{n}\right),$$

which is a well-known formula proved by Euler in his *Introductio in Analysis Infinitorum*, §240.

(b) The formula is obtained at once by logarithmic differentiation of the above formula for $\sin nx$. A direct proof can be found in Euler, *op. cit.*, §§249–250.

Also solved by ARKADY ALT, San Jose, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

All of the solvers pointed out that the formula in (a) *was incorrect. Several then suggested the correct formula and gave either a proof or a reference for it. Alt also proved a formula for the sum on the right side of the published version of* (a)*:*

$$\sum_{k=0}^{n-1} \sin\Big(x + \frac{k\pi}{n}\Big) \;=\; \frac{\cos(x - \frac{\pi}{2n})}{\sin\frac{\pi}{2n}}\,.$$

This is a special case of a more general formula

$$\sum_{k=0}^{n-1} \sin(x + ky) \;=\; \sin\Big(x + \frac{n-1}{2}y\Big)\sin\frac{ny}{2}\csc\frac{y}{2}\,,$$

which is listed in §1.341, p. 29, in I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products. *The correct version of* (a) *is in this same reference, §1.392, p. 33. Zhou noted that the corrected formula in* (a) *and the formula in* (b) *are both in I.E.W. Hobson,* A Treatise on Plane and Advanced Trigonometry, *Dover, 2005, pp. 119–120. The proposer proved his* (correct) *version of* (a) *by defining a polynomial $P(z) = z^n - e^{i(2nx)}$, factoring this polynomial, and then calculating $P(1)$ using both the original form and the factored form of $P(z)$.*

3042. [2005 : 237, 240] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $x_1,\ x_2,\ \ldots,\ x_n$ be positive numbers such that $x_1x_2\cdots x_n = 1$. For $n \geq 3$ and $0 < \lambda \leq (2n-1)/(n-1)^2$, prove that

$$\frac{1}{\sqrt{1+\lambda x_1}} + \frac{1}{\sqrt{1+\lambda x_2}} + \cdots + \frac{1}{\sqrt{1+\lambda x_n}} \;\leq\; \frac{n}{\sqrt{1+\lambda}}\,.$$

Solution by the proposer, expanded by the editor.

Let $y_i = \lambda x_i$ for all $i = 1, 2, \ldots, n$. Then the given condition becomes $\prod_{i=1}^{n} y_i = \lambda^n$ with $\lambda = \left(\prod_{i=1}^{n} y_i\right)^{\frac{1}{n}} \leq \dfrac{2n-1}{(n-1)^2}$, and the inequality to be proved becomes

$$\sum_{i=1}^{n} \frac{1}{\sqrt{1+y_i}} \;\leq\; \frac{n}{\sqrt{1+\lambda}}\,. \tag{1}$$

We first show that (1) is true when $n = 2$ and $\lambda \leq 2$.

Let $y_1 = 2s^2$ and $y_2 = 2t^2$ where $s > 0$ and $t > 0$. Then the condition $y_1y_2 = \lambda^2$ becomes $4s^2t^2 = \lambda^2$, or $\lambda = 2st$, and (1) becomes

$$\frac{1}{\sqrt{1+2s^2}} + \frac{1}{\sqrt{1+2t^2}} \;\leq\; \frac{2}{\sqrt{1+2st}}\,. \tag{2}$$

Let $p = st$ and $q = \left(\frac{1}{2}(s+t)\right)^2$. Then $p \leq q$ and $s^2 + t^2 = 4q - 2p$. Since $\lambda \leq 2$, we also have $p \leq 1$. By squaring both sides, (2) can be rewritten as

$$\frac{1}{1+2s^2}+\frac{1}{1+2t^2}-\frac{2}{1+2p} \;\le\; \frac{2}{1+2p}-\frac{2}{\sqrt{(1+2s^2)(1+2t^2)}}\,. \qquad (3)$$

By straightforward but tedious computations and noting that

$$(1+2s^2)(1+2t^2) \;=\; 1+2(4q-2p)+4p^2 \;=\; 1+4(2q-p)+4p^2\,,$$

we have

$$\begin{aligned}
&\frac{1}{1+2s^2}+\frac{1}{1+2t^2}-\frac{2}{1+2p}\\
&\qquad = \frac{2+2(4q-2p)}{(1+2s^2)(1+2t^2)}-\frac{2}{1+2p}\\
&\qquad = \frac{2+4p+4(1+2p)(2q-p)-2(1+2(4q-2p)+4p^2)}{(1+2s^2)(1+2t^2)(1+2p)}\\
&\qquad = \frac{8(2p-1)(q-p)}{(1+2s^2)(1+2t^2)(1+2p)}\,,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2}{1+2p}-\frac{2}{\sqrt{(1+2s^2)(1+2t^2)}}\\
&\quad = \frac{2\big(\sqrt{(1+2s^2)(1+2t^2)}-(1+2p)\big)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}}\\
&\quad = \frac{-2\big((1+2p)^2-(1+2s^2)(1+2t^2)\big)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\,\Big(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\Big)}\\
&\quad = \frac{-2\big(1+4p+4p^2-(1+2(4q-2p)+4p^2)\big)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\,\Big(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\Big)}\\
&\quad = \frac{16(q-p)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\,\Big(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\Big)}\,.
\end{aligned}$$

Hence, (3) now becomes

$$\begin{aligned}
&\frac{8(2p-1)(q-p)}{(1+2s^2)(1+2t^2)(1+2p)}\\
&\le \frac{16(q-p)}{(1+2p)\sqrt{(1+2s^2)(1+2t^2)}\,\Big(1+2p+\sqrt{(1+2s^2)(1+2t^2)}\Big)}\,. \qquad (4)
\end{aligned}$$

If $p=q$, then (4) clearly holds. If $p\neq q$, then (4) can be rewritten as

$$\frac{2p-1}{\sqrt{(1+2s^2)(1+2t^2)}} \;\le\; \frac{2}{1+2p+\sqrt{(1+2s^2)(1+2t^2)}}\,,$$

or $4p^2 - 1 + (2p-1)\sqrt{(1+2s^2)(1+2t^2)} \le 2\sqrt{(1+2s^2)(1+2t^2)}$. This can be rewritten as $4p^2 - 1 \le (3-2p)\sqrt{8q+(1-2p)^2}$, which is true since

$$\begin{aligned}(3-2p)&\sqrt{8q+(1-2p)^2} - 4p^2 + 1 \\ &\ge\ (3-2p)\sqrt{8p+(1-2p)^2} - 4p^2 + 1 \\ &=\ (3-2p)(1+2p) - 4p^2 + 1 \\ &=\ 4(1+p-2p^2)\ =\ 4(1+2p)(1-p)\ \ge\ 0\,.\end{aligned}$$

Thus, (4) holds with equality if and only if $p = q$.

Hence, (2) is true with equality if and only if $s = t$.

Now, we proceed by induction and assume that (1) is valid for $n-1$ for some $n \ge 3$. Let $x = \sqrt[n-1]{y_1 y_2 \cdots y_{n-1}}$. Without loss of generality, we may assume that $y_1 \le y_2 \le \cdots y_n$. Then $x \le y_n$, which implies that $x^n = x^{n-1}\cdot x \le y_1 y_2 \cdots y_n = \lambda^n$. Thus, $x \le \lambda$. If $n = 3$, then $x \le \frac{5}{4} < 2$ and if $n > 3$, then $x \le \lambda < \frac{2n-1}{(n-1)^2} < \frac{2(n-1)-1}{\left((n-1)-1\right)^2}$, since the last inequality can be easily checked to be equivalent to $2n(n-2)+1 > 0$.

Hence, by the induction hypothesis, we have

$$\frac{1}{\sqrt{1+y_1}} + \frac{1}{\sqrt{1+y_2}} + \cdots + \frac{1}{\sqrt{1+y_{n-1}}}\ \le\ \frac{n-1}{\sqrt{1+x}}\,.$$

Thus, it remains to show that, for $x \le \lambda \le \frac{2n-1}{(n-1)^2}$, we have

$$\frac{1}{\sqrt{1+y_n}} + \frac{n-1}{\sqrt{1+x}}\ \le\ \frac{n}{\sqrt{1+\lambda}}\,. \tag{5}$$

Since $x^{n-1}y_n = \lambda^n$, (5) is equivalent to

$$\sqrt{\frac{x^{n-1}}{x^{n-1}+\lambda^n}} + \frac{n-1}{\sqrt{1+x}}\ \le\ \frac{n}{\sqrt{1+\lambda}}\,. \tag{6}$$

Consider the function $f : [0,\lambda] \to \mathbb{R}$ defined by

$$f(x)\ =\ \sqrt{\frac{x^{n-1}}{x^{n-1}+\lambda^n}} + \frac{n-1}{\sqrt{1+\lambda}}\,.$$

Then $f(\lambda) = \frac{n}{\sqrt{1+\lambda}}$ and thus (6) is equivalent to $f(x) \le f(\lambda)$.

Note also that $\lambda \le \frac{n-1}{(n-1)^2}$ is equivalent to $1+\lambda \le \frac{n^2}{(n-1)^2}$, or $\frac{n}{\sqrt{1+\lambda}} \ge n-1 = f(0)$. Hence, $\lambda \le \frac{n-1}{(n-1)^2}$ is equivalent to $f(0) \le f(\lambda)$.

By direct computations, we find that

$$f'(x)\ =\ \frac{n-1}{2}\left(\frac{x^{(n-3)/2}\lambda^n}{(x^{n-1}+\lambda^n)^{3/2}} - \frac{1}{(x+1)^{3/2}}\right),$$

which has the same sign as the function $g : [0, \lambda] \to \mathbb{R}$ defined by $g(x) = \lambda^{2n/3} x^{(n-3)/3}(x+1) - x^{n-1} - \lambda^n$.

Now,

$$g'(x) = \frac{\lambda^{2n/3}}{3}\left(nx^{(n-3)/3} + (n-3)x^{(n-6)/3}\right) - (n-1)x^{n-2},$$

which has the same sign as the function $h : [0, \lambda] \to \mathbb{R}$ defined by $h(x) = \lambda^{2n/3}(nx + n - 3) - 3(n-1)x^{2n/3}$. Note that

$$h'(x) = n\left(\lambda^{2n/3} - 2(n-1)x^{(2n-3)/3}\right),$$

which has the positive root $x_0 = \lambda\left(\frac{\lambda}{2n-2}\right)^{3/(2n-3)}$.

Since $2n - 2 > 2 > \lambda$, we have $0 < x_0 < \lambda$. Furthermore, $h'(x) > 0$ for $x \in [0, x_0)$ and $h'(x) < 0$ for $x \in (x_0, \lambda]$, which imply that $h(x)$ is strictly increasing on $[0, x_0]$ and strictly decreasing on $[x_0, \lambda]$. Since $h(0) = (n-3)\lambda^{2n/3} \geq 0$ and $h(\lambda) = n\lambda^{2n/3}(\lambda - 2) < 0$, we see that $h(x)$ has a single root x_1 in $(0, \lambda)$ and that $h(x) > 0$ for $x \in (0, x_1)$ and $h(x) < 0$ for $x \in (x_1, \lambda]$.

It follows that $g'(x_1) = 0$, $g'(x) > 0$ for $x \in (0, x_1)$, and $g'(x) < 0$ for $x \in (x_1, \lambda]$. There are two cases to be considered:

(i) If $g(0) < 0$, then, from $g(\lambda) = 0$, we deduce that there is some $x_2 \in (0, \lambda)$ such that $g(x_2) = 0$, $g(x) < 0$ for $x \in [0, x_2)$ and $g(x) > 0$ for $x \in (x_2, \lambda)$. Hence, $f'(x_2) = 0$, $f'(x) < 0$ for $x \in [0, x_2)$ and $f'(x) > 0$ for $x \in (x_2, \lambda)$. Consequently, $f(x)$ is strictly decreasing on $[0, x_2]$ and strictly increasing on $[x_2, \lambda]$. It follows that $f(x) \leq \max\{f(0)\}, f(\lambda)\} = f(\lambda)$.

(ii) If $g(0) \geq 0$, then, from $g(\lambda) = 0$, we deduce that $g(x) > 0$ for $x \in (0, \lambda)$. Hence, $f'(x) > 0$ for $x \in (0, \lambda)$ and it follows that $f(x)$ is strictly increasing on $[0, \lambda]$. Consequently, $f(x) \leq f(\lambda)$.

Our proof is now complete. In (1) equality holds if all the y_is are equal; that is, equality holds in the given inequality if all the x_is are equal.

Also solved by ARKADY ALT, San Jose, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

3043. [2005 : 238, 240] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

For any convex quadrilateral $ABCD$, prove that

$$\begin{aligned} &1 - \cos(A+B)\cos(A+C)\cos(A+D) \\ &\quad \leq\; 2M \sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B+C}{2}\right)\sin\left(\frac{C+A}{2}\right), \end{aligned}$$

where $M = \max\{\sin A, \sin B, \sin C, \sin D\}$.

I. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Repeatedly applying product-to-sum identities, we have

$$\begin{aligned}
&\cos(A+B)\cos(A+C)\cos(A+D)\\
&\quad= \tfrac{1}{2}\Big(\cos(B-C)+\cos(2A+B+C)\Big)\cos(A+D)\\
&\quad= \tfrac{1}{2}\Big(\cos(B-C)+\cos(A-D)\Big)\cos(A+D)\\
&\quad= \tfrac{1}{2}\cos(B-C)\cos(A+D)+\tfrac{1}{2}\cos(A-D)\cos(A+D)\\
&\quad= \tfrac{1}{4}\cos(A-B+C+D)+\tfrac{1}{4}\cos(A+B-c+D)\\
&\qquad\qquad +\tfrac{1}{4}\cos 2A+\tfrac{1}{4}\cos 2D\\
&\quad= \tfrac{1}{4}(\cos 2A+\cos 2B+\cos 2C+\cos 2D)\\
&\quad= 1-\tfrac{1}{2}(\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D)\,.
\end{aligned}$$

Thus, the left side of the proposed inequality is simply

$$\tfrac{1}{2}(\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D)\,.$$

Using a similar method, we get

$$\begin{aligned}
&\sin\left(\tfrac{1}{2}(A+B)\right)\sin\left(\tfrac{1}{2}(B+C)\right)\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{2}\left[\cos\left(\tfrac{1}{2}(A-C)\right)-\cos\left(\tfrac{1}{2}(A+2B+C)\right)\right]\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{2}\left[\cos\left(\tfrac{1}{2}(A-C)\right)-\cos\left(\tfrac{1}{2}(B-D)\right)\right]\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{2}\cos\left(\tfrac{1}{2}(A-C)\right)\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\qquad\qquad +\tfrac{1}{2}\cos\left(\tfrac{1}{2}(B-D)\right)\sin\left(\tfrac{1}{2}(A+C)\right)\\
&\quad= \tfrac{1}{4}\sin A+\tfrac{1}{4}\sin C+\tfrac{1}{4}\sin\left(\tfrac{1}{2}(A+B+C-D)\right)\\
&\qquad\qquad +\tfrac{1}{4}\sin\left(\tfrac{1}{2}(A-B+C+D)\right)\\
&\quad= \tfrac{1}{4}(\sin A+\sin B+\sin C+\sin D)\,.
\end{aligned}$$

Hence, the right side of the proposed inequality is simply

$$\tfrac{1}{4}M(\sin A+\sin B+\sin C+\sin D)\,.$$

Therefore, the proposed inequality is equivalent to

$$\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D \;\le\; M(\sin A+\sin B+\sin C+\sin D)\,.$$

Since the quadrilateral is convex, we have $0 < A, B, C, D < \pi$, which implies that $0 < \sin A, \sin B, \sin C, \sin D \le M$; whence,

$$\begin{aligned}
&\sin^2 A+\sin^2 B+\sin^2 C+\sin^2 D\\
&\qquad\le M\sin A+M\sin B+M\sin C+M\sin D\\
&\qquad= M(\sin A+\sin B+\sin C+\sin D)\,,
\end{aligned}$$

and we are done.

II. *Solution by Michel Bataille, Rouen, France.*

Let $x = \frac{1}{2}(B + C)$, $y = \frac{1}{2}(C + A)$, and $z = \frac{1}{2}(A + B)$. We will use the following known formulas:

$$\begin{aligned} 4 \sin u \sin v \sin w &= \sin(u + v - w) + \sin(v + w - u) \\ &\qquad + \sin(w + u - v) - \sin(u + v + w)\,, \quad (1) \\ 4 \cos u \cos v \cos w &= \cos(u + v - w) + \cos(v + w - u) \\ &\qquad + \cos(w + u - v) + \cos(u + v + w)\,. \quad (2) \end{aligned}$$

Applying (1) and the fact that $A + B + C + D = 2\pi$ yields

$$4 \sin x \sin y \sin z = \sin A + \sin B + \sin C + \sin D\,;$$

applying (2) similarly yields

$$\begin{aligned} 4 \cos 2x \cos 2y \cos 2z &= \cos 2A + \cos 2B + \cos 2C + \cos 2D \\ &= 4 - 2\big(\sin^2 A + \sin^2 B + \sin^2 C + \sin^2 D\big)\,. \end{aligned}$$

Now, the proposed inequality may be rewritten as

$$\sin^2 A + \sin^2 B + \sin^2 C + \sin^2 D \le M(\sin A + \sin B + \sin C + \sin D)\,,$$

and the argument proceeds as in solution I above.

Also solved by IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.