THE OLYMPIAD CORNER

No. 252

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As a first problem set we give the 9th and 10th grades of the Romanian Mathematical Olympiad. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for collecting the set for our use.

ROMANIAN MATHEMATICAL OLYMPIAD

9th Grade

1. Find positive integers a and b such that, for every x, $y \in [a,b]$, we have $\frac{1}{x} + \frac{1}{y} \in [a,b]$.

2. An integer $n \geq 2$ is called *friendly* if there exists a family A_1, A_2, $\ldots$, A_n of subsets of the set $\{1, 2, \ldots, n\}$ such that:

(i) $i \notin A_i$ for every $i \in \{1, 2, \ldots, n\}$;

(ii) $i \in A_j$ if and only if $j \notin A_i$, for every distinct i, $j \in \{1, 2, \ldots, n\}$;

(iii) $A_i \cap A_j$ is non-empty for every i, $j \in \{1, 2, \ldots, n\}$.

Prove: (a) 7 is a friendly number, and (b) n is friendly if and only if $n \geq 7$.

3. Prove that the mid-points of the altitudes of a triangle are collinear if and only if the triangle is right.

4. Let P be a plane. Prove that there exists no function $f : P \to P$ such that for every convex quadrilateral $ABCD$, the points $f(A)$, $f(B)$, $f(C)$, $f(D)$ are the vertices of a concave quadrilateral.

10th Grade

1. Let $OABC$ be a tetrahedron such that $OA \perp OB \perp OC \perp OA$, let r be the radius of its inscribed sphere, and let H be the orthocentre of triangle ABC. Prove that $OH \leq r\left(\sqrt{3}+1\right)$.

2. The complex numbers z_1, z_2, $\ldots$, z_5, have the same non-zero modulus, and $\sum\limits_{i=1}^{5} z_i = \sum\limits_{i=1}^{5} z_i^2 = 0$. Prove that z_1, z_2, $\ldots$, z_5 are the complex coordinates of the vertices of a regular pentagon.

3. Let a, b, c be the complex coordinates of the vertices A, B, C of a triangle. It is known that $|a| = |b| = |c| = 1$ and that there exists $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $a + b\cos\alpha + c\sin\alpha = 0$. Prove that $1 < [ABC] \leq \frac{1+\sqrt{2}}{2}$, where $[XYZ]$ is the area of XYZ.

4. A finite set A of complex numbers has the property that $z \in A$ implies $z^n \in A$ for every positive integer n.

(a) Prove that $\sum_{z \in A} z$ is an integer.

(b) Prove that, for every integer k, there is a set A which fulfills the above condition with $\sum_{z \in A} z = k$.

As a second set of problems we give the problems of the 16th Korean Mathematical Olympiad, written in April, 2003. Thanks go to Andy Liu, Canadian Team Leader, for collecting them for the *Corner*.

16th KOREAN MATHEMATICAL OLYMPIAD

April 12-13, 2003

Time: 9 hours

1. The computers in a computer lab are connected by cables as follows: Each computer is directly connected to exactly three other computers via cables. There is at most one cable joining two computers and any pair of computers in the lab can exchange data. (Two computers A and B can exchange data if there exists a sequence of computers starting from A and ending at B in which two computers next to each other in the sequence are directly joined by a cable.)

Let k be the smallest number of computers in the lab whose removal results in leaving just one computer in the lab or a pair of computers not able to exchange data any more. Let ℓ be the smallest number of cables whose deletion results in the existence of two computers that cannot exchange data any more. Show that $k = \ell$.

2. Let $ABCD$ be a rhombus with $\angle A < 90^\circ$. Let its two diagonals AC and BD meet at a point M. A point O on the line segment MC is selected such that $O \neq M$ and $OB < OC$. The circle centred at O passing through points B and D meets the line AB at point B and a point X (where $X = B$ when the line AB is tangent to the circle) and meets the line BC at point B and a point Y. Let the lines DX and DY meet the line segment AC at P and Q, respectively. Express the value of $\frac{OQ}{OP}$ in terms of t when $\frac{MA}{MO} = t$.

3. Show that there exist no integers x, y, z with $x \neq 0$ satisfying

$$2x^4 + 2x^2y^2 + y^4 = z^2 .$$

4. Suppose that the incircle of $\triangle ABC$ is tangent to the sides AB, BC, CA at points P, Q, R, respectively. Prove the following inequality:

$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6 .$$

5. Answer the following where m is a positive integer.

(a) Prove that if $2^{m+1}+1$ divides $3^{2^m}+1$, then $2^{m+1}+1$ is a prime.

(b) Is the converse of (a) true?

6. Let m and n be relatively prime positive integers satisfying $6 \le 2m < n$. Consider n distinct points on a circle. Join one of these n points, say P, by a line segment to the m^{th} point Q counterclockwise from P, then join Q by a line segment to the m^{th} point R counterclockwise from Q, and so on. Repeat this process until no new line segment is added. Denote by I the number of intersections among these line segments inside the circle (excluding those on the circle).

(a) Find an expression for the maximum of I in terms of m and n when the locations of the n points change.

(b) Show that the inequality $I \ge n$ holds regardless of the locations of the n points. Also show that n points can be located so that $I = n$ if $m = 3$ and n is even.

As a third set we give the X National Mathematical Olympiad of Turkey written in December 2002. Thanks again go to Andy Liu for collecting the problems for our use.

X NATIONAL MATHEMATICAL OLYMPIAD OF TURKEY

Day 1 – 14 December 2002

(Time: 4.5 hours)

1. Let $n \ge 2$ be an integer, and let $(a_1, a_2, \dots, a_n)$ be a permutation of $1, 2, \dots, n$. For each $k \in \{1, 2, \dots, n\}$, a_k apples are placed at the point k on the real axis. Children named A, B, C are assigned respective points x_A, x_B, $x_C \in \{1, 2, \dots, n\}$. For each $k \in \{1, 2, \dots, n\}$, the children whose points are closest to k divide a_k apples equally among themselves. We call (x_A, x_B, x_C) a *stable configuration* if no child's total share can be increased by assigning a new point to this child and not changing the points of the other two. Determine the values of n for which a stable configuration exists for some distribution $(a_1, a_2, \dots, a_n)$ of the apples.

2. Two circles are externally tangent to each other at a point A and internally tangent to a third circle Γ at points B and C. Let D be the mid-point of the secant of Γ which is tangent to the smaller circles at A. Show that A is the incentre of the triangle BCD if the centres of the circles are not collinear.

3. Graph Airlines (GA) operates flights between some of the cities of the Republic of Graphia. There are GA flights between each city and at least three different cities, and it is possible to travel from any city to any other city in the Republic of Graphia using GA flights. GA decides to discontinue some of its flights. Show that this can be done in such a way that it is still possible to travel from any city to any other city using GA flights, yet at least $2/9$ of the cities have only one flight.

Day 2 – 15 December 2002

(Time: 4.5 hours)

4. Find all prime numbers p for which the number of ordered pairs of integers (x, y) with $0 \leq x,\ y < p$ satisfying the condition $y^2 \equiv x^3 - x \pmod{p}$ is exactly p.

5. In an acute triangle ABC with $|BC| < |AC| < |AB|$, the points D on side AB and E on side AC satisfy the condition $|BD| = |BC| = |CE|$. Show that the circumradius of the triangle ADE is equal to the distance between the incentre and the circumcentre of the triangle ABC.

6. Let n be a positive integer and $\mathbb{R}^n$ be the set of ordered n–tuples of real numbers. Let T denote the collection of $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ for which there exists a permutation σ of $1, 2, \ldots, n$ such that $x_{\sigma(i)} - x_{\sigma(i+1)} \geq 1$ for each $i \in \{1, 2, \ldots, n-1\}$. Prove that there is a real number d satisfying the following condition:

For every $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, there exist $(b_1, b_2, \ldots, b_n) \in T$ and $(c_1, c_2, \ldots, c_n) \in T$ such that, for each $i \in \{1, 2, \ldots, n\}$,

$$a_i = \tfrac{1}{2}(b_i + c_i)\,, \quad |a_i - b_i| \leq d\,, \quad \text{and} \quad |a_i - c_i| \leq d\,.$$

Now we turn to the solutions on file for problems of the 2000 Kürschák Contest given in [2004 : 205].

1. For a positive integer n, consider the square in the Cartesian plane whose vertices are $A(0,0)$, $B(n,0)$, $C(n,n)$ and $D(0,n)$. The grid points of the integer lattice inside or on the boundary of this square are coloured either red or green in such a way that every unit square in the lattice has exactly two red vertices. How many such colourings are possible?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Colourings which satisfy the given condition will be called "good". We will prove that the number of good colourings is $2^{n+2} - 2$.

For $n = 1$, the number of good colourings is $\binom{4}{2} = 6 = 2^{n+2} - 2$.

Now consider $n \geq 2$. Let $c_0, c_1, \ldots, c_n$ be the columns of the $(n+1) \times (n+1)$ grid described in the problem, and let $r_0, r_1, \ldots, r_n$ be the rows. Note that in any good colouring, the lower-left unit square contains exactly two red vertices.

Let us form a good colouring. Suppose we colour red the points with coordinates $(0,0)$ and $(1,0)$. The colouring of the two columns c_0 and c_1 is then completely determined. We have two ways to colour the column c_2, depending on which of the points $(2,0)$ and $(2,1)$ we choose to colour red. (Either choice determines the colouring of the whole column c_2 since the colouring of c_1 is already determined). We use the same reasoning for the other columns, so that we obtain exactly 2^{n-1} good colourings in this case.

The same reasoning (with perhaps consideration of the rows instead of the columns) leads to the same number in each of the three other cases for which we colour red two adjacent lattice points in the lower-left unit square.

Now, let a_n be the number of good colourings of the $(n+1) \times (n+1)$ grid for which the two red vertices in the lower-left unit square are $(0,0)$ and $(1,1)$. Let us consider such a good colouring.

(a) If we colour red the point $(2,1)$, then we have two adjacent red vertices at $(1,1)$ and $(2,1)$. A good colouring of the whole grid leads to a good colouring of the $n \times n$ grid obtained by deleting c_0 and r_0, and in this $n \times n$ grid, the lower-left unit square has two adjacent red vertices (namely $(1,1)$ and $(2,1)$ in the 'old' numbering). As seen above, there are 2^{n-2} good colourings for this $n \times n$ grid. Each of them determines the colouring of r_0 (because the point $(2,1)$ is already red) and c_0 (because $(1,3)$ is already red) so as to obtain a good colouring of the original $(n+1) \times (n+1)$ grid.

(b) If we colour red the point $(1,2)$, the same reasoning as in (a) leads to 2^{n-2} good colourings again.

(c) Otherwise, we colour red the point $(2,2)$. This leads to a good colouring of the $n \times n$ grid obtained by deleting c_0 and r_0, where the red points of the lower-left unit square are $(0,0)$ and $(1,1)$ (in the 'new' numbering). There are a_{n-1} such good colourings. Each of these determines the colouring of r_0 and c_0. It follows that $a_n = 2 \times 2^{n-2} + a_{n-1}$. Since $a_1 = 1$, we easily deduce that $a_n = 2^n - 1$.

Similarly, there are $2^n - 1$ good colourings of the $(n+1) \times (n+1)$ grid for which the red vertices in the lower-left unit square are $(1,0)$ and $(0,1)$.

The total number of good colourings is $4 \cdot 2^{n-1} + 2(2^n - 1) = 2^{n+2} - 2$.

2. Let T be a point in the plane of the non-equilateral triangle ABC which is different from the vertices of the triangle. Let the lines AT, BT, and CT meet the circumcircle of the triangle at A_T, B_T, and C_T, respectively. Prove that there are exactly two points P and Q in the plane for which the triangles $A_PB_PC_P$ and $A_QB_QC_Q$ are equilateral. Prove, furthermore, that the line PQ passes through the circumcentre of the triangle ABC.

Solution by Michel Bataille, Rouen, France.

We embed the figure in the complex plane and, for simplicity, denote a point or its complex representation by the same small letter. Without loss of generality, we suppose that a, b, and c lie on the unit circle Γ, so that $a\overline{a} = b\overline{b} = c\overline{c} = 1$. Note that the points p and q we seek cannot lie on Γ.

Let $a' \in \Gamma$ and $m \notin \Gamma$. The line aa' passes through m if and only if $m + aa'\overline{m} = a + a'$; that is, $a' = \overline{T_m(a)}$, where T_m denotes the Möbius transformation defined by $T_m(z) = \dfrac{1 - \overline{m}z}{m - z}$. As a result, for a', b', c' on Γ, the lines aa', bb' and cc' concur at m if and only if $a' = \overline{T_m(a)}$, $b' = \overline{T_m(b)}$ and $c' = \overline{T_m(c)}$.

The statement that $a'b'c'$ is equilateral is successively equivalent to

- $\dfrac{a' - c'}{a' - b} \in \{-\omega,\ -\omega^2\}$, where $\omega = \exp(2\pi i/3)$,
- $\overline{[T_m(m), T_m(a), T_m(b), T_m(c)]} \in \{-\omega,\ -\omega^2\}$, where $[\cdot,\cdot,\cdot,\cdot]$ denotes the cross ratio,
- $[T_m(m), T_m(a), T_m(b), T_m(c)] \in \{-\omega,\ -\omega^2\}$ (since $-\omega$ and $-\omega^2$ are conjugates),
- $[m, a, b, c] \in \{-\omega,\ -\omega^2\}$ (since T_m preserves the cross ratio).

The conclusion follows, since p and q are the two points given by $[p, a, b, c] = -\omega$, $[q, a, b, c] = -\omega^2$. Note that p, $q \neq \infty$ since ABC is not equilateral. Also, from $[q, a, b, c] = \overline{[p, a, b, c]}$, an easy calculation yields $q = 1/p$. Thus, 0, p, and q are collinear (p and q are even inverses in Γ).

Notes.

1. P and Q are called the isodynamic points of $\triangle ABC$. Various properties of these points can be found in R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960, pp. 295–7.

2. A nice reference for the use of complex numbers, cross ratios, and Möbius transformations is L. Hahn, *Complex Numbers and Geometry*, Mathematical Association of America, 1994.

3. For a slightly different solution of the problem, see *American Mathematical Monthly*, 109, December 2002, pp. 926–7.

3. Let k denote a non-negative integer. Assume that the integers $a_1, \ldots, a_n$ give at least $2k$ different remainders when divided by $n+k$. Prove that some of the integers add up to a number divisible by $n + k$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

It is well known that among any n integers we may find some which add up to a number divisible by n. Thus, we will assume that k is positive. In this case, we will prove a stronger version of the problem: If the integers $a_1, \ldots, a_n$ give at least $k + 1$ different remainders when divided by $n + k$, then some of them add up to a number divisible by $n + k$.

First assume that $k > 1$. Without loss of generality, we suppose that $a_1, \ldots, a_{k+1}$ are distinct modulo $n+k$. Let $S = a_1 + \cdots + a_{k+1}$.

Now, we consider the three following groups:

- Type I: $a_1, \ldots, a_{k+1}$.
- Type II: $S - a_1, \ldots, S - a_{k+1}$.
- Type III: S, $S + a_{k+2}$, $S + a_{k+2} + a_{k+3}, \ldots, S + a_{k+2} + \cdots + a_n$.

There are $(k+1) + (k+1) + (n-k) = n + k + 2$ numbers listed in the three groups above. Obviously, if one of these numbers is divisible by $n+k$ we are done. Otherwise, by the Pigeonhole Principle, at least two, say x and y, have the same remainder modulo $n+k$.

Since $a_1, \ldots, a_{k+1}$ are distinct modulo $n+k$, it follows that x and y cannot be both of type I nor both of type II. Clearly, if at least one of x and y has type III, we have the desired conclusion (by considering $y - x$ or $x - y$). Therefore, without loss of generality, we may assume that $x = a_p$ belongs to Type I and $y = S - a_q$ belongs to Type II.

If $p \neq q$, then $y - x$ leads to the desired conclusion (since $k + 1 > 2$, we are sure that $y - x$ is a non-empty sum). If $p = q$, then we actually have three such pairs, say $(a_1, S - a_1)$, $(a_2, S - a_2)$, $(a_3, S - a_3)$. (Remember that we have $n + k + 2$ numbers in the list for only $n + k - 1$ possible remainders and that, according to the previous cases, we may assume that no three numbers have the same remainder modulo $n+k$. Therefore, at most two pairs would lead to at least $n+k$ distinct remainders.) It follows that $2a_1 \equiv 2a_2 \equiv 2a_3 \equiv S \pmod{n+k}$.

Since a_1, a_2, a_3 are distinct modulo $n+k$, we deduce that $n+k$ is even and $a_1 = a_2 = t(n+k)/2$ and $a_2 = a_3 + r(n+k)/2$ for some odd integers t and r. Thus, $a_1 \equiv a_3 \pmod{n+k}$, a contradiction. The conclusion follows.

Now, let us consider the case $k = 1$. If the integers give at least three distinct remainders modulo $n+1$, the proof above can be adapted word for word, since the proof used only the fact that S is a sum of at least three terms. Thus, we only have to consider the case where the given integers give two remainders modulo $n+1$.

Let us assume that, for some positive integer p, we have

$$\begin{aligned} a_1 &\equiv \cdots \equiv a_p \equiv r \pmod{n+1} \\ \text{and} \quad a_{p+1} &\equiv \cdots \equiv a_n \equiv s \pmod{n+1}\,, \end{aligned}$$

with $r \not\equiv s \pmod{n+1}$. If $ir \equiv 0 \pmod{n+1}$ for some $i = 1, \ldots, p$, we are done. Otherwise, it follows that $r, 2r, \ldots, pr$ are distinct modulo $n+1$. We may assume that $js \not\equiv -is \pmod{n+1}$ for all $i = 1, \ldots, p$ and all $j = 1, \ldots, n-p$ (otherwise we are done). Then, there are only $n-p$ available remainders for the $n-p$ numbers $s, 2s, \ldots, (n-p)s$, and one of them is 0. Thus, either one of these numbers is divisible by $n+1$ or at least two have the same remainder modulo $n+1$. In both cases, there exists $j \in \{1, 2, \ldots, np\}$ such that $js \equiv 0 \pmod{n+1}$, and we are done.

Now we move to the September 2004 number of the *Corner* and readers' solutions to some of the problems of the 11th Japanese Mathematical Olympiad, Second Round, given in [2004 : 266–267].

1. An $m \times n$ chessboard is given. Each square is painted black or white in such a way that for every black square, the number of black squares adjacent to it is odd. Prove that the number of black squares is even. (Two squares are *adjacent* if they are different and have a common edge.)

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Consider the graph whose vertices are the black squares, any two joined by an edge if and only if they are adjacent. From the problem statement, each vertex has odd degree. But it is well known that in any finite simple graph the number of vertices with odd degree is even (this follows from the fact that the sum of the degrees is twice the number of edges). Thus, the number of vertices is even, and we are done.

2. A positive integer n is written in decimal notation as $a_m a_{m-1} \cdots a_1$; that is,

$$n = 10^{m-1}a_m + 10^{m-2}a_{m-1} + \cdots + a_1\,,$$

where a_m, a_{m-1}, …, $a_1 \in \{0, 1, \ldots, 9\}$ and $a_m \neq 0$. Find all n such that

$$n = (a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1)\,.$$

Solved by Houda Anoun, Bordeaux, France; and Pierre Bornsztein, Maisons-Laffitte, France. We give the solution by Anoun (en français).

Soit n un entier dont la notation décimale est sous la forme de $a_m a_{m-1} \cdots a_1$ qui satisfait la condition

$$n = (a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1)\,. \qquad (1)$$

Supposons qu'il existe $j \in \{1, \ldots, m\}$ tel que $a_j = 0$. On a $a_i + 1 \le 10$ pour tout $i \in \{1, \ldots, m\}$. Si $a_j = 0$, il s'en suit que

$$(a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1) \le 10^{m-1}\,.$$

C'est à dire que $n \le 10^{m-1}$. Puisque $a_m \neq 0$, on déduit que $a_m = 1$ et $a_i = 0$ pour $i \in \{1, \ldots, m-1\}$. D'après (1) on a aussi $n = 2 \times 1 \cdots \times 1 = 2$. Ceci est absurde. On conclut que $a_i \neq 0$ pour tout $i \in \{1, \ldots, m\}$.

Maintenant supposons qu'il existe $j \in \{1, \ldots, m\}$ tel que $a_j = 9$. Donc $a_j + 1 = 10$ est un diviseur de n (par (1)). D'où $a_1 = 0$, chose qui est contradictoire avec le résultat précédent. On déduit que $a_i \in \{1, \ldots, 8\}$ pour tout $i \in \{1, \ldots, m\}$.

Puis on montre par récurrence que si $n \ge 3$, donc

$$(a_m + 1) \times (a_{m-1} + 1) \times \cdots \times (a_1 + 1) < 10^{m-1}a_m + 10^{m-2}a_{m-1} + \cdots + a_1\,. \qquad (2)$$

On considère d'abord le cas $m = 3$. On a $(a_2+1)\times(a_1+1) \leq 10a_2+a_1$, car cette inégalité est équivalent à $(9 - a_1) \times a_2 \geq 1$, qui est vrai. Or

$$\begin{aligned}(a_3+1)\times(a_2+1)\times(a_1+1) &\leq (a_3+1)\times(10a_2+a_1)\\ &= (10a_2+a_1)\times a_3 + 10a_2 + a_1\\ &< 10^2a_3 + 10a_2 + a_1 .\end{aligned}$$

Ceci vérifie (2) pour $m = 3$.

Puis supposons que (2) est appliqué pour $m = k$, où $k \in \{3, 4, \ldots, \}$. On a alors

$$\begin{aligned}&(a_{k+1}+1)\times(a_k+1)\times\cdots\times(a_1+1)\\ &\quad< (a_{k+1}+1)\times\left(10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1\right)\\ &\quad= \left(10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1\right)\times a_{k+1}\\ &\qquad\qquad + 10^{k-1}a_k + 10^{k-2}a_{k-1} + \cdots + a_1\\ &\quad< 10^k a_{k+1} + 10^{k-1}a_k + \cdots + a_1 .\end{aligned}$$

La preuve par récurrence est complète.

Si n vérifie la condition (1), alors il n'est pas composé au plus de deux chiffres. Il ne peut être composé d'un seul chiffre a_1, car sinon on aura $a_1 = a_1 + 1$. Cherchons finalement les entiers n de la forme $10a_2 + a_1$ tels que $n = (a_2+1)\times(a_1+1)$. On a alors:

$$10a_2 + a_1 = (a_2+1)\times(a_1+1) .$$

D'où $(9 - a_1)\times a_2 = 1$, ceci implique que $a_1 = 8$ et $a_2 = 1$.

L'unique solution au problème est donc l'entier 18.

3. Three real numbers a, b, $c \geq 0$ satisfy

$$a^2 \leq b^2 + c^2, \qquad b^2 \leq c^2 + a^2, \qquad c^2 \leq a^2 + b^2 .$$

Prove the inequality

$$(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3) \geq 4(a^6+b^6+c^6) .$$

When does equality hold?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Vedula N. Murty, Dover, PA, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA. We use Zhou's presentation.

By the Cauchy-Schwarz Inequality, we have

$$(a+b+c)(a^3+b^3+c^3) \geq (a^2+b^2+c^2)^2 .$$

Thus,

$$(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3) \geq (a^2+b^2+c^2)^3 .$$

Therefore, it will suffice to prove that

$$(a^2+b^2+c^2)^3 \ \geq\ 4(a^6+b^6+c^6)\,.$$

We have

$$\begin{aligned}(a^2+b^2+c^2)^3 &= (a^6+b^6+c^6)+6(abc)^2\\ &\quad +3(a^2b^4+b^2c^4+c^2a^4+a^4b^2+b^4c^2+c^4a^2)\\ &= 4(a^6+b^6+c^6)+12(abc)^2\\ &\quad +3(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)\\ &\geq 4(a^6+b^6+c^6)\,,\end{aligned}$$

since $a^2 \leq b^2+c^2$, $b^2 \leq c^2+a^2$, and $c^2 \leq a^2+b^2$. Equality holds if and only if one of a, b, c is 0 and the other two are equal.

4. Let p be a prime number and m a positive integer. Show that there exists a positive integer n such that the decimal representation of p^n contains a string of m consecutive 0s.

Solutiond by Pierre Bornsztein, Maisons-Laffitte, France.

Case 1. $p \notin \{2, 5\}$.

Then p is coprime to 10^{m+1}. Hence, by the Euler-Fermat Theorem, there exists $n > 0$ such that $p^n \equiv 1 \pmod{10^{m+1}}$ (for example, $n = \phi(10^{m+1})$). It follows that the decimal expansion of p^n has the form $\overline{q}0\ldots01$, where $\overline{q}$ is the decimal expansion of some positive integer q and there are m consecutive 0s between $\overline{q}$ and 1.

Case 2. $p = 2$.

As above, there exists $a > 0$ such that $2^a \equiv 1 \pmod{5^{2m}}$. It follows that $2^{a+2m} - 2^{2m} \equiv 0 \pmod{10^{2m}}$. Thus, $2^{a+2m} - 2^{2m}$ contains a string of $2m$ consecutive 0s.

But, since $2^{2m} = 4^m < 10^m$, the decimal expansion of 2^{2m} does not use more than m digits. It follows that the decimal expansion of 2^{a+2m} contains a string of m consecutive 0s, as desired.

Case 3. $p = 5$.

As above, there exists $a > 0$ such that $5^a \equiv 1 \pmod{2^{4m}}$. It follows that $5^{a+4m} - 5^{4m} \equiv 0 \pmod{10^{4m}}$. Thus, $5^{a+4m} - 5^{4m}$ contains a string of $4m$ consecutive 0s.

But, since $5^{4m} < 10^{3m}$, the decimal expansion of 5^{4m} does not use more than $3m$ digits. It follows that the decimal expansion of 5^{a+4m} contains a string of m consecutive 0s, as desired.

Next we turn to readers' solutions to problems of the 14th Mexican Mathematical Olympiad given in [2004 : 267–268].

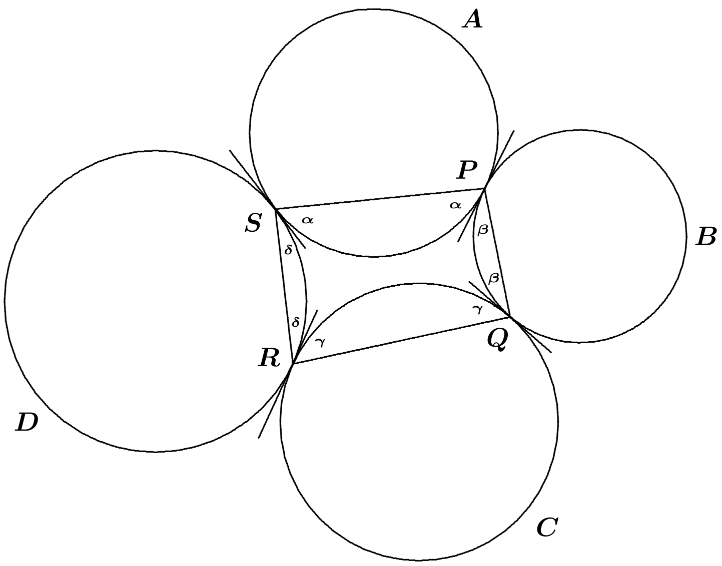
1. Let A, B, C, and D be circles such that (i) A and B are externally tangent at P, (ii) B and C are externally tangent at Q, (iii) C and D are externally tangent at R, and (iv) D and A are externally tangent at S. Assume that A and C do not intersect and that B and D do not intersect.

(a) Prove that P, Q, R, and S lie on a circle.

(b) Assume further that A and C have radius 2, B and D have radius 3, and the distance between the centres of A and C is 6. Determine the area of $PQRS$.

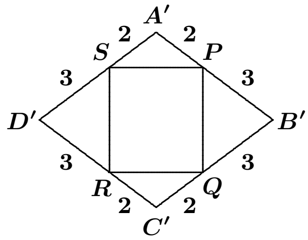
Solution by Geoffrey A. Kandall, Hamden, CT, USA.

(a)



Draw the common internal tangents at P, Q, R, and S, and label angles α, β, γ, and δ as shown in the diagram above. Since the sum of the internal angles in quadrilateral $PQRS$ is $360°$, it follows that $\alpha+\beta+\gamma+\delta = 180°$. Hence, opposite angles of $PQRS$ are supplementary. Thus, $PQRS$ is cyclic.

(b) Let A', B', C', D' be the centres of the circles A, B, C, D, and let $A'C'$ and $B'D'$ meet at E. Since $A'B'C'D'$ is a rhombus, the diagonals $A'C'$ and $B'D'$are the perpendicular bisectors of one another. Then, since the distance between A' and C' is 6, we have $A'E = 3$. Hence, $D'E = 4$ and $D'B' = 8$. Since $SP \parallel D'B' \parallel RQ$ and $SR \parallel A'C' \parallel PQ$, it follows that $PQRS$ is a rectangle.



We have

$$\frac{SP}{D'B'} = \frac{2}{5} \quad \text{and} \quad \frac{SR}{A'C'} = \frac{3}{5}.$$

Therefore, $SP = \frac{2}{5} \cdot 8 = \frac{16}{5}$ and $SR = \frac{3}{5} \cdot 6 = \frac{18}{5}$. Finally,

$$[PQRS] = SP \cdot SR = \frac{288}{25}.$$

2. A triangle like the one shown is constructed with the numbers from 1 to 2000 in the first row. Each number in the triangle, except those in the first row, is the sum of the two numbers above it. What number occupies the lowest vertex of the triangle? (Write your final answer as a product of primes.)

$$\begin{array}{ccccccccc} 1 & & 2 & & 3 & & 4 & & 5 \\ & 3 & & 5 & & 7 & & 9 & \\ & & 8 & & 12 & & 16 & & \\ & & & 20 & & 28 & & & \\ & & & & 48 & & & & \end{array}$$

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let p be the number of integers written on the first row. Let a_n be the first number (on the left) appearing on the n^{th} row. An easy induction shows that the n^{th} row is

$$a_n,\ a_n + 2^{n-1},\ a_n + 2 \cdot 2^{n-1},\ \ldots,\ a_n + (p-n)2^{n-1}.$$

It follows immediately that $a_{n+1} = 2a_n + 2^{n-1}$. Since $a_1 = 1$, we deduce that $a_n = (n+1)2^{n-2}$. The problem asks for a_{2000}. We have

$$a_{2000} = 2001 \times 2^{1998} = 2^{1998} \times 3 \times 23 \times 29.$$

3. Given a set A of positive integers, a set A' is constructed consisting of all elements of A as well as all positive integers that can be obtained as follows: some elements of A are chosen, without repetition, and for each of them a sign ($+$ or $-$) is chosen; the signed numbers are then added and the result is placed in A'. For example, if $A = \{2, 8, 13, 20\}$, then two elements of A' are 8 and 14 (since 8 belongs to A and $14 = 20 + 2 - 8$). From A', a set A'' is constructed in the same fashion as A' is constructed from A. What is the minimum number of elements that A must have if A'' is to contain all integers from 1 to 40 (including 1 and 40)?

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The minimum is 3 and it is achieved for $A = \{1, 5, 25\}$ (for example). Indeed, in that case, $A' = \{1, 4, 5, 6, 19, 20, 21, 24, 25, 26, 29, 30, 31\}$. From that, it is routine to verify that $\{1, 2, \ldots, 40\} \subset A''$.

Now, let $0 < x < y$ and $A = \{x, y\}$. Thus, $A' = \{x, y-x, y, y+x\}$ (there may be some repetitions). Note that if we select $k \geq 1$ elements from A', we may construct 2^k sums from them, not necessary positive. More precisely, it is clear that if we may obtain the sum s from these k elements, then we may also obtain $-s$ by reversing all the signs.

Since $|A'| \le 4$, we deduce that we can construct at most

$$2 \times \binom{4}{1} + 4 \times \binom{4}{2} + 8 \times \binom{4}{3} + 16 \times \binom{4}{4} = 80$$

sums, not necessarily positive or distinct. From above, we see that there are at most 40 positive sums. But these sums are not pairwise distinct, since x can be obtained from $\{x\}$ and from $\{y + x, y\}$. Thus, there are fewer than 40 positive sums, so that we cannot obtain all the integers from 1 to 40.

It follows that 3 cannot be improved.

5. An $n \times n$ square is divided into unit squares and painted black and white in a checkerboard pattern. The following operation may be performed on the board: choose a sub-rectangle whose side lengths are both odd or both even, but not both 1, and reverse the colours of the unit squares in this rectangle (that is, black squares become white and white squares become black).

Find all values of n for which it is possible to make all unit squares the same colour by a finite sequence of operations.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

We will prove that this is possible for all positive n except $n = 2$.

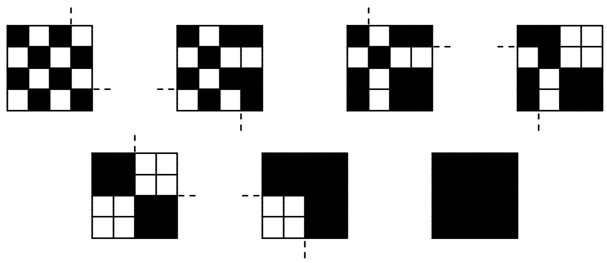
There is nothing to do for $n = 1$. It is impossible for $n = 2$, since the only subrectangle available is the whole square itself, which does not lead to the desired colouring.

Assume first that $n \ge 3$ is odd. By choosing $1 \times n$ rectangles corresponding to the columns c_1, c_3, ..., c_n (from left to right), each of the rows of the whole square becomes monochromatic. Next, choosing appropriate rows as subrectangles leads to a monochromatic whole square. It follows that each odd $n \ge 3$ is a solution of the problem.

Now suppose that n is even. Let $n = 2^a b$, where $a \ge 1$ and $b \ge 3$, with b odd. By dividing the $n \times n$ square into 2^{2a} subsquares of size $b \times b$, we may use the result above to give the same colour, say black, to all these subsquares. This leads to a monochromatic $n \times n$ square. Thus, n is a solution.

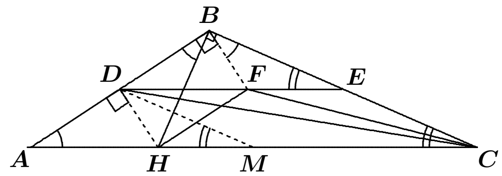
Following the same reasoning, we may prove that if $n = 4$ is a solution, then $n = 2^a$ is also a solution for all $a \ge 2$. Therefore, it only remains to prove that $n = 4$ is a solution.

But the following sequence of 6 operations does the job, where we have used dotted lines to mark the rectangles involved at each stage:



6. Let ABC be a triangle with $\angle B > 90^\circ$ such that, for some point H on AC, we have $AH = BH$, and BH is perpendicular to BC. Let D and E be the mid-points of AB and BC, respectively. Through H a parallel to AB is drawn, intersecting DE at F. Prove that $\angle BCF = \angle ACD$.

Solved by Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give Seimiya's treatment.



Since D and E are the mid-points of AB and BC, respectively, we have $DE \parallel AC$. Let M be the mid-point of AC; then $DM \parallel BC$. These imply that

$$\angle BED = \angle BCA = \angle DMA. \qquad (1)$$

Since $DF \parallel AH$ and $HF \parallel AD$, quadrilateral $DAHF$ is a parallelogram. Thus, $HF = AD = DB$; hence, quadrilateral $BDHF$ is a parallelogram, and we have $BF \parallel DH$. Since $AH = BH$ and $AD = BD$, we get $HD \perp AB$. Thus, $\angle DBF = \angle ADH = 90^\circ$.

Since $\angle HBE = 90^\circ$, we have $\angle DBF = \angle HBE$; that is,

$$\angle DBH + \angle HBF = \angle HBF + \angle FBE.$$

Thus, $\angle DBH = \angle FBE$. Since $AH = BH$, we have $\angle DAH = \angle DBH$, and hence,

$$\angle FBE = \angle DAH. \qquad (2)$$

It follows from (1) and (2) that $\triangle FBE \sim \triangle DAM$. Thus,

$$EF : DM = BE : AM = EC : MC. \qquad (3)$$

Since $\angle FEC = 180^\circ - \angle BEF = 180^\circ - \angle DMA = \angle DMC$, it follows (in view of (3)) that $\triangle FEC \sim \triangle DMC$. Hence, $\angle ECF = \angle MCD$; that is, $\angle BCF = \angle ACD$.

That completes the *Corner* for this issue. This is now Olympiad Season. Send me your Olympiad materials, as well as your nice solutions and generalizations to problems featured in the *Corner*.