Problem of the Month

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A couple of years ago, while fiddling with some problems in preparation for our annual June Mathematics Contest Seminar here in Waterloo, I came across the following problem:

Solve the following system of equations:

$$\begin{aligned} x^2 + xy &= 12\,, \\ 2xy + 3y^2 &= -5\,. \end{aligned}$$

Luckily, my good friend (and former Mayhem columnist) Paul Ottaway happened by. We set to work. With our combined wisdom, we managed to remember that the usual method for solving a system of equations was to eliminate one of the variables.

Following this idea, we solved the first equation for y to get $y = x - \dfrac{12}{x}$, which we then substituted into the second equation, obtaining

$$2x\left(x - \frac{12}{x}\right) + 3\left(x - \frac{12}{x}\right)^2 \;=\; -5\,.$$

After expanding and clearing out denominators, we ended up with a quartic equation, which was hardly appetizing. It was also apparent that solving the second equation for x and substituting into the first equation was not going to be any better.

Thus, the standard technique of eliminating one of the variables was not working. Then one of us had the clever idea to try eliminating the constants instead. (Since the idea was clever, odds are it was Paul's idea, not mine!)

We multiplied the first equation by 5 (obtaining $5x^2 + 5xy = 60$) and the second equation by 12 (obtaining $24xy + 36y^2 = -60$) and added the equations to obtain

$$5x^2 + 29xy + 36y^2 \;=\; 0\,,$$

which we were then able to factor to give

$$(x + 4y)(5x + 9y) \;=\; 0\,,$$

yielding $x = -4y$ or $x = -\frac{9}{5}y$.

Substituting $x = -4y$ into $x^2 + xy = 12$ gives $16y^2 - 4y^2 = 12$; that is, $y^2 = 1$. Hence, $y = \pm 1$, and $(x, y) = (-4, 1)$ or $(x, y) = (4, -1)$.

Substituting $x = -\frac{9}{5}y$ into $x^2 + xy = 12$ gives $\frac{81}{25}y^2 - \frac{9}{5}y^2 = 12$; that is, $\frac{36}{25}y^2 = 12$. Hence, $y = \pm\frac{5}{\sqrt{3}}$, and $(x, y) = \left(-\frac{9}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right)$ or $(x, y) = \left(\frac{9}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right)$.

We found a neat way to tackle certain systems of equations. Much to my delight, as I was flipping through a recent journal looking for problems, I found the following problem from a European competition of which I had never heard:

Problem. (2005 Mathematical Duel)
Determine all integer solutions of the system of equations

$$\begin{aligned} x^2z + y^2z + 4xy &= 40\,, \\ x^2 + y^2 + xyz &= 20\,. \end{aligned}$$

I attempted this problem for a few minutes by the usual method of eliminating one of the variables. This met with little success. Then I remembered Paul's trick!

Solution: If we subtract 2 times the second equation from the first, the constants are eliminated, and we obtain

$$x^2z + y^2z + 4xy - 2x^2 - 2y^2 - 2xyz = 0\,.$$

Where to now? Our only real hope is to try to factor this equation somehow. If we rearrange the terms to get

$$x^2z - 2x^2 + y^2z - 2y^2 - 2xyz + 4xy = 0\,,$$

the light at the end of the tunnel starts to appear. We can now do some factoring to get

$$x^2(z-2) + y^2(z-2) - 2xy(z-2) = 0\,.$$

Thus, there is a common factor of $z - 2$, which we can factor out, giving

$$(z-2)(x^2 - 2xy + y^2) = 0\,,$$

or

$$(z-2)(x-y)^2 = 0\,.$$

Therefore, $z = 2$ or $x = y$.

If $z = 2$, then the second equation gives $x^2 + 2xy + y^2 = 20$, or $(x+y)^2 = 20$. But we are looking for integer solutions (unlike the first problem). There are no solutions here, since 20 is not a perfect square.

If $x = y$, the second equation gives $2x^2 + zx^2 = 20$, or $x^2(z+2) = 20$. (The first equation reduces to this same equation as well.) Since x and z are integers and the only perfect squares which are divisors of 20 are 1 and 4, the possibilities for x are 1, -1, 2, and -2, giving values for z of 18, 18, 3, and 3, respectively.

Therefore, the integer solutions for (x, y, z) are $(1, 1, 18)$, $(-1, -1, 18)$, $(2, 2, 3)$, $(-2, -2, 3)$.

We have discovered a new technique for solving old problems—always a satisfying feeling. Problems involving systems of equations occur frequently on contests and in real life, and it is important to have a wide variety of different techniques in our toolbox.