Mayhem Solutions

M163. Corrected. *Proposed by the Mayhem Staff.*

Show that it is possible to put non-negative integer values on the faces of two dice (not necessarily the same on both dice) so that, when the dice are tossed, the outcomes 1, 2, 3, ..., 12 are equally probable.

Solution by Geoffrey Siu, London Central Secondary School, London, ON.

There are many ways in which this can be done. For example,

Die 1	Die 2
0 0 0 6 6 6	1 2 3 4 5 6
0 0 0 1 1 1	1 3 5 7 9 11
1 1 1 2 2 2	0 2 4 6 8 10
1 1 1 7 7 7	0 1 2 3 4 5
0 6 13 14 15 16	1 2 3 4 5 6
6 6 6 6 6 6	7 7 7 7 7 7

It can be verified that the first four examples above yield pairs where 1, 2, 3, ..., 12 all occur with probability $\frac{1}{12}$. In the fifth example above, 1, 2, 3, ..., 12 all occur with probability $\frac{1}{36}$; while in the sixth example, they occur with probability 0, which fits the requirements of the problem.

Also solved by the Austrian IMO team; Roger He, Prince of Wales Collegiate, St. Johns, NL; Titu Zvonaru, Comanesti, Romania.

M164. *Proposed by the Mayhem Staff.*

Consider the following procedure for dividing the three-digit number 375 by 8. Write down the number formed by the first two digits, namely, 37. Multiply this by 2 to get 74. Add to this the units digit of 375 (the original number), obtaining $74 + 5 = 79$. Then divide by 8 to get 9 with a remainder of 7. Add this result (9, remainder 7) to the number 37 (the first two digits of 375) to get your answer: 46, remainder 7. Thus, 375 divided by 8 equals 46 with a remainder of 7.

Does this method always work for three-digit numbers? Why, or why not?

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

The answer to the question is yes.

Let $100a + 10b + c$ be a three-digit number (a, b, and c are the digits). Then we have

$$\frac{100a + 10b + c}{8} = \frac{80a + 8b}{8} + \frac{20a + 2b + c}{8} = 10a + b + \frac{2(10a + b) + c}{8}.$$

The calculation on the right side is exactly the "method" described in the problem.

Notice that this algorithm can be extended quite naturally to an n–digit number, since

$$\frac{\sum_{i=0}^{n} a_i 10^i}{8} = \frac{a_0 + \sum_{i=1}^{n} (8+2) a_i 10^{i-1}}{8} = \sum_{i=1}^{n} a_i 10^{i-1} + \frac{a_0 + 2\sum_{i=1}^{n} a_i 10^{i-1}}{8}.$$

M165. *Proposed by Babis Stergiou, Chalkida, Greece.*

If a, $b > 0$, prove that

(a) $\sqrt{ab} \ \geq \ \dfrac{2}{1/a + 1/b}$.

(b) $a^6 + b^6 + 8a^3 + 8b^3 + 2a^3b^3 + 16 \ \geq \ 36ab$.

Solution by Mark Yang, St. Paul's Junior High School, St. John's, NL.

(a) By the AM–GM Inequality, we know that

$$\frac{a^3b + a^2b^2 + a^2b^2 + ab^3}{4} \ \geq \ \sqrt[4]{a^8b^8}\,;$$

that is, $ab\big(a^2 + 2ab + b^2\big) \geq 4a^2b^2$. Since a, $b > 0$, we rearrange and take the square root of both sides to get

$$\sqrt{ab} \ \geq \ \frac{2ab}{a+b} \ = \ \frac{2}{1/a + 1/b}\,.$$

(b) By the AM–GM Inequality, we know that

$$\frac{a^6 + b^6 + a^3 + b^3 + \cdots + a^3 + b^3 + a^3b^3 + a^3b^3 + 1 + \cdots + 1}{36} \ \geq \ ab\,,$$

where the numerator on the left side contains 8 terms each of a^3 and b^3, and 16 terms of 1. The result follows immediately.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Vedula N. Murty, Dover, PA, USA.

M166. *Proposed by the Mayhem Staff.*

(a) Simplify

$$(3n)^2 + (4n-1)^2 - (5n-1)^2\,, \qquad (3n+2)^2 + (4n)^2 - (5n+1)^2\,.$$

(b) Using (a) or otherwise, prove that all positive integers can be represented in the form $a^2 + b^2 - c^2$ where a, b, c are integers and $0 < a < b < c$.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

(a) Note that

$$(3n)^2 + (4n-1)^2 - (5n-1)^2 \\ = \; 9n^2 + 16n^2 - 8n + 1 - 25n^2 + 10n - 1 \; = \; 2n\,.$$

Similarly,

$$(3n+2)^2 + (4n)^2 - (5n+1)^2 \\ = \; 9n^2 + 12n + 4 + 16n^2 - 25n^2 - 10n - 1 \; = \; 2n+3\,.$$

(b) Let m be a positive integer. Assume first that m is odd. For the first several odd integers, observe that $1 = 4^2 + 7^2 - 8^2$, $3 = 4^2 + 6^2 - 7^2$, $5 = 4^2 + 5^2 - 6^2$, and $7 = 10^2 + 14^2 - 17^2$. Now let $m = 2n + 3$ with $n > 2$. Then $m = (3n+2)^2 + (4n)^2 - (5n+1)^2$, and since $n > 2$, we have $0 < 3n + 2 < 4n < 5n + 1$.

Next, assume m is even. We note first that $2 = 5^2 + 11^2 - 12^2$. Now, for $n > 1$, let $m = 2n$. Then $m = 2n = (3n)^2 + (4n-1)^2 - (5n-1)^2$, and since $n > 1$, we have $0 < 3n < 4n - 1 < 5n - 1$.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.

M167. *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Solve the following inequality:

$$(2+\cos x)(1+\sin x-\cos x) \;\geq\; \cos x(1+2\sin x-\cos x)\,.$$

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

The inequality we are to solve is equivalent to

$$2(1-\cos x) + \sin x(2-\cos x) \;\geq\; 0\,. \tag{1}$$

Since the inequality is periodic with period 2π, it will suffice to solve it in an interval of length 2π.

Consider any $x \in (-\pi, \pi)$. Then $-\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}$. Let $t = \tan\left(\frac{x}{2}\right)$ (or equivalently, $\frac{x}{2} = \arctan t$). Using the formulas

$$\begin{aligned} \cos x &= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{1-t^2}{1+t^2} \\ \text{and} \qquad \sin x &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2t}{1+t^2}\,, \end{aligned}$$

the inequality (1) can be stated equivalently as

$$2t(t+1)(2t^2+t+1) \;\geq\; 0\,,$$

which holds for all t such that $t \leq -1$ or $t \geq 0$.

Therefore, the original inequality is valid for all $x \in (-\pi, \pi)$ such that $\tan\left(\frac{x}{2}\right) \leq -1$ or $\tan\left(\frac{x}{2}\right) \geq 0$; that is, for all x such that $-\pi < x \leq -\frac{\pi}{2}$ or

$0 \le x < \pi$. By continuity (or by direct calculation), the inequality must also hold when $x = \pm\pi$.

Our results can be stated more simply if we use the interval $[0, 2\pi)$ in place of $[-\pi, \pi)$. In $[0, 2\pi)$, the inequality holds for $0 \le x \le \frac{3\pi}{2}$. The complete solution set is

$$\left\{x : 2k\pi \le x \le 2k\pi + \tfrac{3\pi}{2} \quad \text{for some integer } k\right\} .$$

One incorrect solution was received.

M168. *Proposed by Neven Jurič, Zagreb, Croatia.*

How many different 3×3 arrays of non-negative integers is it possible to construct so that each of the three horizontal sums and each of the three vertical sums is equal to 7, the first diagonal sum is equal to 10, and the second diagonal sum is equal to 9? (Two arrays which may be transformed into one another by rotations and/or reflections are not considered to be different.)

Here is an example of such an array:

$$\begin{array}{ccc} 2 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 0 & 4 \end{array}$$

Solution by Geneviève Lalonde, Massey, ON.

Let the entries in the array be $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$. Then the conditions may be written as a system of equations:

$$\begin{aligned} a_1 + a_2 + a_3 &= 7\,, & (1)\\ a_4 + a_5 + a_6 &= 7\,, & (2)\\ a_7 + a_8 + a_9 &= 7\,, & (3)\\ a_1 + a_4 + a_7 &= 7\,, & (4)\\ a_2 + a_5 + a_8 &= 7\,, & (5)\\ a_3 + a_6 + a_9 &= 7\,, & (6)\\ a_1 + a_5 + a_9 &= 10\,, & (7)\\ a_3 + a_5 + a_7 &= 9\,. & (8) \end{aligned}$$

(There is no loss of generality here, since the array may be reflected or rotated.) By adding equations (1), (2), and (3), we get

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \;=\; 21\,. \qquad (9)$$

On the other hand, if we add (2), (5), (7), and (8), we obtain

$$a_1 + a_2 + a_3 + a_4 + 4a_5 + a_6 + a_7 + a_8 + a_9 \;=\; 33\,. \qquad (10)$$

Subtracting (9) from (10) yields $3a_5 = 12$, from which we get $a_5 = 4$. We may assume, without loss of generality, that $a_1 \ge a_9$ and $a_3 \ge a_7$. From (7) we see that $a_1 \geq 3$, and from (8) we see that $a_3 \geq 3$. Then, from (1) we see that there are only three possibilities for a_1 and a_3, namely $a_1 = a_3 = 3$, $a_1 = 3$ and $a_3 = 4$, or $a_1 = 4$ and $a_3 = 3$. Each of these lead to a unique array satisfying (1) to (8) (the details are left to the reader):

$$\begin{bmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 4 \\ 3 & 4 & 0 \\ 1 & 3 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 2 \\ 2 & 3 & 2 \end{bmatrix}.$$

The last array above is equivalent to the one given as an example in the problem statement.

M169. *Proposed by the Mayhem Staff.*

Prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2003}-\frac{1}{2004} = \frac{1}{1003}+\frac{1}{1004}+\frac{1}{1005}+\cdots+\frac{1}{2003}+\frac{1}{2004}.$$

Solution by Geneviève Lalonde, Massey, ON.

We claim that, for all positive integers n,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n} = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

(The problem posed is simply the case where $n = 1002$.)

To prove our claim, we apply Mathematical Induction. For $n = 1$, the statement is $1 - \frac{1}{2} = \frac{1}{2}$, which is clearly true. Assume that, for some positive integer k, the statement is true; that is,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k} = \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}.$$

Let us now examine $n = k + 1$. Starting with the left side, and using the above assumption, we have

$$\begin{aligned} & 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2} \\ &= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right) \\ &= \left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right) \\ &= \frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\left(\frac{2}{2k+2}-\frac{1}{2k+2}\right) \\ &= \frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}, \end{aligned}$$

which means that the the result is true for $n = k + 1$. Thus, it is true for $n = k + 1$ whenever it is true for $n = k$. Therefore, it is true for all n.

Also solved by Mihály Bencze, Brasov, Romania. Bencze noted that the identity proved above is due to Catalan.

M170. *Proposed by the Mayhem Staff.*

Evaluate $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 90^\circ$.

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

Let S denote the given sum. Since $\cos 90^\circ = 0$, we have

$$S \;=\; \cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 89^\circ\,.$$

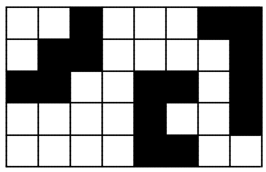
Then, since $\cos x = \sin(90^\circ - x)$ and $\sin^2 x = 1 - \cos^2 x$, we get

$$\begin{aligned} S &= \sin^2(90^\circ - 1^\circ) + \sin^2(90^\circ - 2^\circ) + \cdots + \sin^2(90^\circ - 89^\circ) \\ &= \sin^2 89^\circ + \sin^2 88^\circ + \cdots + \sin^2 1^\circ \\ &= (1 - \cos^2 89^\circ) + (1 - \cos^2 88^\circ) + \cdots + (1 - \cos^2 1^\circ) \\ &= 89 - S\,. \end{aligned}$$

Thus, $2S = 89$, which implies that $S = 44.5$.

Also solved by Mihály Bencze, Brasov, Romania.

M171. *Proposed by Neven Jurič, Zagreb, Croatia.*



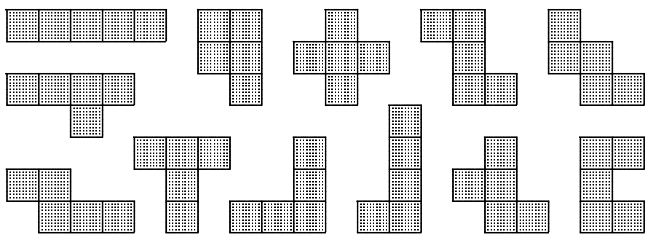
There are 12 distinct (non-congruent) pentominoes, 3 of which are shown to the right. Each pentomino covers an area of 5 square units. (Note: *Pentominoes* is a registered trademark of Solomon W. Golomb.)

1. Find the remaining 9 pentominoes.
2. Arrange all 12 pentominoes on the 60 numbered cells in the diagram to the right, so that each pentomino covers numbers whose sum is 10.

2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

Solution by Roger He, Prince of Wales Collegiate, St. John's, NL.

First, we list the 12 possible pentominoes.



Next we need to fit them in and have their blocks sum to 10. Note that only the numbers 1, 2, 3, 4, 5 are used in the given grid. Looking at all the ways to use five positive integers to add to 10 we get: $5+2+1+1+1$, $4+3+1+1+1$, $4+2+2+1+1$, $3+3+2+1+1$, $3+2+2+2+1$, and

$2+2+2+2+2$. We quickly note that there is no grouping in the given grid that can make the last sum; so we need only consider the others. Counting the number of occurrences of each number in the puzzle, we find that there are 5 fives, 5 fours, 8 threes, 9 twos, and 33 ones. Since 5 only appears in the first pattern above, we can block off around the 5s the cells that could possibly go with it (one region is marked below left).

Piecing together the conditions for the other numbers, we come up with the solution below right.

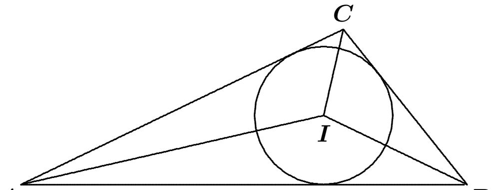
2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

M172. *Proposed by Mihály Bencze, Brasov, Romania.*

Let I denote the centre of the inscribed circle in triangle ABC. Prove that if one of the triangles AIB, BIC, or CIA is similar to triangle ABC, then the angles of triangle ABC are in geometric progression.

Solution by the proposer.



Without loss of generality, we may suppose that $\triangle BIC$ is the triangle which is similar to $\triangle ABC$. Then the angles of $\triangle BIC$ will match up with the angles in $\triangle ABC$ in some order. The angles in $\triangle BIC$ are $\frac{1}{2}B$, $\frac{1}{2}C$, and $\frac{\pi}{2}+\frac{1}{2}A$.

If $\frac{\pi}{2}+\frac{1}{2}A = A$, then $A = \pi$, which is impossible. If $\frac{1}{2}B = B$, then $B = 0$, which is impossible. We can similarly rule out the case where $\frac{1}{2}C = C$.

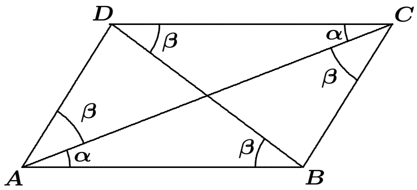
Thus, we have two possibilities:

$$A = \tfrac{1}{2}B\,, \quad B = \tfrac{1}{2}C\,, \quad \text{and} \quad C = \tfrac{\pi}{2}+\tfrac{1}{2}A$$
$$\text{or} \quad A = \tfrac{1}{2}C\,, \quad B = \tfrac{\pi}{2}+\tfrac{1}{2}A\,, \quad \text{and} \quad C = \tfrac{1}{2}B\,,$$

both of which lead to $\triangle ABC$ having angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$, which are in geometric progression.

M173. *Proposed by K.R.S. Sastry, Bangalore, India.*

Suppose that the diagonals AC and BD of a parallelogram $ABCD$ determine angles α and β as shown in the diagram below.



1. Prove that such an arrangement of angles is possible if and only if the diagonals are proportional to the sides.
2. Use trigonometry to express β in terms of α.

Combination of solutions by Bruce Crofoot, Thompson Rivers University, Kamloops, BC; and the proposer.

1. Let E denote the intersection point of the diagonals AC and BD. The given arrangement of angles occurs if and only if $\triangle ABE \sim \triangle ACB$ (or equivalently, $\triangle CDE \sim \triangle CAD$). We will prove that this is true if and only if the diagonals are proportional to the sides (meaning that there is some $k > 0$ such that $AC = k\,AB$ and $BD = k\,BC$).

First suppose that $\triangle ABE \sim \triangle ACB$. Then

$$\frac{AE}{AB} = \frac{AB}{AC} \quad \text{and} \quad \frac{BE}{AE} = \frac{BC}{AB}\,.$$

Since $AE = \frac{1}{2}AC$ and $BE = \frac{1}{2}BD$, we have

$$\frac{\frac{1}{2}AC}{AB} = \frac{AB}{AC} \quad \text{and} \quad \frac{BD}{AC} = \frac{BC}{AB}\,.$$

The first equation implies that $AC = \sqrt{2}\,AB$. Using this result in the second equation, we obtain $BD = \sqrt{2}\,BC$. Thus, the diagonals are proportional to the sides with proportionality constant $\sqrt{2}$.

Conversely, suppose that $AC = k\,AB$ and $BD = k\,BC$ for some $k > 0$. We apply the Parallelogram Law (which holds for any parallelogram):

$$AC^2 + BD^2 = 2AB^2 + 2BC^2\,.$$

Substituting $AC = k\,AB$ and $BD = k\,BC$ on the left, we find that $k = \sqrt{2}$. Then $AC^2 = 2AB^2$, or equivalently, $\frac{AE}{AB} = \frac{AB}{AC}$. This implies that $\triangle ABE \sim \triangle ACB$.

2. Applying the Law of Sines to $\triangle ABC$, we have

$$\frac{a}{\sin\beta} = \frac{b}{\sin\alpha} = \frac{\sqrt{2}a}{\sin(\alpha+\beta)}\,.$$

From the first and third expressions above, we have $\sin(\alpha+\beta)=\sqrt{2}\sin\beta$, from which we deduce that

$$\tan\beta \;=\; \frac{\sin\alpha}{\sqrt{2}-\cos\alpha}\,.$$

M174. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let x denote the measure of an angle of a non-degenerate triangle. Determine x, given that

$$\frac{1}{\sin x} \;=\; \frac{1}{\sin 2x}+\frac{1}{\sin 3x}\,.$$

Solution by Miguel Marañón, IES Sagasta, Logroño, Spain.

First we rewrite the given condition successively as

$$\begin{aligned}\frac{1}{\sin x} &= \frac{\sin 2x+\sin 3x}{\sin 2x\sin 3x}\,,\\ \sin 2x\sin 3x &= \sin x(\sin 2x+\sin 3x)\,,\\ 2\sin x\cos x\sin 3x &= \sin x(\sin 2x+\sin 3x)\,.\end{aligned}$$

We now divide by $\sin x$ (since the given condition makes sense only for $\sin x\neq 0$) to get

$$2\cos x\sin 3x \;=\; \sin 2x+\sin 3x\,.$$

Using the trigonometric identity

$$\sin A+\sin B \;=\; 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

with $A=4x$ and $B=2x$, we get $\sin 4x+\sin 2x=2\sin 3x\cos x$. Then we can rewrite our equation above as

$$\sin 4x+\sin 2x \;=\; \sin 2x+\sin 3x\,,$$

or $\sin 4x = \sin 3x$. This has solutions of the form $4x = 3x+2k\pi$ or $4x = \pi+2k\pi-3x$ for $k\in\mathbb{Z}$; that is, $x = 2k\pi$ or $x = (2k+1)\frac{\pi}{7}$. The former implies that $\sin x=0$, which we have already ruled out. For the same reason, we can rule out the case where $2k+1$ is a multiple of 7 in the latter case.

Therefore, the solutions of the proposed equation are $x=(2k+1)\frac{\pi}{7}$ where k is an integer and $k\not\equiv 3 \pmod 7$.

Also solved by Mihály Bencze, Brasov, Romania.