MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Please send your solutions to the problems in this edition by **1 June 2006**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M226. *Proposed by John Ciriani, Kamloops, BC.*

Antonino has a drawer full of identical black socks and identical white socks. If he were to select two socks at random from his drawer, the probability that they match would be $\frac{1}{2}$. How many of each colour of sock does Antonino have? (There is more than one answer.)

M227. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, ON.*

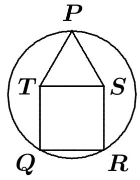
Let N be a positive integer such that N leaves a remainder of 2 or 4 when divided by 6 and there are integers x and y such that $N = x^2 + 27y^2$. Prove that there exist integers a and b with $N = a^2 + 3b^2$ where b is not divisible by 3.

M228. *Proposed by K.R.S. Sastry, Bangalore, India.*

(a) The zeros of the polynomial $P(x) = x^2 - 5x + 2$ are precisely the dimensions of a rectangle in centimetres. Determine the perimeter and the area of the rectangle.

(b) The zeros of the polynomial $P(x) = x^3 - 70x^2 + 1629x - 12600$ are precisely the inner dimensions of a rectangular room in metres. Find the total surface area and the volume of the interior of the room (when doors and windows are closed).

M229. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*



An equilateral triangle sits atop a square as in the diagram. All sides have length 1. A circle passes through points P, Q and R. What is the radius of the circle?

M230. *Proposed by the Mayhem Staff.*

Al, Betty, Cecil, Dora, and Eugene are going to divide n coins among themselves knowing that:

1. Everyone receives at least one coin.
2. Al gets fewer coins than Betty, who gets fewer than Cecil, who gets fewer than Dora, who gets fewer than Eugene.
3. Each person knows only the total n and how many coins he or she got.

What is the smallest possible value of n such that nobody can deduce the number of coins received by each of the others without more information?

M231. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Cordelia and Kent play the following game. Cordelia goes first and they take alternate turns. Each selects a number from 1 to 6 inclusive that has not already been selected; the game ends in six moves. At the end of each move, the player making the move takes the sum of all the numbers selected by either player up to that point and claims all of its positive divisors. When the game is over, the score of each player is the highest number k for which the player has claimed all the consecutive numbers 1, 2, 3, $\ldots$, k from 1 to k inclusive. The winner is the player with the highest score; if both have the same score, neither wins and the game is a draw. For example, suppose the six moves are as follows: C:2; K:4; C:1; K:3; C:5; K:6. The respective claims by C are 1, 2; 1, 7; 1, 3, 5, 15; and by K are 1, 2, 3, 6; 1, 2, 5, 10; 1, 3, 7, 21. Cordelia and Kent have the same score, 3, and the game is a draw. The example does not demonstrate very good play. Is there any way that Cordelia can be prevented from winning assuming she is playing as an expert?

. . . .

M226. *Proposé par John Ciriani, Kamloops, BC.*

Antonino a un tiroir plein de chaussettes noires identiques et de chaussettes blanches identiques. S'il devait choisir deux chaussettes au hasard, la probabilité qu'elles soient de même couleur serait $\frac{1}{2}$. Combien de chaussettes de chaque couleur Antonino possède-t-il ? (Il y a plus d'une réponse.)

M227. *Proposé par Kenneth S. Williams, Université Carleton, Ottawa, ON.*

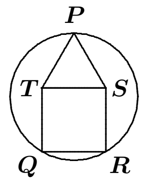
Soit N un nombre entier positif tel que N donne un reste de 2 ou 4 lorsque divisé par 6 et il y a des entiers x et y tels que $N = x^2 + 27y^2$. Montrer qu'il existe des entiers a et b avec $N = a^2 + 3b^2$ où b n'est pas divisible par 3.

M228. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

(a) Les zéros du polynôme $P(x) = x^2 - 5x + 2$ donnent en centimètres les dimensions d'un rectangle. Déterminer le périmètre et l'aire du rectangle.

(b) Les zéros du polynôme $P(x) = x^3 - 70x^2 + 1629x - 12600$ donnent en mètres les dimensions intérieures d'une chambre rectangulaire. Trouver la surface totale et le volume de l'intérieur de la chambre (les portes et fenêtres étant fermées).

M229. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*



On dessine un triangle équilatéral à partir du côté supérieur d'un carré suivant la figure ci-contre. Tous les côtés sont de longueur 1. Si un cercle passe par les points P, Q et R, quel est son rayon ?

M230. *Proposé par l'Equipe de Mayhem.*

Alex, Berthe, Carole, Denise et Eugène se partagent n jetons de sorte que :

1. Chacun reçoît au moins un jeton.
2. Par ordre alphabétique, chacun en reçoît strictement moins que le suivant.
3. À part le total des n jetons, chaque personne ne connaît que le nombre de jetons qu'elle a reçus.

Quelle est la plus petite valeur possible de n de sorte que personne ne puisse en déduire, sans information supplémentaire, combien les autres ont reçu de jetons ?

M231. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Catherine et Patrick jouent le jeu suivant. On joue tour à tour et c'est Catherine qui commence. Chacun choisit un nombre, de 1 à 6 inclusivement, qui n'a pas encore été sélectionné ; le jeu s'arrête après six tours. À la fin de chaque tour, le joueur à qui c'est le tour fait la somme de tous les nombres choisis jusqu'ici par chacun des joueurs et marque tous les diviseurs positifs de celle-ci. À la fin du jeu, la marque de chaque jouer est le plus grand nombre k pour lequel le joueur a marqué tous les nombres consécutifs $1, 2, 3, \ldots, k$ de 1 à k inclusivement. Le gagnant est le joueur avec la plus haute marque ; il y a un match nul si les deux joueurs obtiennent la même marque. Par exemple, supposons que les six tours donnent : C :2 ; P :4 ; C :1 ; P :3 ; C :5 ; P :6. Les marques respectives de C sont $1, 2$; $1, 7$; $1, 3, 5, 15$; et celles de P sont$1, 2, 3, 6$; $1, 2, 5, 10$; et $1, 3, 7, 21$. Catherine et Patrick ont la même marque, 3, et c'est match nul. Cet exemple ne représente pas la meilleure manière de jouer. Y a-t-il une manière d'empêcher Catherine de gagner en supposant qu'elle joue comme un expert ?

Mayhem Solutions

M163. Corrected. *Proposed by the Mayhem Staff.*

Show that it is possible to put non-negative integer values on the faces of two dice (not necessarily the same on both dice) so that, when the dice are tossed, the outcomes 1, 2, 3, $\dots$, 12 are equally probable.

Solution by Geoffrey Siu, London Central Secondary School, London, ON.

There are many ways in which this can be done. For example,

Die 1	Die 2
0 0 0 6 6 6	1 2 3 4 5 6
0 0 0 1 1 1	1 3 5 7 9 11
1 1 1 2 2 2	0 2 4 6 8 10
1 1 1 7 7 7	0 1 2 3 4 5
0 6 13 14 15 16	1 2 3 4 5 6
6 6 6 6 6 6	7 7 7 7 7 7

It can be verified that the first four examples above yield pairs where 1, 2, 3, $\dots$, 12 all occur with probability $\frac{1}{12}$. In the fifth example above, 1, 2, 3, $\dots$, 12 all occur with probability $\frac{1}{36}$; while in the sixth example, they occur with probability 0, which fits the requirements of the problem.

Also solved by the Austrian IMO team; Roger He, Prince of Wales Collegiate, St. Johns, NL; Titu Zvonaru, Comanesti, Romania.

M164. *Proposed by the Mayhem Staff.*

Consider the following procedure for dividing the three-digit number 375 by 8. Write down the number formed by the first two digits, namely, 37. Multiply this by 2 to get 74. Add to this the units digit of 375 (the original number), obtaining $74 + 5 = 79$. Then divide by 8 to get 9 with a remainder of 7. Add this result (9, remainder 7) to the number 37 (the first two digits of 375) to get your answer: 46, remainder 7. Thus, 375 divided by 8 equals 46 with a remainder of 7.

Does this method always work for three-digit numbers? Why, or why not?

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

The answer to the question is yes.

Let $100a + 10b + c$ be a three-digit number (a, b, and c are the digits). Then we have

$$\frac{100a + 10b + c}{8} = \frac{80a + 8b}{8} + \frac{20a + 2b + c}{8} = 10a + b + \frac{2(10a + b) + c}{8}.$$

The calculation on the right side is exactly the "method" described in the problem.

Notice that this algorithm can be extended quite naturally to an n–digit number, since

$$\frac{\sum_{i=0}^{n} a_i 10^i}{8} = \frac{a_0 + \sum_{i=1}^{n}(8+2)a_i 10^{i-1}}{8} = \sum_{i=1}^{n} a_i 10^{i-1} + \frac{a_0 + 2\sum_{i=1}^{n} a_i 10^{i-1}}{8}\,.$$

M165. *Proposed by Babis Stergiou, Chalkida, Greece.*

If a, $b > 0$, prove that

(a) $\sqrt{ab} \;\geq\; \dfrac{2}{1/a + 1/b}$.

(b) $a^6 + b^6 + 8a^3 + 8b^3 + 2a^3b^3 + 16 \;\geq\; 36ab$.

Solution by Mark Yang, St. Paul's Junior High School, St. John's, NL.

(a) By the AM–GM Inequality, we know that

$$\frac{a^3b + a^2b^2 + a^2b^2 + ab^3}{4} \;\geq\; \sqrt[4]{a^8b^8}\,;$$

that is, $ab\big(a^2 + 2ab + b^2\big) \geq 4a^2b^2$. Since a, $b > 0$, we rearrange and take the square root of both sides to get

$$\sqrt{ab} \;\geq\; \frac{2ab}{a+b} \;=\; \frac{2}{1/a + 1/b}\,.$$

(b) By the AM–GM Inequality, we know that

$$\frac{a^6 + b^6 + a^3 + b^3 + \cdots + a^3 + b^3 + a^3b^3 + a^3b^3 + 1 + \cdots + 1}{36} \;\geq\; ab\,,$$

where the numerator on the left side contains 8 terms each of a^3 and b^3, and 16 terms of 1. The result follows immediately.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Vedula N. Murty, Dover, PA, USA.

M166. *Proposed by the Mayhem Staff.*

(a) Simplify

$$(3n)^2 + (4n-1)^2 - (5n-1)^2\,, \qquad (3n+2)^2 + (4n)^2 - (5n+1)^2\,.$$

(b) Using (a) or otherwise, prove that all positive integers can be represented in the form $a^2 + b^2 - c^2$ where a, b, c are integers and $0 < a < b < c$.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

(a) Note that

$$\begin{aligned}(3n)^2 + (4n-1)^2 - (5n-1)^2 \\ = 9n^2 + 16n^2 - 8n + 1 - 25n^2 + 10n - 1 = 2n\,.\end{aligned}$$

Similarly,

$$\begin{aligned}(3n+2)^2 + (4n)^2 - (5n+1)^2 \\ = 9n^2 + 12n + 4 + 16n^2 - 25n^2 - 10n - 1 = 2n+3\,.\end{aligned}$$

(b) Let m be a positive integer. Assume first that m is odd. For the first several odd integers, observe that $1 = 4^2 + 7^2 - 8^2$, $3 = 4^2 + 6^2 - 7^2$, $5 = 4^2 + 5^2 - 6^2$, and $7 = 10^2 + 14^2 - 17^2$. Now let $m = 2n+3$ with $n > 2$. Then $m = (3n+2)^2 + (4n)^2 - (5n+1)^2$, and since $n > 2$, we have $0 < 3n+2 < 4n < 5n+1$.

Next, assume m is even. We note first that $2 = 5^2 + 11^2 - 12^2$. Now, for $n > 1$, let $m = 2n$. Then $m = 2n = (3n)^2 + (4n-1)^2 - (5n-1)^2$, and since $n > 1$, we have $0 < 3n < 4n-1 < 5n-1$.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.

M167. *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Solve the following inequality:

$$(2+\cos x)(1+\sin x-\cos x) \;\geq\; \cos x(1+2\sin x-\cos x)\,.$$

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

The inequality we are to solve is equivalent to

$$2(1-\cos x) + \sin x(2-\cos x) \;\geq\; 0\,. \tag{1}$$

Since the inequality is periodic with period 2π, it will suffice to solve it in an interval of length 2π.

Consider any $x \in (-\pi, \pi)$. Then $-\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}$. Let $t = \tan\left(\frac{x}{2}\right)$ (or equivalently, $\frac{x}{2} = \arctan t$). Using the formulas

$$\begin{aligned}\cos x &= \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{1-t^2}{1+t^2} \\ \text{and} \qquad \sin x &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = \frac{2t}{1+t^2}\,,\end{aligned}$$

the inequality (1) can be stated equivalently as

$$2t(t+1)(2t^2+t+1) \;\geq\; 0\,,$$

which holds for all t such that $t \leq -1$ or $t \geq 0$.

Therefore, the original inequality is valid for all $x \in (-\pi, \pi)$ such that $\tan\left(\frac{x}{2}\right) \leq -1$ or $\tan\left(\frac{x}{2}\right) \geq 0$; that is, for all x such that $-\pi < x \leq -\frac{\pi}{2}$ or

$0 \leq x < \pi$. By continuity (or by direct calculation), the inequality must also hold when $x = \pm\pi$.

Our results can be stated more simply if we use the interval $[0, 2\pi)$ in place of $[-\pi, \pi)$. In $[0, 2\pi)$, the inequality holds for $0 \leq x \leq \frac{3\pi}{2}$. The complete solution set is

$$\left\{x : 2k\pi \leq x \leq 2k\pi + \tfrac{3\pi}{2} \quad \text{for some integer } k\right\} .$$

One incorrect solution was received.

M168. *Proposed by Neven Jurič, Zagreb, Croatia.*

How many different 3×3 arrays of non-negative integers is it possible to construct so that each of the three horizontal sums and each of the three vertical sums is equal to 7, the first diagonal sum is equal to 10, and the second diagonal sum is equal to 9? (Two arrays which may be transformed into one another by rotations and/or reflections are not considered to be different.)

Here is an example of such an array:

2	3	2
2	4	1
3	0	4

Solution by Geneviève Lalonde, Massey, ON.

Let the entries in the array be $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$. Then the conditions may be written as a system of equations:

$$\begin{aligned} a_1 + a_2 + a_3 &= 7\,, & (1)\\ a_4 + a_5 + a_6 &= 7\,, & (2)\\ a_7 + a_8 + a_9 &= 7\,, & (3)\\ a_1 + a_4 + a_7 &= 7\,, & (4)\\ a_2 + a_5 + a_8 &= 7\,, & (5)\\ a_3 + a_6 + a_9 &= 7\,, & (6)\\ a_1 + a_5 + a_9 &= 10\,, & (7)\\ a_3 + a_5 + a_7 &= 9\,. & (8) \end{aligned}$$

(There is no loss of generality here, since the array may be reflected or rotated.) By adding equations (1), (2), and (3), we get

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \;=\; 21\,. \qquad (9)$$

On the other hand, if we add (2), (5), (7), and (8), we obtain

$$a_1 + a_2 + a_3 + a_4 + 4a_5 + a_6 + a_7 + a_8 + a_9 \;=\; 33\,. \qquad (10)$$

Subtracting (9) from (10) yields $3a_5 = 12$, from which we get $a_5 = 4$. We may assume, without loss of generality, that $a_1 \geq a_9$ and $a_3 \geq a_7$. From (7) we see that $a_1 \geq 3$, and from (8) we see that $a_3 \geq 3$. Then, from (1) we see that there are only three possibilities for a_1 and a_3, namely $a_1 = a_3 = 3$, $a_1 = 3$ and $a_3 = 4$, or $a_1 = 4$ and $a_3 = 3$. Each of these lead to a unique array satisfying (1) to (8) (the details are left to the reader):

$$\begin{bmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 4 \\ 3 & 4 & 0 \\ 1 & 3 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 2 \\ 2 & 3 & 2 \end{bmatrix}.$$

The last array above is equivalent to the one given as an example in the problem statement.

M169. *Proposed by the Mayhem Staff.*

Prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2003}-\frac{1}{2004} = \frac{1}{1003}+\frac{1}{1004}+\frac{1}{1005}+\cdots+\frac{1}{2003}+\frac{1}{2004}.$$

Solution by Geneviève Lalonde, Massey, ON.

We claim that, for all positive integers n,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n} = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

(The problem posed is simply the case where $n = 1002$.)

To prove our claim, we apply Mathematical Induction. For $n = 1$, the statement is $1-\frac{1}{2}=\frac{1}{2}$, which is clearly true. Assume that, for some positive integer k, the statement is true; that is,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k} = \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}.$$

Let us now examine $n = k+1$. Starting with the left side, and using the above assumption, we have

$$\begin{aligned} 1&-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2} \\ &= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right) \\ &= \left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right) \\ &= \frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\left(\frac{2}{2k+2}-\frac{1}{2k+2}\right) \\ &= \frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}, \end{aligned}$$

which means that the the result is true for $n = k+1$. Thus, it is true for $n = k+1$ whenever it is true for $n = k$. Therefore, it is true for all n.

Also solved by Mihály Bencze, Brasov, Romania. Bencze noted that the identity proved above is due to Catalan.

M170. *Proposed by the Mayhem Staff.*

Evaluate $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 90^\circ$.

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

Let S denote the given sum. Since $\cos 90^\circ = 0$, we have

$$S \;=\; \cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 89^\circ\,.$$

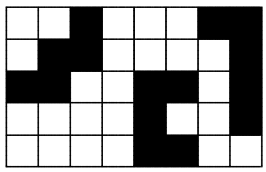
Then, since $\cos x = \sin(90^\circ - x)$ and $\sin^2 x = 1 - \cos^2 x$, we get

$$\begin{aligned} S &= \sin^2(90^\circ - 1^\circ) + \sin^2(90^\circ - 2^\circ) + \cdots + \sin^2(90^\circ - 89^\circ) \\ &= \sin^2 89^\circ + \sin^2 88^\circ + \cdots + \sin^2 1^\circ \\ &= (1 - \cos^2 89^\circ) + (1 - \cos^2 88^\circ) + \cdots + (1 - \cos^2 1^\circ) \\ &= 89 - S\,. \end{aligned}$$

Thus, $2S = 89$, which implies that $S = 44.5$.

Also solved by Mihály Bencze, Brasov, Romania.

M171. *Proposed by Neven Jurič, Zagreb, Croatia.*



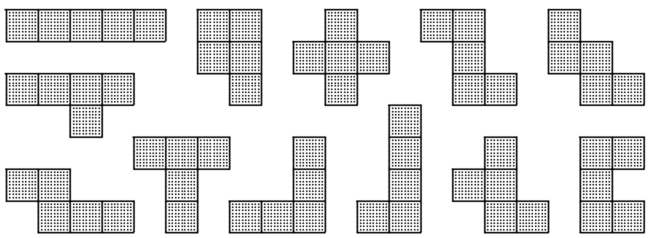
There are 12 distinct (non-congruent) pentominoes, 3 of which are shown to the right. Each pentomino covers an area of 5 square units. (Note: *Pentominoes* is a registered trademark of Solomon W. Golomb.)

1. Find the remaining 9 pentominoes.
2. Arrange all 12 pentominoes on the 60 numbered cells in the diagram to the right, so that each pentomino covers numbers whose sum is 10.

2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

Solution by Roger He, Prince of Wales Collegiate, St. John's, NL.

First, we list the 12 possible pentominoes.



Next we need to fit them in and have their blocks sum to 10. Note that only the numbers 1, 2, 3, 4, 5 are used in the given grid. Looking at all the ways to use five positive integers to add to 10 we get: $5+2+1+1+1$, $4+3+1+1+1$, $4+2+2+1+1$, $3+3+2+1+1$, $3+2+2+2+1$, and

$2+2+2+2+2$. We quickly note that there is no grouping in the given grid that can make the last sum; so we need only consider the others. Counting the number of occurrences of each number in the puzzle, we find that there are 5 fives, 5 fours, 8 threes, 9 twos, and 33 ones. Since 5 only appears in the first pattern above, we can block off around the 5s the cells that could possibly go with it (one region is marked below left).

Piecing together the conditions for the other numbers, we come up with the solution below right.

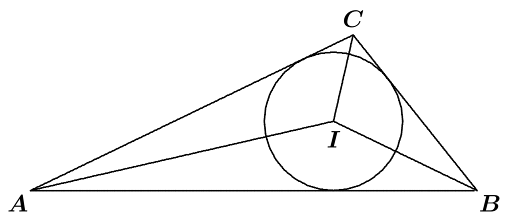
2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

M172. *Proposed by Mihály Bencze, Brasov, Romania.*

Let I denote the centre of the inscribed circle in triangle ABC. Prove that if one of the triangles AIB, BIC, or CIA is similar to triangle ABC, then the angles of triangle ABC are in geometric progression.

Solution by the proposer.



Without loss of generality, we may suppose that $\triangle BIC$ is the triangle which is similar to $\triangle ABC$. Then the angles of $\triangle BIC$ will match up with the angles in $\triangle ABC$ in some order. The angles in $\triangle BIC$ are $\frac{1}{2}B$, $\frac{1}{2}C$, and $\frac{\pi}{2}+\frac{1}{2}A$.

If $\frac{\pi}{2}+\frac{1}{2}A = A$, then $A = \pi$, which is impossible. If $\frac{1}{2}B = B$, then $B = 0$, which is impossible. We can similarly rule out the case where $\frac{1}{2}C = C$.

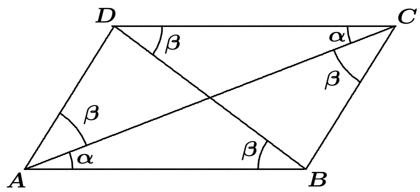
Thus, we have two possibilities:

$$A = \tfrac{1}{2}B\,, \quad B = \tfrac{1}{2}C\,, \quad \text{and} \quad C = \tfrac{\pi}{2}+\tfrac{1}{2}A$$
$$\text{or} \quad A = \tfrac{1}{2}C\,, \quad B = \tfrac{\pi}{2}+\tfrac{1}{2}A\,, \quad \text{and} \quad C = \tfrac{1}{2}B\,,$$

both of which lead to $\triangle ABC$ having angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$, which are in geometric progression.

M173. *Proposed by K.R.S. Sastry, Bangalore, India.*

Suppose that the diagonals AC and BD of a parallelogram $ABCD$ determine angles α and β as shown in the diagram below.



1. Prove that such an arrangement of angles is possible if and only if the diagonals are proportional to the sides.

2. Use trigonometry to express β in terms of α.

Combination of solutions by Bruce Crofoot, Thompson Rivers University, Kamloops, BC; and the proposer.

1. Let E denote the intersection point of the diagonals AC and BD. The given arrangement of angles occurs if and only if $\triangle ABE \sim \triangle ACB$ (or equivalently, $\triangle CDE \sim \triangle CAD$). We will prove that this is true if and only if the diagonals are proportional to the sides (meaning that there is some $k > 0$ such that $AC = k\,AB$ and $BD = k\,BC$).

First suppose that $\triangle ABE \sim \triangle ACB$. Then

$$\frac{AE}{AB} = \frac{AB}{AC} \quad \text{and} \quad \frac{BE}{AE} = \frac{BC}{AB}\,.$$

Since $AE = \frac{1}{2}AC$ and $BE = \frac{1}{2}BD$, we have

$$\frac{\frac{1}{2}AC}{AB} = \frac{AB}{AC} \quad \text{and} \quad \frac{BD}{AC} = \frac{BC}{AB}\,.$$

The first equation implies that $AC = \sqrt{2}\,AB$. Using this result in the second equation, we obtain $BD = \sqrt{2}\,BC$. Thus, the diagonals are proportional to the sides with proportionality constant $\sqrt{2}$.

Conversely, suppose that $AC = k\,AB$ and $BD = k\,BC$ for some $k > 0$. We apply the Parallelogram Law (which holds for any parallelogram):

$$AC^2 + BD^2 = 2AB^2 + 2BC^2\,.$$

Substituting $AC = k\,AB$ and $BD = k\,BC$ on the left, we find that $k = \sqrt{2}$. Then $AC^2 = 2AB^2$, or equivalently, $\frac{AE}{AB} = \frac{AB}{AC}$. This implies that $\triangle ABE \sim \triangle ACB$.

2. Applying the Law of Sines to $\triangle ABC$, we have

$$\frac{a}{\sin\beta} = \frac{b}{\sin\alpha} = \frac{\sqrt{2}a}{\sin(\alpha+\beta)}\,.$$

From the first and third expressions above, we have $\sin(\alpha+\beta) = \sqrt{2}\sin\beta$, from which we deduce that

$$\tan\beta \;=\; \frac{\sin\alpha}{\sqrt{2}-\cos\alpha}\,.$$

M174. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let x denote the measure of an angle of a non-degenerate triangle. Determine x, given that

$$\frac{1}{\sin x} \;=\; \frac{1}{\sin 2x} + \frac{1}{\sin 3x}\,.$$

Solution by Miguel Marañón, IES Sagasta, Logroño, Spain.

First we rewrite the given condition successively as

$$\begin{aligned}\frac{1}{\sin x} &= \frac{\sin 2x + \sin 3x}{\sin 2x \sin 3x}\,,\\ \sin 2x \sin 3x &= \sin x(\sin 2x + \sin 3x)\,,\\ 2\sin x \cos x \sin 3x &= \sin x(\sin 2x + \sin 3x)\,.\end{aligned}$$

We now divide by $\sin x$ (since the given condition makes sense only for $\sin x \neq 0$) to get

$$2\cos x \sin 3x \;=\; \sin 2x + \sin 3x\,.$$

Using the trigonometric identity

$$\sin A + \sin B \;=\; 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

with $A = 4x$ and $B = 2x$, we get $\sin 4x + \sin 2x = 2\sin 3x \cos x$. Then we can rewrite our equation above as

$$\sin 4x + \sin 2x \;=\; \sin 2x + \sin 3x\,,$$

or $\sin 4x = \sin 3x$. This has solutions of the form $4x = 3x + 2k\pi$ or $4x = \pi + 2k\pi - 3x$ for $k \in \mathbb{Z}$; that is, $x = 2k\pi$ or $x = (2k+1)\frac{\pi}{7}$. The former implies that $\sin x = 0$, which we have already ruled out. For the same reason, we can rule out the case where $2k+1$ is a multiple of 7 in the latter case.

Therefore, the solutions of the proposed equation are $x = (2k+1)\frac{\pi}{7}$ where k is an integer and $k \not\equiv 3 \pmod 7$.

Also solved by Mihály Bencze, Brasov, Romania.

Problem of the Month

Ian VanderBurgh, University of Waterloo

A couple of years ago, while fiddling with some problems in preparation for our annual June Mathematics Contest Seminar here in Waterloo, I came across the following problem:

Solve the following system of equations:

$$\begin{aligned} x^2 + xy &= 12\,, \\ 2xy + 3y^2 &= -5\,. \end{aligned}$$

Luckily, my good friend (and former Mayhem columnist) Paul Ottaway happened by. We set to work. With our combined wisdom, we managed to remember that the usual method for solving a system of equations was to eliminate one of the variables.

Following this idea, we solved the first equation for y to get $y = x - \dfrac{12}{x}$, which we then substituted into the second equation, obtaining

$$2x\left(x - \frac{12}{x}\right) + 3\left(x - \frac{12}{x}\right)^2 \;=\; -5\,.$$

After expanding and clearing out denominators, we ended up with a quartic equation, which was hardly appetizing. It was also apparent that solving the second equation for x and substituting into the first equation was not going to be any better.

Thus, the standard technique of eliminating one of the variables was not working. Then one of us had the clever idea to try eliminating the constants instead. (Since the idea was clever, odds are it was Paul's idea, not mine!)

We multiplied the first equation by 5 (obtaining $5x^2 + 5xy = 60$) and the second equation by 12 (obtaining $24xy + 36y^2 = -60$) and added the equations to obtain

$$5x^2 + 29xy + 36y^2 \;=\; 0\,,$$

which we were then able to factor to give

$$(x + 4y)(5x + 9y) \;=\; 0\,,$$

yielding $x = -4y$ or $x = -\frac{9}{5}y$.

Substituting $x = -4y$ into $x^2 + xy = 12$ gives $16y^2 - 4y^2 = 12$; that is, $y^2 = 1$. Hence, $y = \pm 1$, and $(x, y) = (-4, 1)$ or $(x, y) = (4, -1)$.

Substituting $x = -\frac{9}{5}y$ into $x^2 + xy = 12$ gives $\frac{81}{25}y^2 - \frac{9}{5}y^2 = 12$; that is, $\frac{36}{25}y^2 = 12$. Hence, $y = \pm\frac{5}{\sqrt{3}}$, and $(x, y) = \left(-\frac{9}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right)$ or $(x, y) = \left(\frac{9}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right)$.

We found a neat way to tackle certain systems of equations. Much to my delight, as I was flipping through a recent journal looking for problems, I found the following problem from a European competition of which I had never heard:

Problem. (2005 Mathematical Duel)
Determine all integer solutions of the system of equations

$$\begin{aligned} x^2z + y^2z + 4xy &= 40\,, \\ x^2 + y^2 + xyz &= 20\,. \end{aligned}$$

I attempted this problem for a few minutes by the usual method of eliminating one of the variables. This met with little success. Then I remembered Paul's trick!

Solution: If we subtract 2 times the second equation from the first, the constants are eliminated, and we obtain

$$x^2z + y^2z + 4xy - 2x^2 - 2y^2 - 2xyz = 0\,.$$

Where to now? Our only real hope is to try to factor this equation somehow. If we rearrange the terms to get

$$x^2z - 2x^2 + y^2z - 2y^2 - 2xyz + 4xy = 0\,,$$

the light at the end of the tunnel starts to appear. We can now do some factoring to get

$$x^2(z-2) + y^2(z-2) - 2xy(z-2) = 0\,.$$

Thus, there is a common factor of $z - 2$, which we can factor out, giving

$$(z-2)(x^2 - 2xy + y^2) = 0\,,$$

or

$$(z-2)(x-y)^2 = 0\,.$$

Therefore, $z = 2$ or $x = y$.

If $z = 2$, then the second equation gives $x^2 + 2xy + y^2 = 20$, or $(x + y)^2 = 20$. But we are looking for integer solutions (unlike the first problem). There are no solutions here, since 20 is not a perfect square.

If $x = y$, the second equation gives $2x^2 + zx^2 = 20$, or $x^2(z+2) = 20$. (The first equation reduces to this same equation as well.) Since x and z are integers and the only perfect squares which are divisors of 20 are 1 and 4, the possibilities for x are 1, -1, 2, and -2, giving values for z of 18, 18, 3, and 3, respectively.

Therefore, the integer solutions for (x, y, z) are $(1, 1, 18)$, $(-1, -1, 18)$, $(2, 2, 3)$, $(-2, -2, 3)$.

We have discovered a new technique for solving old problems—always a satisfying feeling. Problems involving systems of equations occur frequently on contests and in real life, and it is important to have a wide variety of different techniques in our toolbox.