SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2923. [2004 : 108, 111; 2005 : 124–125] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that $x, y \geq 0$ ($x, y \in \mathbb{R}$) and $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

Generalization by Li Zhou, Polk Community College, Winter Haven, FL, USA.

The ingenious Solution I by Chip Curtis [2005 : 124–125] can be easily modified to prove an infinite chain of inequalities.

Theorem. Suppose that x_i, a_i, and b_i are real numbers and that $x_i > 0$ for all i, $1 \leq i \leq n$. Let $c_i \in \{a_i, b_i\}$. If $x_1^{a_1} + \cdots + x_n^{a_n} \geq x_1^{b_1} + \cdots + x_n^{b_n}$, then, for $k = 0, 1, 2, \ldots$, we have

$$x_1^{c_1+k(a_1-b_1)} + x_2^{c_n+k(a_n-b_n)} \leq x_1^{c_1+(k+1)(a_1-b_1)} + \cdots + x_n^{c_n+(k+1)(a_n-b_n)} .$$

Proof: For each $k = 0, 1, 2, \ldots$, set

$$S_k = x_1^{c_1+k(a_1-b_1)} + \cdots + x_n^{c_n+k(a_n-b_n)} .$$

Then, by the Cauchy–Schwarz Inequality, the assumption, and the AM–GM Inequality, we have

$$\begin{aligned} S_0 &= \left\langle x_1^{\frac{b_1}{2}}, \ldots, x_n^{\frac{b_n}{2}} \right\rangle \cdot \left\langle x_1^{c_1-\frac{b_1}{2}}, \ldots, x_n^{c_n-\frac{b_n}{2}} \right\rangle \\ &\leq \sqrt{x_1^{b_1} + \cdots + x_n^{b_n}}\sqrt{x_1^{2c_1-b_1} + \cdots + x_n^{2c_n-b_n}} \\ &\leq \sqrt{x_1^{a_1} + \cdots + x_n^{a_n}}\sqrt{x_1^{2c_1-b_1} + \cdots + x_n^{2c_n-b_n}} \\ &\leq \tfrac{1}{2}\left(x_1^{a_1} + \cdots + x_n^{a_n} + x_1^{2c_1-b_1} + \cdots + x_n^{2c_n-b_n}\right) . \end{aligned}$$

Note that if $c_i = a_i$, then $2c_i - b_i = c_i + (a_i - b_i)$; and if $c_i = b_i$, then $2c_i - b_i = c_i$ and $a_i = c_i + (a_i - b_i)$. In either case,

$$x_i^{a_i} + x_i^{2c_i-b_i} = x_i^{c_i} + x_i^{c_i+(a_i-b_i)} .$$

Hence, $S_0 \leq \frac{1}{2}(S_0 + S_1)$; that is, $S_0 \leq S_1$.

As an induction hypothesis, assume that $S_{k-1} \leq S_k$ for some $k \geq 1$. Then, by the Cauchy–Schwarz Inequality again, we have

$$\begin{aligned} S_k &= \left\langle x_1^{\frac{c_1+(k-1)(a_1-b_1)}{2}}, \ldots, x_n^{\frac{c_n+(k-1)(a_n-b_n)}{2}} \right\rangle \\ &\qquad \cdot \left\langle x_1^{\frac{c_1+(k+1)(a_1-b_1)}{2}}, \ldots, x_n^{\frac{c_n+(k+1)(a_n-b_n)}{2}} \right\rangle \\ &\leq \sqrt{S_{k-1}}\sqrt{S_{k+1}} \;\leq\; \sqrt{S_k}\sqrt{S_{k+1}}\,. \end{aligned}$$

Hence, $S_k \leq S_{k+1}$, and the induction is complete. ∎

The following related inequalities may also be of interest to the reader.

Theorem. Suppose that u_i, v_i, and t_i are real numbers and that $u_i > 0$ and $v_i > 0$ for all i, $1 \leq i \leq n$. Let $w_i \in \{u_i, v_i\}$. If

$$u_1^{t_1} + \cdots + u_n^{t_n} \;\geq\; v_1^{t_1} + \cdots + v_n^{t_n}\,,$$

then, for $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned} &\left[w_1\left(\frac{u_1}{v_1}\right)^k\right]^{t_1} + \cdots + \left[w_n\left(\frac{u_n}{v_n}\right)^k\right]^{t_n} \\ &\qquad\leq\; \left[w_1\left(\frac{u_1}{v_1}\right)^{k+1}\right]^{t_1} + \cdots + \left[w_n\left(\frac{u_n}{v_n}\right)^{k+1}\right]^{t_n}. \end{aligned}$$

Proof: Let $x_i = e^{t_i}$, $a_i = \ln u_i$, $b_i = \ln v_i$, and $c_i = \ln w_i$. It follows that $u_i^{t_i} = x_i^{a_i}$, $v_i^{t_i} = x_i^{b_i}$, $\left[w_i\left(\frac{u_i}{v_i}\right)^k\right]^{t_i} = x_i^{c_i+k(a_i-b_i)}$, and all the inequalities are transformed into those of the previous theorem. ∎

2984. [2004 : 431, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)} \;=\; 2\sum_{n=1}^{\infty}\frac{1}{n^3}\,.$$

Ed: At the end of the solution to 2984 [2005 : 480], we included a conjecture by the proposer: If

$$P(k) \;=\; \sum_{i_1=1}^{\infty}\cdots\sum_{i_k=1}^{\infty}\frac{1}{i_1 i_2\cdots i_k(i_1+i_2+\cdots+i_k)}\,,$$

for $k = 1, 2, 3, \ldots$, then $P(k) = k\zeta(k+1)$, where $\zeta(k+1)$ is the Riemann Zeta function evaluated at $k+1$. [The conjecture, as it originally appeared, was that $\zeta(k) = k\zeta(k+1)$, which was obviously incorrect.]

David Bradley, University of Maine, Orono, ME, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA, have each determined that the conjecture is false, but can be easily fixed, namely $P(k) = k!\zeta(k+1)$.

Bradley states that Mordell proved this in the 1950s. See equation (5) of L.J. Mordell, "On the evaluation of some multiple series", *J. London Math Soc.* (2) 33 (1958), 368–371. His equation (10) is the case $k = 2$, which is the published version of the problem. Mordell actually proved the generalization

$$\sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{i_1 i_2 \cdots i_k (i_1 + i_2 + \cdots + i_k + a)} \;=\; k! \sum_{i=0}^{\infty} \frac{(-1)^i \binom{a-1}{i}}{(i+1)^{k+1}}\,,$$

which reduces to the above formula for $P(k)$ when $a = 0$.

Furdui and Zhou both provided a proof of the corrected conjecture. Their combined proof is given below.

Proof of corrected conjecture.

We first notice that

$$\begin{aligned} P(k) &= \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{i_1 i_2 \cdots i_k} \int_0^1 x^{i_1+i_2+\cdots+i_k-1}\,dx \\ &= \int_0^1 \frac{1}{x} \biggl(\sum_{i_1=1}^{\infty} \frac{x^{i_1}}{i_1} \biggr) \cdots \biggl(\sum_{i_k=1}^{\infty} \frac{x^{i_k}}{i_k} \biggr)\,dx \\ &= \int_0^1 (-1)^k \frac{\bigl(\ln(1-x)\bigr)^k}{x}\,dx\,. \end{aligned}$$

We now use the substitution $t = 1 - x$ in the above integral to get

$$\begin{aligned} P(k) &= (-1)^k \int_0^1 \frac{(\ln t)^k}{1-t}\,dt \;=\; (-1)^k \int_0^1 (\ln t)^k \biggl(\sum_{n=0}^{\infty} t^n \biggr)\,dt \\ &= (-1)^k \sum_{n=0}^{\infty} \int_0^1 t^n (\ln t)^k\,dt\,. \end{aligned}$$

In view of the integration by parts formulas, we see that

$$\int_0^1 t^n (\ln t)^k\,dt \;=\; (-1)^k \frac{k!}{(n+1)^{k+1}}\,.$$

Therefore,

$$\begin{aligned} P(k) &= (-1)^k \sum_{n=0}^{\infty} (-1)^k \frac{k!}{(n+1)^{k+1}} \\ &= k! \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k+1}} \;=\; k!\zeta(k+1)\,. \end{aligned}$$

3001. [2005 : 43, 46] *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Given a, b, c, d, $e > 0$ such that $a^2+b^2+c^2+d^2+e^2 \geq 1$, prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c} \;\geq\; \frac{\sqrt{5}}{3}\,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.

We first show that the above inequality holds for a, b, c, d, $e > 0$ such that $a^2+b^2+c^2+d^2+e^2 = 1$.

By the Cauchy-Schwarz Inequality we have

$$\begin{aligned} 1 \;=\; \Big(\sum_{\text{cyclic}} a^2\Big)^2 \;&=\; \left(\sum_{\text{cyclic}} \frac{a}{\sqrt{b+c+d}}\left(a\sqrt{b+c+d}\right)\right)^2 \\ &\leq\; \left(\sum_{\text{cyclic}} \frac{a^2}{b+c+d}\right)\left(\sum_{\text{cyclic}} a^2(b+c+d)\right). \end{aligned}$$

Thus, it suffices to show that

$$\sum_{\text{cyclic}} a^2(b+c+d) \;\leq\; \frac{3}{\sqrt{5}}\,. \qquad (1)$$

Again, by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \left(\sum_{\text{cyclic}} a^2(b+c+d)\right)^2 \;&\leq\; \left(\sum_{\text{cyclic}} a^2\right)\left(\sum_{\text{cyclic}} a^2(b+c+d)^2\right) \\ &=\; \sum_{\text{cyclic}} a^2(b+c+d)^2\,. \end{aligned}$$

Hence, (1) follows if we can show that

$$\sum_{\text{cyclic}} a^2(b+c+d)^2 \;\leq\; \frac{9}{5}\,.$$

The Cauchy-Schwarz Inequality yields the following two inequalities:

$$(b+c+d)^2 \;\leq\; (1^2+1^2+1^2)(b^2+c^2+d^2) \;=\; 3(b^2+c^2+d^2)$$

and

$$\left(\sum_{\text{cyclic}}(a^2+e^2)\right)^2 \leq (1^2+1^2+1^2+1^2+1^2)\sum_{\text{cyclic}}(a^2+e^2)^2 = 5\sum_{\text{cyclic}}(a^2+e^2)^2.$$

These, together with the fact that $2\sum\limits_{\text{cyclic}}(a^4+a^2e^2) = \sum\limits_{\text{cyclic}}(a^2+e^2)^2$, imply that

$$
\begin{aligned}
\sum_{\text{cyclic}} a^2(b+c+d)^2 &\le 3\sum_{\text{cyclic}} a^2(b^2+c^2+d^2) = 3\sum_{\text{cyclic}} a^2(1-a^2-e^2) \\
&= 3\sum_{\text{cyclic}} a^2 - 3\sum_{\text{cyclic}}(a^4+a^2e^2) \\
&= 3 - \frac{3}{2}\sum_{\text{cyclic}}(a^2+e^2)^2 \\
&\le 3 - \frac{3}{10}\Big(\sum_{\text{cyclic}}(a^2+e^2)\Big)^2 = 3 - \frac{3}{10}(2)^2 = \frac{9}{5}\,.
\end{aligned}
$$

The original inequality follows from the proven inequality if we replace (a,b,c,d,e) with $\left(\frac{a}{\sqrt{r}},\frac{b}{\sqrt{r}},\frac{c}{\sqrt{r}},\frac{d}{\sqrt{r}},\frac{e}{\sqrt{r}}\right)$ where $r=a^2+b^2+c^2+d^2+e^2$.

Also solved by KEE-WAI LAU, Hong Kong, China; MARIAN TETIVA, Bîrlad, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Janous has proposed the following generalization: Determine the set of all positive exponents α such that

$$\sum_{\text{cyclic}} \frac{a^\alpha}{b+c+d} \ge \frac{5^{(1-\gamma)}}{3}\left(a^\beta+b^\beta+c^\beta+d^\beta+e^\beta\right)^\gamma ,$$

where $\beta=\beta(\alpha)$ and $\gamma=\gamma(\alpha)$ are suitable.

3002. [2005 : 43, 46] *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let r, $s\in\mathbb{R}$ with $0<r<s$, and let a, b, $c\in(r,s)$. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \le \frac{3}{2}+\frac{(r-s)^2}{2r(r+s)} ,$$

and determine when equality occurs.

Similar solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let

$$F(a,b,c) = \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

with domain $[r,s]$. Then $\frac{\partial^2 f}{\partial a^2} = \frac{2b}{(c+a)^3}+\frac{2c}{(a+b)^3} > 0$. By symmetry, we conclude that F is a strictly convex function in each one of its three variables. Then F attains its maximum at the vertices of the box $[r,s]^3$. By symmetry, we need only check four of the vertices:

$$F(r,r,r) = \frac{3}{2} = F(s,s,s)$$

$$\text{and} \qquad F(r,s,s) = \frac{3}{2}+\frac{(r-s)^2}{2s(r+s)} < \frac{3}{2}+\frac{(r-s)^2}{3r(r+s)} = F(r,r,s)\,.$$

Thus, $F(a,b,c) \leq \frac{3}{2} + \frac{(r-s)^2}{2r(r+s)}$. Equality occurs if and only if two of the three numbers are equal to r while the third is equal to s.

Also solved by the AUSTRIAN IMO-TEAM 2005; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; MARIAN TETIVA, Bîrlad, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was also one incorrect solution.

3003. [2005 : 43, 46] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle with $AB \neq AC$. Let AD be the altitude from A to BC and let BE and CF be the internal angle bisectors of $\angle B$ and $\angle C$, respectively, with E on AC and F on AB. Let B' and C' be the points of intersection of AD with BE and CF, respectively, and let A' be the point where BE intersects CF.

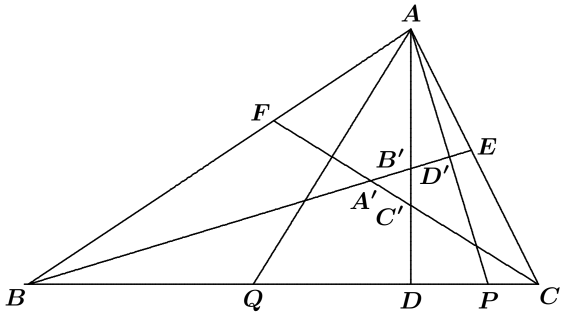
Construct the point Q on BC on the same side of C as B such that $QC = AC$, and construct the point P on BC on the same side of B as C such that $PB = AB$.

Prove that $\triangle A'B'C'$ is similar to $\triangle AQP$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let D' be the intersection point of AP and BE.

Since BE bisects $\angle ABP$ of the isosceles triangle ABP, we have $BE \perp AP$. Consequently, $PDB'D'$ is a cyclic quadrilateral, and we get $\angle APQ = \angle A'B'C'$. Similarly, $\angle AQP = \angle A'C'B'$, completing the proof.



Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ROBERT BILINSKI, Collège Montmorency, Laval, QC; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; IMO-TEAM AUSTRIA 2005; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Several solvers had similar proofs, but this editor felt that Zhou's solution was the neatest (in keeping with his love of pure geometric arguments).

3004. [2005 : 43, 46] *Proposed by Mihály Bencze, Brasov, Romania.*

Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Prove that

$$\frac{\left(\sqrt{a}-\sqrt{b}\right)^2+\left(\sqrt{b}-\sqrt{c}\right)^2+\left(\sqrt{c}-\sqrt{a}\right)^2}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2} \;\le\; \frac{4}{9}\left(\frac{R}{r}-2\right).$$

Solution and generalization by Gabriel Dospinescu, Paris, France, and Marian Tetiva, Bîrlad, Romania, adapted by the editor.

We shall establish the much stronger result that $D \le \frac{1}{16}\left(\frac{R}{r}-2\right)$, where D denotes the expression on left side of the inequality above.

Let $x=s-a$, $y=s-b$, and $z=s-c$, where $s=\frac{1}{2}(a+b+c)$ denotes the semiperimeter of $\triangle ABC$, and let F denote the area of $\triangle ABC$.

From the well-known formulas $R=\dfrac{abc}{4F}$ and $F=rs$, we have

$$\frac{R}{r} \;=\; \frac{abcs}{4F^2} \;=\; \frac{abcs}{4s(s-a)(s-b)(s-c)} \;=\; \frac{(x+y)(y+z)(z+x)}{4xyz}.$$

However,

$$\begin{aligned}\frac{(x+y)(y+z)(z+x)}{4xyz} &= \frac{(x+y)(y+z)}{4yz}+\frac{(x+y)(y+z)}{4xy}\\ &\ge \frac{x+y}{y+z}+\frac{y+z}{x+y}=\frac{c}{a}+\frac{a}{c},\end{aligned}$$

since $\dfrac{y+z}{4yz}\ge\dfrac{1}{y+z}$ and $\dfrac{x+y}{4xy}\ge\dfrac{1}{x+y}$. Thus, $\dfrac{R}{r}\ge\dfrac{c}{a}+\dfrac{a}{c}$, which implies that

$$\begin{aligned}\frac{R}{r}-2 &\ge \frac{c^2+a^2}{ac}-2 = \frac{(a-c)^2}{ac} = \frac{(\sqrt{a}+\sqrt{c})^2(\sqrt{a}-\sqrt{c})^2}{ac}\\ &\ge \frac{4\sqrt{ac}(\sqrt{a}-\sqrt{c})^2}{ac}. \qquad (1)\end{aligned}$$

Due to the symmetry in D, we may assume, without loss of generality, that $a\ge b\ge c$. Since

$$\begin{aligned}(\sqrt{a}-\sqrt{c})^2 &= \left((\sqrt{a}-\sqrt{b})+(\sqrt{b}-\sqrt{c})\right)^2\\ &\ge (\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2,\end{aligned}$$

and since $(\sqrt{a}+\sqrt{b}+\sqrt{c})^2\ge(\sqrt{a}+2\sqrt{c})^2\ge 8\sqrt{ac}$, we have

$$\begin{aligned}D &= \frac{(\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{a})^2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\\ &\le \frac{2(\sqrt{a}-\sqrt{c})^2}{8\sqrt{ac}} = \frac{4\sqrt{ac}(\sqrt{a}-\sqrt{c})^2}{16ac}. \qquad (2)\end{aligned}$$

From (1) and (2), we obtain $\dfrac{R}{r}-2\ge 16D$, and hence $D\le\dfrac{1}{16}\left(\dfrac{R}{r}-2\right)$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions)*; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.*

Arslanagić pointed out that in the paper "Another Inequality Strengthening Euler's Inequality $R-2r \geq 0$, Octogon Mathematical Magazine*, Vol. 11, No. 2, October, 2003, p. 746, Janous had obtained the stronger result $D \leq \frac{4}{27}\left(\frac{R}{r}-2\right)$, which is of course weaker than the one featured above. Janous himself also submitted a solution of his result. Dospinescu and Tetiva asked the natural question: what is the largest constant $k > 0$ for which $kD \leq \frac{R}{r} - 2$ holds for all triangles?*

3005. [2005 : 44, 46] *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Let h_a, h_b, h_c be the lengths of the altitudes of $\triangle ABC$ issuing from A, B, C, respectively, and let w_a, w_b, w_c be the lengths of the interior angle bisectors of A, B, C, respectively. Prove that

$$\frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \;\geq\; 1 + \frac{4r}{R}\,.$$

Solution by Michel Bataille, Rouen, France, modified slightly by the editor.

We will prove the stronger inequality

$$\frac{h_a^2}{w_a^2} + \frac{h_b^2}{w_b^2} + \frac{h_c^2}{w_c^2} \;\geq\; 1 + \frac{4r}{R}\,.$$

First we note that if AH is the altitude and AW the interior bisector of $\angle CAB$, then $\angle HAW = \frac{1}{2}\left|\angle B - \angle C\right|$, and hence, $\frac{h_a}{w_a} = \cos\frac{B-C}{2} \leq 1$. Similarly, $\frac{h_b}{w_b} = \cos\frac{C-A}{2} \leq 1$ and $\frac{h_c}{w_c} = \cos\frac{A-B}{2} \leq 1$. Thus,

$$\begin{aligned}
\frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} &\geq \frac{h_a^2}{w_a^2} + \frac{h_b^2}{w_b^2} + \frac{h_c^2}{w_c^2} \\
&= \cos^2\frac{B-C}{2} + \cos^2\frac{C-A}{2} + \cos^2\frac{A-B}{2} \\
&= \tfrac{3}{2} + \tfrac{1}{2}\left[\cos(B-C) + \cos(C-A) + \cos(A-B)\right]\,.
\end{aligned}$$

Using the inequality

$$\cos(A-B) + \cos(B-C) + \cos(C-A) \;\geq\; 8(\cos A + \cos B + \cos C) - 9$$

(***CRUX with MAYHEM*** 2760 [2002 : 396; 2003 : 342]), and the well-known identities

$$\cos A + \cos B + \cos C \;=\; 1 + 4\sin\left(\tfrac{1}{2}A\right)\sin\left(\tfrac{1}{2}B\right)\sin\left(\tfrac{1}{2}C\right) \;=\; 1 + \frac{r}{R}\,,$$

we obtain

$$\begin{aligned}
&\tfrac{3}{2}+\tfrac{1}{2}\left[\cos(B-C)+\cos(C-A)+\cos(A-B)\right] \\
&\qquad\ge\ \tfrac{3}{2}+\tfrac{1}{2}\left[8(\cos A+\cos B+\cos C)-9\right] \\
&\qquad=\ \tfrac{3}{2}+\tfrac{1}{2}\left[8\left(1+4\sin\left(\tfrac{1}{2}A\right)\sin\left(\tfrac{1}{2}B\right)\sin\left(\tfrac{1}{2}C\right)\right)-9\right] \\
&\qquad=\ \tfrac{3}{2}+4\left(1+r/R\right)-\tfrac{9}{2}\ =\ 1+4r/R\,,
\end{aligned}$$

which completes the proof. Equality holds if and only if the triangle is equilateral.

Also solved by SCOTT BROWN, Auburn University, Montgomery, AL, USA; JOHN G. HEUVER, Grande Prairie, AB (2 solutions)*; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; MARIAN TETIVA, Bîrlad, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.*

The stronger inequality from the above solution was also proved by Janous, Murty, Tetiva, and the proposer.

3006. [2005 : 44, 47] *Proposed by Luis V. Dieulefait, Centre de Recerca Matemática, Ballaterra, Spain.*

An old man willed that, upon his death, his three sons would receive the u^{th}, v^{th}, and w^{th} parts of his herd of camels respectively. He had N camels in the herd when he died, where $N+1$ is a common multiple of u, v, and w. Since the three sons could not divide N exactly into u, v, or w parts, they approached a distinguished ***CRUX*** problem solver for help. He rode over on his own camel, which he added to the herd. The herd was then divided up according to the old man's wishes. Our ***CRUX*** problem solver then took back the one camel that remained, which was, of course, his own.

(a) Find all solutions (u, v, w, N).

(b)$\star$ Solve the same problem if there are four sons.

(c)$\star$ Let there be k sons. Find an upper bound $f(k)$ on N for the problem to have a solution.

[*Ed*: This is a generalization of Problem 2226 [1997 : 166; 1998 : 186].]

Combination of solutions by Michel Bataille, Rouen, France; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA, for parts (a) *and* (b). *Only Curtis solved part* (c).

(a) In the case of three sons, we seek an integer $N \ge 3$ and positive integers u, v, and w such that u, v, and w each divide $N+1$ (but not N), and $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)(N+1)=N$, or equivalently,

$$\frac{1}{u}+\frac{1}{v}+\frac{1}{w} \;=\; \frac{N}{N+1}\,.$$

Without loss of generality, we may assume that $u \le v \le w$. The assumption $u \ge 5$ would yield $\frac{N}{N+1} \le \frac{3}{5}$, contradicting $N \ge 3$. Thus, $2 \le u \le 4$.

• If $u = 4$, then $\frac{1}{v}+\frac{1}{w} = \frac{3N-1}{4N+4}$ and, arguing as above, we see that $v \ge 5$ is impossible. Since $v \ge u$, we must have $v = 4$. Hence, $\frac{1}{w} = \frac{N-1}{2N+2}$ and $w = \frac{2N+2}{N-1} = 2 + \frac{4}{N-1}$. It follows that $N-1$ divides 4. Therefore, $N = 3$ or $N = 5$. Now $N = 4$ leads to $w = 3 < v$, a contradiction. Thus, $N = 3$ and $w = 4$, which yields $(u,v,w,N) = (4,4,4,3)$. Conversely, this is obviously suitable (with $a = b = c = 1$).

• If $u = 3$, then a similar argument shows that we must have $v = 4$ or $v = 3$. As above, $v = 4$ leads to $w = \frac{12N+12}{5N-7}$, implying that $5N-7$ must divide $12N+12$. Then $5N-7$ must divide $5(12N+12) - 12(5N-7) = 144$. Since $N+1$ is a multiple of both 3 and 4, we see that $N \ge 11$. It is easily seen that the only possibility is $5N-7 = 48$ and $N = 11$. But this gives $w = 3 < v$, a contradiction. Similarly, $v = 3$ leads to $N = 5$ or 11, and to the solutions $(3,3,6,5)$ and $(3,3,4,11)$.

• If $u = 2$, the same method calls for the examination of the cases $v = 6$, 5, 4, or 3 and leads to 9 more solutions.

In conclusion, there are 12 solutions which are displayed in the chart below.

N	(u,v,w)
3	$(4,4,4)$
5	$(2,6,6)$
5	$(3,3,6)$
7	$(2,4,8)$
9	$(2,5,5)$
11	$(2,3,12)$
11	$(2,4,6)$
11	$(3,3,4)$
17	$(2,3,9)$
19	$(2,4,5)$
23	$(2,3,8)$
41	$(2,3,7)$

(b) For the case of four sons, we seek an integer $N \geq 4$ and positive integers u, v, w, x, such that u, v, w, and x each divide $N+1$ (but none divide N), and $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}+\frac{1}{x}\right)(N+1)=N$, or equivalently,

$$\frac{1}{u}+\frac{1}{v}+\frac{1}{w}+\frac{1}{x} = \frac{N}{N+1}.$$

Without loss of generality, we may again assume that $u \leq v \leq w \leq x$. Arguing in a similar vein to part (a) we find the solutions in the chart below.

N	(u,v,w,x)
4	$(5,5,5,5)$
5	$(3,6,6,6)$
7	$(4,4,4,8)$
7	$(2,8,8,8)$
8	$(3,3,9,9)$
9	$(2,5,10,10)$
11	$(4,4,4,6)$
11	$(3,4,4,12)$
11	$(3,4,6,6)$
11	$(3,3,6,12)$
11	$(2,4,12,12)$
11	$(2,6,6,12)$
13	$(2,7,7,7)$
14	$(3,5,5,5)$
14	$(3,3,5,15)$
15	$(2,4,8,16)$
17	$(3,3,6,9)$
17	$(2,3,18,18)$
17	$(2,6,6,9)$
19	$(4,4,4,5)$
19	$(2,4,10,10)$
19	$(2,5,5,20)$
20	$(3,3,7,7)$
23	$(3,4,4,8)$
23	$(3,3,4,24)$
23	$(3,3,6,8)$
23	$(2,3,12,24)$
23	$(2,4,6,24)$
23	$(2,4,8,12)$
23	$(2,6,6,8)$
27	$(2,4,7,14)$
29	$(3,3,5,10)$
29	$(2,3,10,30)$
29	$(2,3,15,15)$
29	$(2,5,5,15)$
29	$(2,5,6,10)$
35	$(3,3,4,18)$
35	$(2,3,9,36)$
35	$(2,3,12,18)$
35	$(2,4,6,18)$
35	$(2,4,9,9)$
39	$(2,4,5,40)$
39	$(2,4,8,10)$
41	$(3,3,6,7)$
41	$(2,3,14,14)$
41	$(2,6,6,7)$
44	$(3,3,5,9)$
47	$(3,3,4,16)$
47	$(2,3,8,48)$
47	$(2,3,12,16)$
47	$(2,4,6,16)$
53	$(2,3,9,27)$
59	$(3,4,5,5)$
59	$(3,3,4,15)$
59	$(2,3,10,20)$
59	$(2,3,12,15)$
59	$(2,4,5,30)$
59	$(2,4,6,15)$
59	$(2,5,5,12)$
69	$(2,5,7,7)$
71	$(2,4,8,9)$
71	$(2,3,8,36)$
71	$(2,3,9,24)$
77	$(2,3,13,13)$
83	$(3,3,4,14)$
83	$(2,3,7,84)$
83	$(2,3,12,14)$
83	$(2,4,6,14)$
89	$(2,3,10,18)$
95	$(2,3,8,32)$
99	$(2,4,5,25)$
109	$(2,5,5,11)$
119	$(3,3,5,8)$
119	$(2,3,8,30)$
119	$(2,4,5,24)$
119	$(2,5,6,8)$
125	$(2,3,7,63)$
125	$(2,3,9,21)$
139	$(2,4,7,10)$
155	$(3,3,4,13)$
155	$(2,3,12,13)$
155	$(2,4,6,13)$
167	$(2,3,7,56)$
167	$(2,3,8,28)$
179	$(2,3,9,20)$
215	$(2,3,8,27)$
219	$(2,4,5,22)$
239	$(2,3,10,16)$
293	$(2,3,7,49)$
311	$(2,3,8,26)$
335	$(2,3,7,48)$
341	$(2,3,9,19)$
419	$(2,4,5,21)$
599	$(2,3,8,25)$
629	$(2,3,7,45)$
923	$(2,3,7,44)$
1805	$(2,3,7,43)$

(c) For k sons, we claim that the maximal number of camels for which there is a solution is exactly given by

$$f(k) \; = \; a(k+1) - 2\,,$$

where $a(k)$ is *Sylvester's sequence* (also called *Euclid numbers*), defined recursively (see sequence A000058 in [1]) by $a(0) = 2$ and, for $k \geq 0$,

$$a(k+1) \; = \; a(k)^2 - a(k) + 1\,.$$

To see why, note first that N is maximal if and only if $\frac{N}{N+1}$ is as close to 1 as possible. But (again, see [1]) $\{a(k)\}$ is a "greedy sequence"; that is, $a(k+1)$ is the smallest integer greater than $a(n)$ such that $\sum\limits_{j=1}^{k+1} \frac{1}{a(k)}$ does not exceed 1. Moreover,

$$\sum_{j=1}^{k} \frac{1}{a(k)} \; = \; \frac{a(k+1)-2}{a(k+1)-1}\,,$$

implying that the sum is of the form $\frac{N}{N+1}$. In fact, setting

$$\frac{a(k+1)-2}{a(k+1)-1} \; = \; \frac{N}{N+1}$$

gives $N = a(k+1) - 2$, as claimed.

In particular, the formula gives the following:

# of sons	Maximum number of camels
1	1
2	5
3	41
4	1805
5	3263441
6	10650056950805
7	113423713055421844361000441
8	12864938683278671740537145998360961546653259485195805

Reference

[1] N.J.A. Sloane, "The On-line Encyclopedia of Integer Sequences", http://www.research.att.com/ njas/sequences.

Parts (a) *and* (b) *also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.*

3007. [2005 : 44, 47; corrected 2005 : 173, 176] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle, and let $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$ such that
$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k > 0\,.$$

1. Prove that the segments AA_1, BB_1, CC_1 are the sides of a triangle.

Let T_k denote this triangle. Let R_k and r_k be the circumradius and inradius of T_k. Prove that:

2. $P(T_k) < P(ABC)$, where $P(T)$ denotes the perimeter of triangle T;
3. $[T_k] = \dfrac{k^2+k+1}{(k+1)^2}[ABC]$, where $[T]$ denotes the area of triangle T;
4. $R_k \geq \dfrac{k\sqrt{k}\,P(ABC)}{(k+1)(k^2+k+1)}$;
5. $r_k > \dfrac{k^2+k+1}{(k+1)^2}\,r$, where r is the inradius of $\triangle ABC$.

[Editor: The problem was originally stated with equality in parts 4 and 5. This was the fault of the editor, not the proposer.]

Solution by Titu Zvonaru, Comănești, Romania.

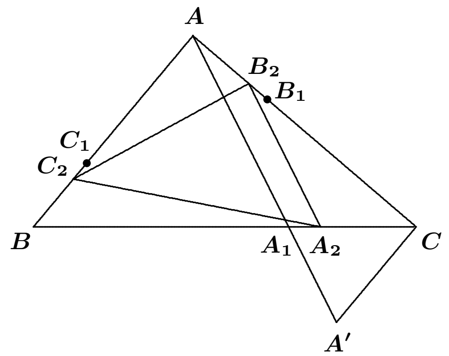
As usual, let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Since
$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k\,,$$
it follows that $BA_1 = \dfrac{ak}{k+1}$ and $A_1C = \dfrac{a}{k+1}$, with analogous expressions for CB_1, B_1A, AC_1, and C_1B. By the Law of Cosines, we obtain
$$\begin{aligned} AA_1^2 &= c^2 + \frac{a^2k^2}{(k+1)^2} - \frac{k}{k+1}\cdot 2ac\cos B \\ &= \frac{c^2(k+1)^2 + a^2k^2 - (k^2+k)(a^2+c^2-b^2)}{(k+1)^2} \\ &= \frac{b^2(k^2+k) + c^2(k+1) - a^2k}{(k+1)^2}\,. \end{aligned}$$

1. Let $A_2 \in BC$, $B_2 \in CA$, and $C_2 \in AB$ such that
$$\frac{BA_2}{A_2C} = \frac{CB_2}{B_2A} = \frac{AC_2}{C_2B} = k+1\,.$$

Then $A_2C = \dfrac{a}{k+2}$ and $CB_2 = \dfrac{b(k+1)}{k+2}$. By the Law of Cosines,

$$\begin{aligned} A_2B_2^2 &= \frac{a^2}{(k+2)^2} + \frac{b^2(k+1)^2}{(k+2)^2} - \frac{k+1}{(k+2)^2}\cdot 2ab\cos C \\ &= \frac{a^2+b^2(k+1)^2-(k+1)(a^2+b^2-c^2)}{(k+2)^2} \\ &= \frac{b^2(k^2+k)+c^2(k+1)-a^2k}{(k+2)^2} = \frac{(k+1)^2}{(k+2)^2}\cdot AA_1^2\,. \end{aligned}$$

Hence, $AA_1 = \dfrac{k+2}{k+1}\cdot A_2B_2$. It follows that AA_1, BB_1, and CC_1 are the sides of a triangle similar to triangle $A_2B_2C_2$.



2. The line through C parallel to AB intersects AA_1 at a point A'. By similitude, $A'A_1 = AA_1/k$ and $CA' = c/k$. In $\triangle ACA'$, we have $AA' < AC + CA'$. Equivalently, $AA_1 + \dfrac{AA_1}{k} < b + \dfrac{c}{k}$, or $AA_1 < \dfrac{bk+c}{k+1}$. Hence,

$$P(T_k) = AA_1+BB_1+CC_1 < \frac{bk+c+ck+a+ak+b}{k+1} = P(ABC)\,.$$

We note an alternative way to prove the inequality $AA_1 < \dfrac{bk+c}{k+1}$. This inequality is equivalent to $b^2(k^2+k)+c^2(k+1)-a^2k < (bk+c)^2$, or (by simplification) $|b-c| < a$, which is the Triangle Inequality in $\triangle ABC$.

3. We have

$$[CA_2B_2] = \frac{A_2C\cdot B_2C\cdot\sin C}{2} = \frac{ab(k+1)\sin C}{2(k+2)^2} = \frac{k+1}{(k+2)^2}[ABC]\,.$$

Since the sides of $\triangle A_2B_2C_2$ are in the ratio $k+2 : k+1$ to the sides of T_k, we deduce that

$$\begin{aligned} [T_k] = \frac{(k+2)^2}{(k+1)^2}[A_2B_2C_2] &= \frac{(k+2)^2}{(k+1)^2}\left([ABC] - 3\frac{k+1}{(k+2)^2}[ABC]\right) \\ &= \frac{(k^2+k+1)}{(k+1)^2}[ABC]\,. \end{aligned}$$

4. We have

$$\begin{aligned} AA_1 &= \frac{\sqrt{b^2k^2 + c^2 + b^2k + c^2k - a^2k}}{k+1} \\ &\geq \frac{\sqrt{2bck + b^2k + c^2k - a^2k}}{k+1} = \frac{\sqrt{k}\sqrt{(b+c)^2 - a^2}}{k+1} \\ &= \frac{\sqrt{k}\sqrt{(b+c+a)(b+c-a)}}{k+1} = \frac{2\sqrt{k}\sqrt{s(s-a)}}{k+1}. \end{aligned}$$

Since $R_k = \dfrac{AA_1 \cdot BB_1 \cdot CC_1}{4[T_k]}$, it follows that

$$\begin{aligned} R_k &\geq \frac{8k\sqrt{k}\sqrt{s^3(s-a)(s-b)(s-c)}}{(k+1)^3 \cdot 4 \cdot \dfrac{k^2+k+1}{(k+1)^2}[ABC]} \\ &= \frac{2k\sqrt{k} \cdot s \cdot [ABC]}{(k+1)(k^2+k+1)[ABC]} = \frac{k\sqrt{k}\,P(ABC)}{(k+1)(k^2+k+1)}, \end{aligned}$$

where we have used Heron's Formula $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$. Equality holds if and only if $bk = c$, $ck = a$, and $ak = b$; that is, if and only if $abck^3 = abc$, which means that $k = 1$ and $a = b = c$.

5. Since $r_k = \dfrac{2[T_k]}{P(T_k)}$ and $r = \dfrac{2[ABC]}{P([ABC])}$, the inequality follows from parts 2 and 3.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain (parts 1–3 and 5 only); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.