Some Inversion Formulas for Sums of Quotients

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In this note we establish some formulas for certain sums of quotients of a positive integer n, which are closely related to an identity established by Préville-Ratelle in Problem M40 of the April 2003 issue of this journal [1]. We also establish some elementary facts that are not well known about quotients and remainders. Our main result is the following theorem.

Theorem 1. Let n and k be any positive integers with $k \le n$. Then

$$\sum_{d=1}^{k} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=\lfloor \frac{n}{k} \rfloor + 1}^{n} \left\lfloor \frac{n}{d} \right\rfloor = k \left\lfloor \frac{n}{k} \right\rfloor . \qquad (F_k)$$

The first sum is clearly the sum of the quotients of n from $\lfloor n/1 \rfloor$ through $\lfloor n/k \rfloor$. We show below that the second sum is the sum of the quotients of n that are equal to one of $\{1, \ldots, k-1\}$. The "inversion" aspect of the formula is that the quotient sums are being taken in opposite directions.

As an illustration, here are the quotients for $n = 15$:

divisor d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
quotient $\lfloor n/d \rfloor$	15	7	5	3	3	2	2	1	1	1	1	1	1	1	1

Theorem 1 is trivial for $k = 1$. When $k = 2$, for example, the formula says that $15 + 7 - (1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) = 2\lfloor \frac{15}{2} \rfloor$; whereas for $k = 4$, the formula gives $15+7+5+3-(1+1+1+1+1+1+1+1+2+2+3+3) = 4\lfloor \frac{15}{4} \rfloor$.

In order to prove Theorem 1, we use the following result.

Lemma 1. If d and i are positive integers not exceeding n, then

$$\left\lfloor \frac{n}{i} \right\rfloor = d \quad \Longleftrightarrow \quad \left\lfloor \frac{n}{d+1} \right\rfloor < i \le \left\lfloor \frac{n}{d} \right\rfloor .$$

Proof: We first suppose that $\lfloor n/i \rfloor = d$. By definition of the floor function, d is the unique integer such that $d \le n/i < d + 1$. Inverting the inequality yields $n/(d + 1) < i \le n/d$. Certainly, $\lfloor n/(d + 1) \rfloor \le n/(d + 1)$. On the other hand, since i is an integer, we get $i \le \lfloor n/d \rfloor$ from $i \le n/d$. Hence, $\lfloor n/(d + 1) \rfloor < i \le \lfloor n/d \rfloor$. Since these steps are reversible the proof is complete. ∎

Proof of Theorem 1: From Lemma 1, we get at once

$$\sum_{d=\lfloor \frac{n}{k+1}\rfloor+1}^{\lfloor \frac{n}{k}\rfloor} \left\lfloor \frac{n}{d} \right\rfloor = k\left(\left\lfloor \frac{n}{k}\right\rfloor - \left\lfloor \frac{n}{k+1}\right\rfloor\right).$$

Letting $Q(n,k) = \lfloor n/k \rfloor - \lfloor n/(k+1) \rfloor$, we have

$$\sum_{d=\lfloor \frac{n}{k}\rfloor+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{j=1}^{k-1} \sum_{d=\lfloor \frac{n}{j+1}\rfloor+1}^{\lfloor \frac{n}{j}\rfloor} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{j=1}^{k-1} jQ(n,j)\,. \tag{1}$$

On the other hand, it is easy to verify that, for $d = 2, 3, \ldots, k$, we have

$$(d-1)Q(n,d-1) + d\left\lfloor \frac{n}{d} \right\rfloor - (d-1)\left\lfloor \frac{n}{d-1} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor.$$

Part of the left side of this equation telescopes when we sum d from 2 to k. Therefore, using (1), we obtain

$$\sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=1}^{k-1} dQ(n,d) + k\left\lfloor \frac{n}{k} \right\rfloor - n = \sum_{d=\lfloor \frac{n}{k}\rfloor+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor + k\left\lfloor \frac{n}{k} \right\rfloor - n\,.$$

Adding n to each side of this expression completes the proof. ■

Remark. Expression (1) and Lemma 1 show that the second sum in the statement of Theorem 1 is, in fact, the sum of the quotients of n that are equal to one of $\{1, 2, \ldots, k-1\}$.

Next we establish some facts about quotients that are a direct consequence of Lemma 1.

Corollary 1. Let d and i be positive integers not exceeding n.

(a) $d \le \left\lfloor \dfrac{n}{\lfloor n/d \rfloor} \right\rfloor$.

(b) $\left\lfloor \dfrac{n}{\lfloor n/\lfloor n/d \rfloor \rfloor} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor$.

(c) $\left\lfloor \dfrac{n}{\lfloor n/i \rfloor} \right\rfloor = \left\lfloor \dfrac{n}{\lfloor n/d \rfloor} \right\rfloor$ if and only if $\left\lfloor \dfrac{n}{i} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor$.

(d) $\left\lfloor \dfrac{n}{i} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor$ if and only if $\left\lfloor \dfrac{n}{\lfloor n/d \rfloor + 1} \right\rfloor < i \le \left\lfloor \dfrac{n}{\lfloor n/d \rfloor} \right\rfloor$.

(e) $\left\lfloor \dfrac{n}{\lfloor n/i \rfloor} \right\rfloor = d$ if and only if $\left\lfloor \dfrac{n}{i} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor > \left\lfloor \dfrac{n}{d+1} \right\rfloor$.

Proof: We use freely the fact that $\lfloor n/d \rfloor \geq \lfloor n/(d+1) \rfloor$ for any $d \in \mathbb{N}$.

(a) Let $k = \lfloor n/\lfloor n/d \rfloor \rfloor$. By Lemma 1, we see that $\lfloor n/(k+1) \rfloor < \lfloor n/d \rfloor$. Thus, $k+1 > d$; that is, $k \geq d$.

(b) We use (a). Replacing d by $\lfloor n/d \rfloor$, we get $\lfloor n/\lfloor n/\lfloor n/d \rfloor \rfloor \rfloor \geq \lfloor n/d \rfloor$. On the other hand, we have $\lfloor n/\lfloor n/\lfloor n/d \rfloor \rfloor \rfloor \leq \lfloor n/d \rfloor$, because $\lfloor n/d \rfloor$ is a decreasing function of d.

(c) This follows at once from (b).

(d) This follows immediately from Lemma 1 by replacing d with $\lfloor n/d \rfloor$.

(e) Case $i = d$ follows from Lemma 1 by replacing i with $\lfloor n/d \rfloor$. Now we prove the general case. Suppose that $\lfloor n/\lfloor n/i \rfloor \rfloor = d$. Thus,

$$\left\lfloor \frac{n}{\lfloor n/\lfloor n/i \rfloor \rfloor} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor .$$

From (b) we have $\lfloor n/i \rfloor = \lfloor n/d \rfloor$, which implies that $\lfloor n/\lfloor n/d \rfloor \rfloor = d$. Hence, from the case $i = d$, we get $\lfloor n/d \rfloor > \lfloor n/(d+1) \rfloor$. Now suppose that $\lfloor n/i \rfloor = \lfloor n/d \rfloor > \lfloor n/(d+1) \rfloor$. Then $\lfloor n/\lfloor n/i \rfloor \rfloor = \lfloor n/\lfloor n/d \rfloor \rfloor$. Using the case $i = d$ again, we obtain $\lfloor n/\lfloor n/d \rfloor \rfloor = d$. Hence, $\lfloor n/\lfloor n/i \rfloor \rfloor = d$. ■

We can also reformulate Corollary 1 in terms of $n \bmod d = n - d\lfloor n/d \rfloor$, the remainder on division of n by d. For example, reformulation of (a) and case $i = d$ of (e) yields the following result:

Corollary 2. Let d and n be positive integers with $d \leq n$. Then

(a) $n \bmod \lfloor n/d \rfloor \leq n \bmod d$.

(b) $n \bmod \lfloor n/d \rfloor < n \bmod d$ if and only if $\lfloor n/d \rfloor = \lfloor n/(d+1) \rfloor$.

Furthermore, from Theorem 1 and Lemma 1, we get some unusual expressions for $n \bmod k$ and, hence, a criterion for divisibility of n by k.

Corollary 3. Let d and n be positive integers with $d \leq n$. Then

(a) $$n \bmod k = \sum_{d=k+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{\lfloor \frac{n}{k} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=\lfloor \frac{n}{k} \rfloor + 1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor .$$

(b) $n \bmod k = \frac{1}{2}(F(k) + F(\lfloor n/k \rfloor))$, where

$$F(k) = \sum_{d=k+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor .$$

Moreover, $n \bmod k = F(k)$ if and only if $\lfloor n/(k+1) \rfloor < k = \lfloor n/k \rfloor$.

(c) $k \mid n$ if and only if $$\frac{n}{k} = \sum_{d=\lfloor \frac{n}{k} \rfloor + 1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{k-1} \left\lfloor \frac{n}{d} \right\rfloor .$$

Proof: (a) We get the second identity by partitioning the sum $\sum_{d=2}^{n} \left\lfloor \frac{n}{d} \right\rfloor$ in two obvious ways. To prove the first identity, we add and subtract $\sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor$ on the left side of (F_k) to obtain the following equivalent formula:

$$\sum_{d=1}^{\lfloor n/k \rfloor} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=k+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor = k \left\lfloor \frac{n}{k} \right\rfloor .$$

Thus, replacing $k\lfloor n/k \rfloor$ by $n - n \bmod k$, cancelling n, and multiplying both sides of the equation by -1, we complete the proof of (a).

(b) The formula for $n \bmod k$ holds because the right member is the arithmetic mean of the second and third member of (a). From this, we have $n \bmod k = F(k)$ if and only if $F(k) = F(\lfloor n/k \rfloor)$. Partitioning the sum $\sum_{d=2}^{n} \left\lfloor \frac{n}{d} \right\rfloor$ as was done in (a), we get $\sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=2}^{\lfloor \frac{n}{k} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor$. Since each quotient is positive, we have $\lfloor n/k \rfloor = k$. Since these steps are reversible we have proved that $n \bmod k = F(k)$ if and only if $\lfloor n/k \rfloor = k$. Then, from Lemma 1, after we replace i and d by k, the proof of (b) is complete.

(c) This follows at once from (a) using the rightmost expression for $n \bmod k$. ∎

Next we establish a generalization of Theorem 1 that clearly shows the process of inversion of the sums involved.

Corollary 4. Let n, j, and k be positive integers with $j \leq k \leq n$. Then

$$\sum_{d=j+1}^{k} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=\lfloor n/k \rfloor+1}^{\lfloor n/j \rfloor} \left\lfloor \frac{n}{d} \right\rfloor = k \left\lfloor \frac{n}{k} \right\rfloor - j \left\lfloor \frac{n}{j} \right\rfloor . \qquad (F_{j,k})$$

Proof: This follows at once from Theorem 1 by subtracting (F_j) from (F_k). ∎

Remark. Theorem 1 and Corollary 4 are logically equivalent, because (F_k) follows from $(F_{1,k})$.

We have now generalized Préville-Ratelle's identity, since it is precisely Corollary 4 for the case when j and k both divide n.

Corollary 5. Let n, j, and k be positive integers with $j \leq k \leq n$. Then

$$\sum_{d=j+1}^{k} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=\lfloor n/k \rfloor+1}^{\lfloor n/j \rfloor} \left\lfloor \frac{n}{d} \right\rfloor \iff n \bmod j = n \bmod k .$$

Proof: If one replaces $k\lfloor n/k \rfloor - j\lfloor n/j \rfloor$ with $(n \bmod j) - (n \bmod k)$ in $(F_{j,k})$, then the result follows at once from Corollary 4. ∎

Concluding remark. Préville-Ratelle's solution to Problem M40 gives a nice graphical interpretation of $(F_{j,k})$ for the case when j and k are divisors of n. Can the reader generalize that graphical approach to prove $(F_{j,k})$ for arbitrary j and k with $j \leq k$?

References

[1] Louis-François Préville-Ratelle, Solution to Problem M40, *Crux Mathematicorum with Mathematical Mayhem* 29:3 (2003), pp. 140–141.

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