SKOLIAD No. 91

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 August, 2006***. A copy of MATHEMATICAL MAYHEM Vol. 2 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our problems come from a contest organized by Collège Montmorency for the secondary school boards in the Laval region of Quebec.

Je voudrais remercier André Labelle du Collège Montmorency pour nous avoir fourni gracieusement une copie de ce concours.

Montmorency Contest 2003–04

Grade 11, November 2003

1. A magician says he can quickly calculate the square of any number between 50 and 59 inclusive.

You tell him to calculate 57^2.

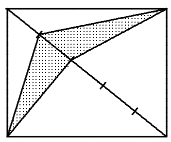
He answers: "Abracadabra! $5^2 = 25$ and $25 + 7 = 32$".

He continues with: "Abracadabra! $7^2 = 49$" (Hint: if the second number had been a 2, then we would have $2^2 = 04$).

He ends with: "The square you seek is 3249!".

He is right! Prove it algebraically.

2. Consider a rectangle with width 8 and length 10. Cutting one of its diagonals in five equal parts, calculate the area of the shaded region.



3. (a) Show that, for any pair of real numbers $a > 0$ and $b > 0$, we have $\frac{a}{b} + \frac{b}{a} \geq 2$.

(b) Deduce, from the result of part (a), that for $x > 0$, $y > 0$, and $z > 0$, we always have $(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) > 8$.

4. Given a rectangular plot having area 4000 m^2. Using two straight line cuts parallel to the sides of the plot, we want to cut it into four small rectangular plots A, B, C, and D.

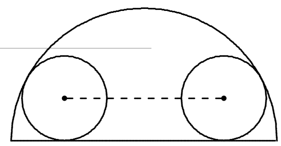
Is it possible to do it in such a way that the area of $A = 2000$ m^2, the area of $B = 1000$ m^2, the area of $C = 600$ m^2, and the area of $D = 400$ m^2?

A	B
C	D

5. During a power outage, two candles of the same length are lit at 6:00 pm. The first candle takes 6 hours to burn completely, and the second takes 8 hours. At a certain time both candles were extinguished and it was observed that the first one was exactly half as long as the second. What was the exact time when this happened?

6. Two circles of radius 8 are placed inside a semicircle of radius 25. The two circles are each tangent to the diameter and to the semicircle.

What is the distance between the centres of the two circles?

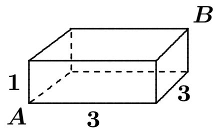


7. Evaluate the very long product that follows:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2003^2}\right).$$

(Hint: $\left(1-\frac{1}{n^2}\right)=\frac{n^2-1}{n^2}=\cdots$)

8. A rectangular box has a base of 3×3 and a height of 1. Find the minimal length of the path a spider could follow, along the surface, to get from corner A to the opposite corner B.



Concours Montmorency 2003-04

Sec V, novembre 2003

1. Un magicien vous propose de calculer le carré de n'importe quel nombre entre 50 et 59 inclusivement.

Vous lui proposez de calculer 57^2.

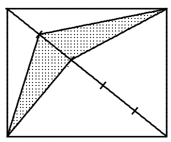
Il répond : «Abracadabra : $5^2=25$ et $25+7=32$».

Il continue : «Abracadabra : $7^2 = 49$». (Rem : si le deuxième chiffre avait été 2, il aurait écrit $2^2=04$).

Il ajoute finalement : «Le carré est 3249».

Il a raison ! Justifiez-le algébriquement.

2. Considérons un rectangle de largeur 8 et de longueur 10. En séparant la diagonale en cinq parties égales, calculer l'aire de la zone hachurée.



3. (a) Montrer que, pour toute paire de nombres réels $a>0$ et $b>0$, on a : $\frac{a}{b}+\frac{b}{a}\geq 2$.

(b) Déduire, grâce au résultat obtenu en (a), que pour $x>0$, $y>0$ et $z>0$, on a toujours $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)>8$.

4. On veut partager un terrain rectangulaire de 4000 m^2 à l'aide de deux lignes droites parallèles aux côtés, en quatre petits terrains rectangulaires A, B, C et D.

Est-il possible de le faire de telle manière que l'aire de $A = 2000$ m^2, l'aire de $B = 1000$ m^2, l'aire de $C = 600$ m^2 et l'aire de $D = 400$ m^2 ?

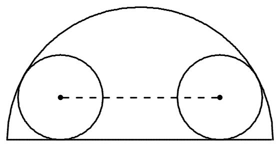
A	B
C	D

5. Durant une panne d'électricité, deux chandelles de même longueur sont allumées à 18 :00 h. La première chandelle se consume en 6 heures et la seconde, en 8 heures. À une certaine heure, on éteint les deux chandelles et on observe que la première est exactement deux fois plus courte que la seconde.

À quelle heure exactement, a-t-on éteint les deux chandelles ?

6. Deux cercles de rayon 8 sont à l'intérieur d'un demi-cercle de rayon 25. Ces deux cercles sont à la fois tangents au diamètre et à la circonférence du demi-cercle.

Quelle est la distance entre les deux centres des cercles intérieurs ?

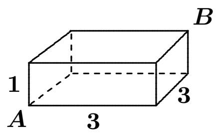


7. Évaluer le très long produit suivant :

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2003^2}\right).$$

(Indice : $\left(1-\frac{1}{n^2}\right) = \dfrac{n^2-1}{n^2} = \cdots$)

8. Un parallélépipède rectangle a une base 3×3 et une hauteur 1. Trouver la longueur minimale d'un chemin qu'une araignée pourrait suivre, le long de la surface, pour se rendre du sommet A au sommet opposé B.



Next we give the solutions to the 2005 BC Junior High School Mathematics Contests (Preliminary and Final rounds). [2005 : 261–270].

BC Colleges High School Mathematics Contest 2005
Junior Preliminary Round
Wednesday, March 2, 2005

1. A wire is cut into two parts in the ratio $3 : 2$. Each part is bent to form a square. The ratio of the perimeter of the larger square to the perimeter of the smaller square is :

(A) $3 : 2$ (B) $9 : 4$ (C) $5 : 3$ (D) $5 : 2$ (E) $12 : 5$

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Si le rapport de deux longueurs est de $3 : 2$, alors le rapport du périmètre des carrés formés à partir de ces longueurs sera également de $3 : 2$. La réponse est A.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

2. Given the following

I. even II. odd III. a perfect square IV. a multiple of 5

then it is true that the product $21 \times 35 \times 15$ is:

(A) II & IV (B) I & IV (C) II & III (D) III & I (E) II, III, & IV

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

We have $21 \times 35 \times 15 = 7 \times 3 \times 7 \times 5 \times 5 \times 3 = 7^2 \times 5^2 \times 3^2$. From this, we can conclude that the number is odd, the number is a multiple of 5, and the number is a perfect square. Hence, the answer is E.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

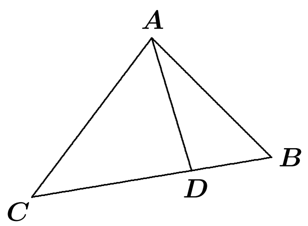
3. The radius of the largest sphere that can fit entirely inside a rectangular box with dimensions 5 cm $\times$7 cm $\times$11 cm is:

(A) 2 cm (B) $\frac{5}{2}$ cm (C) 3 cm (D) $\frac{23}{6}$ cm (E) $\frac{9}{2}$ cm

Identical solutions by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON; and Jean-François Désilets, student, Collège Montmorency, Laval, QC.

Since the smallest dimension of the box is 5 cm, the diameter of the sphere can only be a maximum of 5 cm. Therefore, the radius is 2.5 cm. The answer is B.

4. In the diagram, the area of the triangle ABC is 60. If DB is one third of CB, then the area of triangle ACD is:



(A) 20 (B) 30 (C) 40

(D) 45 (E) 50

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

L'aire d'un triangle est égal à $\frac{1}{2}b \cdot h$. Dans un produit, si on réduit un facteur d'un tiers, le produit est réduit d'un tiers. Ainsi, on a que l'aire du triangle ADB est $\frac{1}{3} \cdot 60 = 20$. La réponse est A.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

5. If $\dfrac{1}{n+5} = 4$, then $\dfrac{1}{n+6}$ equals:

(A) 3 (B) $\frac{1}{5}$ (C) $\frac{5}{4}$ (D) $\frac{4}{5}$ (E) None of these

Identical solutions by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON; and Jean-François Désilets, student, Collège Montmorency, Laval, QC.

From the given data, $n+5 = \dfrac{1}{4}$; thus, $n+6 = \dfrac{5}{4}$. Hence, $\dfrac{1}{n+6} = \dfrac{4}{5}$. The answer is D.

6. A standard 6-sided die is tossed twice. The probability of obtaining a sum of 5 is:

(A) $\frac{1}{12}$ (B) $\frac{1}{9}$ (C) $\frac{5}{36}$ (D) $\frac{1}{6}$ (E) $\frac{2}{9}$

I. *Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.*

Au premier lancer, il faut avoir un chiffre entre 1 et 4 pour que l'addition soit possible. Au deuxième lancer, il faut obtenir le chiffre qui additionné à notre premier chiffre donnera 5. Donc, $P = \frac{4}{6} \cdot \frac{1}{6} = \frac{4}{36} = \frac{1}{9}$. La réponse est B.

II. *Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.*

Since the 6-sided die is tossed twice, there are 36 possible outcomes. The only combinations which add up to 5 are $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$. Therefore, the probability of obtaining a sum of 5 is $\frac{4}{36} = \frac{1}{9}$. The answer is B.

7. A rectangle has dimensions 20 cm $\times$ 50 cm. If the length is increased by 20% and the width is decreased by 20%, then the change in the area is:

(A) an 8% increase (B) a 4% increase (C) a 0% increase

(D) a 4% decrease (E) an 8% decrease

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

Let us denote the length of the original rectangle by ℓ and the width by w. The area of the rectangle is ℓw. Increasing ℓ by 20% and decreasing w by 20%, we get 1.2ℓ and $0.8w$ for the new rectangle. Hence, the area of the new rectangle is $0.96\ell w$. Therefore, the area has decreased by $0.04\ell w$ or 4%. The answer is D.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

8. The greatest prime factor of 21831 is:

(A) 435 (B) 57 (C) 783 (D) 383 (E) 10917

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON, modified by the editor.

Let us take all the answers given and see if they are prime or not. The answer 435 is not prime because its last digit is 5, which means that it is divisible by 5. Answers 57, 783, and 10917 are not prime because $5+7 = 12$, $7 + 8 + 3 = 18$, and $1 + 0 + 9 + 1 + 7 = 18$, which means they are each divisible by 3. The only number left in the list is 383. The answer is D.

[*Ed:* While this argument shows that 383 is the only feasible answer, for completeness one should still verfiy that 383 is prime and is a divisor of 21831, both of which can be checked easily.]

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

9. The number of houses sold in Kamloops in 2004 is exactly 40% more than the number sold in 2003. Assuming that at least one house was sold in 2004, the smallest possible number of houses sold in Kamloops in 2004 is:

(A) 5 (B) 7 (C) 14 (D) 70 (E) 140

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Puisqu'on ne peut pas vendre de demi-maison, le plus petit nombre n pour lequel $0,4n$ est entier est 5. Pour savoir le nombre de maisons vendues en 2004, il suffit de faire $1,4 \times 5 = 7$. La réponse est B.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

10. The minimum number of students that must be in a room to ensure that at least 10 are boys or at least 10 are girls is:

(A) 10 (B) 11 (C) 18 (D) 19 (E) 20

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Il peut y avoir 9 filles et 9 garçons dans 18 étudiants. Le $19^{\text{ième}}$ fait qu'il doit y avoir au moins 10 garçons ou 10 filles. La réponse est D.

Une solution incorrecte a été soumise.

11. The number of integers that satisfy the inequality $\frac{3}{7} < \frac{n}{14} < \frac{2}{3}$ is:

(A) 0 (B) 2 (C) 3 (D) 4 (E) 5

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

The inequality can be written as $\frac{6}{14} < \frac{n}{14} < \frac{28}{42}$. Therefore we can easily conclude that n starts at 7. To find the maximum value for n, let us consider $n = 9$. We get $\frac{9}{14} = \frac{27}{42}$. Therefore, $n = 9$ is possible. For $n = 10$, we have $\frac{10}{14} = \frac{30}{42} > \frac{28}{42}$. Hence, $n = 10$ is not possible. Therefore, n can be 7, 8, or 9. The answer is C.

An incorrect solution was submitted.

12. Given that $20! = 20 \times 19 \times 18 \times \cdots \times 2 \times 1$ and 2^n is a factor of $20!$, then the largest possible value of n is:

(A) 10 (B) 12 (C) 18 (D) 20 (E) 24

Une solution identique soumise par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC; et Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

Pour trouver n, il suffit de regarder combien de fois 2 entre dans chaque facteur de 20!. Nous placerons dans le tableau les facteurs pairs de 20! (les seuls contenant des 2) et le nombre de deux qu'ils contiennent.

Facteur	2	4	6	8	10	12	14	16	18	20	Total
Nombre de deux	1	2	1	3	1	2	1	4	1	2	18

La réponse est C.

13. Terry has \$28.00 in nickels, dimes, and quarters. The value of the dimes is twice the value of the quarters, and it is half the value of the nickels. The total number of coins that Terry has is:

(A) 72 (B) 264 (C) 416 (D) 560 (E) 632

Official solution.

Let Q be the value of the quarters; then the value of the dimes is $2Q$, and, since this is half the value of the nickels, the value of the nickels is $4Q$. Hence, $Q + 2Q + 4Q = 2800$, which means that $Q = 400$. Thus, there must be \$4 in quarters, \$8 in dimes, and \$16 in nickels.

Finally, since there are 4 quarters per dollar, 10 dimes per dollar, and 20 nickels per dollar, the number of coins is $4 \times 4 + 10 \times 8 + 20 \times 16 = 416$. The answer is C.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC; and Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

14. The number 2005 can be written in the form $a^2 - b^2$, where a and b are integers that are greater than one, in exactly one way. The value of $a^2 + b^2$ is:

(A) 160825 (B) 160801 (C) 80418 (D) 80413 (E) 80406

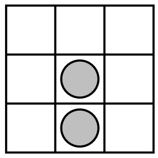
Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

We can write $a^2 - b^2 = (a + b)(a - b)$, and we can factor 2005 either as 2005×1 or as 401×5.

Suppose we let $a + b = 2005$ and $a - b = 1$. Solving these two equations, we get $a = 1003$ and $b = 1002$. However, this solution produces a value for $a^2 + b^2$ which is exceeds every possible solution in the list.

Now let $a + b = 401$ and $a - b = 5$. Then we get $a = 203$ and $b = 198$. Hence, $a^2 = 41209$ and $b^2 = 39204$. Thus, $a^2 + b^2 = 80413$. The answer is D.

15. The game of Solitaire JumpIt is played on a 3×3 grid with two identical game discs. If the two discs are adjacent horizontally, vertically, or diagonally, one disc can jump the other by moving onto the open space opposite the other disc. The disc that is jumped is removed. (See the diagram). The number of ways to place two identical game discs on the grid so that no jump is possible is:



(A) 16 (B) 20 (C) 24 (D) 32 (E) 40

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC, adapté par le rédacteur.

Nous allons identifier toutes les solutions possibles avec méthode. Nous évaluons toutes les possibilités pour chaque carré ayant un O dedans. Les carrés ayant des X représentent les possibilités pour le second jeton. De plus, les carrés ayant déjà contenu des O ne pourront plus être utilisés par la suite pour éviter le double comptage d'une solution déjà comptée.

O		X
		X
X	X	X

	X	X
O		X
	X	X

	X	X
		X
O		X

	X	X
		X
	O	

	X	X
		O

	X	
		O

Il y a vingt X dans les tableaux, donc 20 solutions possibles. La réponse est B.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

That brings us to the end of another issue. This month's winners of a past Volume of *Mathematical Mayhem* are Jean-François Désilets and Khartik Natarajan. Congratulations, Jean-François and Khartik! Please continue to send in your contests and solutions.