

THE OLYMPIAD CORNER

No. 251

R.E. Woodrow

Here it is—the start of another year and a new volume of *CRUX with MAYHEM*. It is appropriate to look back over the 2005 numbers of the *Corner* and thank all those who provided us with problems, comments, and solutions:

Mohammed Aassila	Bob Serkey
Arkady Alt	Achilleas Sinefakopoulos
Miguel Amengual Covas	Skotidas Sotirios
Michelle Bataille	Chris Small
Robert Bilinski	D.J. Smeenk
Pierre Bornsztein	Mike Spivey
Christopher J. Bradley	The Samford University
Yuming Chen	Problem Solving Group
José Luis Díaz-Barrero	Stan Wagon
George Evagelopoulos	Tracy Walker
Geoffrey A. Kandall	Edward T.H. Wang
Andy Liu	Kaiming Zhao
Bill Sands	Yufei Zhao
Toshio Seimiya	Titu Zvonaru

I also want to thank Joanne Canape (néé Longworth) who continues to work miracles with my scribbles, turning them into high-quality \LaTeX files.

As a first set of problems to warm up your problem solving capacity for 2006 we give the problems of the 2003 Vietnamese Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for collecting the problems for our use.

2003 VIETNAMESE MATHEMATICAL OLYMPIAD

1. Let ABC be a triangle inscribed in a circle with centre O , and let M and N be two points on the line AC such that $\overline{MN} = \overline{AC}$. Let D be the orthogonal projection of M onto the line BC , and E that of N onto AB .

- (i) Prove that the orthocentre H of triangle ABC lies on the circumcircle with centre O' of triangle BED .
- (ii) Prove that the mid-point of segment AN is symmetric to B with respect to the mid-point of segment OO' .

2. Let Γ_1 and Γ_2 be two circles in the plane tangent to each other at M . Let O_1 and O_2 be their respective centres, and let R_1 and R_2 be their respective radii. Suppose that $R_2 > R_1$. Let A be a point on Γ_2 which is not on the line O_1O_2 . Let AB and AC be the tangent lines to Γ_1 where B and C are the points of tangency. The lines MB and MC meet Γ_2 again at E and F , respectively. Let D be the point of intersection of the line EF and the tangent to Γ_2 at A . Prove that D moves on a fixed line as A moves along Γ_2 as long as the three points O_1 , O_2 , and A are not collinear.

3. Find all polynomials $P(x)$ with real coefficients, satisfying the relation

$$(x^3 + 3x^2 + 3x + 2)P(x - 1) = (x^3 - 3x^2 + 3x - 2)P(x)$$

for every real number x .

4. Let $P(x) = 4x^3 - 2x^2 - 15x + 9$ and $Q(x) = 12x^3 + 6x^2 - 7x + 1$.

(i) Prove that each of these polynomials has three distinct real roots.

(ii) Let α and β be the greatest roots of $P(x)$ and $Q(x)$, respectively. Prove that $\alpha^2 + 3\beta^2 = 4$.

5. Find the greatest positive integer n such that the following system of equations has an integral solution $(x, y_1, y_2, \dots, y_n)$:

$$\begin{aligned} (x + 1)^2 + y_1^2 &= (x + 2)^2 + y_2^2 = \dots \\ &= (x + k)^2 + y_k^2 = \dots = (x + n)^2 + y_n^2. \end{aligned}$$

6. Let f be a function defined on the set of real numbers \mathbb{R} , taking values in \mathbb{R} , and satisfying the condition $f(\cot x) = \sin 2x + \cos 2x$ for every x belonging to the open interval $(0, \pi)$. Find the least and the greatest values of the function $g(x) = f(x) \cdot f(1 - x)$ on the closed interval $[-1, 1]$.

7. Let α be a real number, $\alpha \neq 0$. Consider the sequence of real numbers $\{x_n\}$, $n = 1, 2, 3, \dots$, defined by $x_1 = 0$ and $x_{n+1}(x_n + \alpha) = \alpha + 1$ for $n = 1, 2, 3, \dots$.

(i) Find the general term of the sequence $\{x_n\}$.

(ii) Prove that the sequence $\{x_n\}$ has a finite limit when $n \rightarrow +\infty$. Find this limit.

8. Let \mathbb{R}^+ denote the set of all positive real numbers, and let F be the set of all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying the condition

$$f(3x) \geq f(f(2x)) + x$$

for every $x \in \mathbb{R}^+$. Find the greatest real number α such that $f(x) \geq \alpha x$ for all $f \in F$ and $x \in \mathbb{R}^+$.

9. Consider an integer $n > 1$. Colour all natural numbers red and blue so that the following conditions are simultaneously satisfied:

- (i) Every number is coloured red or blue, and there are infinitely many numbers coloured red and infinitely many coloured blue.
- (ii) The sum of n distinct red numbers is coloured red and the sum of n distinct blue numbers is coloured blue.

Is it possible to colour in such a manner when (a) $n = 2002$? (b) $n = 2003$?

10. For each integer $n > 1$, denote by s_n the number of permutations (a_1, a_2, \dots, a_n) of the first n positive integers such that each permutation satisfies the condition $1 \leq |a_k - k| \leq 2$ for $k = 1, 2, \dots, n$. Prove that $1.75 \cdot s_{n-1} < s_n < 2 \cdot s_{n-1}$ for all integers $n > 6$.

Next we give problems of the XXIX Russian Mathematical Olympiad V (Final) Round. Thanks again go to Andy Liu for collecting the set.

XXIX RUSSIAN MATHEMATICAL OLYMPIAD

V (Final) Round — 10th Form

1. (N. Agakhanov) Let M be a set containing 2003 different positive real numbers, such that for any 3 different elements a, b, c from M the number $a^2 + bc$ is rational. Prove that it is possible to choose a natural number n such that for each a from M the number $a\sqrt{n}$ is rational.

2. (S. Berlov) Diagonals of the inscribed quadrilateral $ABCD$ intersect at point O . Let S_1 and S_2 be the circumcircles of triangles ABO and CDO , respectively, and let K be the second point of intersection of S_1 and S_2 . Straight lines passing through O parallel to AB and CD intersect S_1 and S_2 again at points L and M , respectively. Points P and Q are chosen on segments OL and OM , respectively, so that $OP : PL = MQ : QO$. Prove that points O, K, P , and Q lie on the same circle.

3. (V. Dolnikov) A tree is given on $n \geq 2$ vertices (that is, a graph on n vertices and $n - 1$ edges in which it is possible to pass from one vertex to any other vertex by edges, and there is no cyclical path passing through edges). Numbers x_1, x_2, \dots, x_n are placed on the tree's vertices and the product of numbers at the ends of an edge is written on that edge. Let S be the sum of numbers on all the edges. Prove that

$$\sqrt{n-1}(x_1^2 + x_2^2 + \dots + x_n^2) \geq 2S.$$

4. (V. Dolnikov, R. Karasev) Let X be a finite set of points in the plane, and let T be an equilateral triangle in the same plane. Suppose that, if X' is any subset of X consisting of no more than 9 points, then X' can be covered by two translations of the triangle T . Prove that the entire set X can be covered by two translations of T .

5. (O. Podlipsky) There are N cities in a country. Between any two cities there is either a highway or a railroad. A tourist wants to travel around the country, visiting each city exactly once, and to return to the city where he started his journey. Prove that the tourist can choose a city with which to begin his journey, as well as the path, in such a way that he will have to change the type of transportation no more than once.

6. (A. Khrabrov) Starting with some natural number a_0 , the sequence of natural numbers a_n is created in the following way: $a_{n+1} = a_n/5$, if a_n is divisible by 5, and $a_{n+1} = \lfloor \sqrt{5}a_n \rfloor$ if a_n is not divisible by 5. Prove that, for some number N , the sequence $\{a_n\}_{n \geq N}$ is increasing.

7. (P. Kozhevnikov) In triangle ABC let O and I be the circumcentre and incentre, respectively. Let ω_a be the excircle which touches the extensions of sides AB and AC at points K and M , respectively, and touches side BC at point N . If the mid-point P of the segment KM lies on the circumcircle of triangle ABC , prove that O , N , and I are collinear.

8. (D. Khramtsov) Find the greatest natural number N such that, for any arrangement of the natural numbers from 1 to 400 in the cells of a square table of size 20×20 , there can be found two numbers located in the same row or in the same column, the difference of which is not less than N .

11th Form

1. (N. Agakhanov, A. Golovanov, V. Senderov) Let α , β , γ , and τ be positive numbers such that, for all x ,

$$\sin \alpha x + \sin \beta x = \sin \gamma x + \sin \tau x.$$

Prove that $\alpha = \gamma$ or $\alpha = \tau$.

2. [Ed: Same problem as #2 in the previous set.]

3. (A. Khrabrov) Let $f(x)$ and $g(x)$ be polynomials with non-negative integer coefficients, and let m be the greatest coefficient of $f(x)$. Suppose that there are natural numbers $a < b$ such that $f(a) = g(a)$ and $f(b) = g(b)$. Prove that if $b > m$, then $f(x)$ and $g(x)$ are the same polynomial.

4. (E. Cherepanov) Originally, Anna and Boris each had a long sheet of paper. The letter A was written on one sheet of paper, and the letter B on the other. Every minute either Anna or Boris, not necessarily taking turns, adds (on the right or the left) to the word on his or her sheet of paper a word from the other sheet of paper. Prove that in 24 hours the word from Anna's sheet of paper could be cut into two parts which could be switched in such a way as to obtain the same word written backwards.

- 5.** (N. Agakhanov) The lengths of the sides of a triangle are roots of a cubic equation with rational coefficients. Prove that the lengths of the altitudes of the triangle are roots of an equation of degree 5 with rational coefficients.
- 6.** (S. Berlov) Is it possible to arrange natural numbers in the cells of an infinite checkerboard in such a way that for any natural numbers $m > 100$ and $n > 100$ the sum of the numbers in any $m \times n$ rectangle is divisible by $m + n$?
- 7.** (I. Ivanov) There are 100 cities in a country; some pairs of cities are connected by roads. For each set of four cities there are at least two roads between them. Suppose that there is no path that passes through each city exactly once. Prove that one could choose two cities in such a way that each of the remaining cities would be connected by a road with at least one of the two chosen cities.
- 8.** (F. Bakharev) A sphere inscribed in the tetrahedron $ABCD$ touches its four faces ABC , ABD , ACD , and BCD at points D_1 , C_1 , B_1 , and A_1 , respectively. Consider the plane equidistant from point A and from the plane $B_1C_1D_1$, and the three planes analogous to it. Prove that the tetrahedron formed by these four planes has the same centre for the circumscribed sphere as the tetrahedron $ABCD$.

Next we turn to solutions from our readers to problems of the XXXVI Spanish Mathematical Olympiad National Round given [2004 : 202–203].

- 1.** Let $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ and $Q(x) = x^4 + cx^3 + bx^2 + ax + 1$, with a, b, c real numbers and $a \neq c$. Find conditions on a, b , and c so that $P(x)$ and $Q(x)$ have two common roots. In this case, solve the equations $P(x) = 0, Q(x) = 0$.

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Maisons-Laffitte, France; Robert Bilinski, Collège Montmorency, Laval, QC; Christopher J. Bradley, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Skotidas Sotirios, Karditso, Greece. We give the write-up by Díaz-Barrero.

The common roots of $P(x)$ and $Q(x)$ are among the roots of the polynomial

$$P(x) - Q(x) = (a - c)x(x^2 - 1).$$

Thus, they are among $-1, 0$, and 1 . Since $P(0) = Q(0) = 1$, the common roots must be -1 and 1 . Substituting these values of x into the equations $P(x) = 0$ and $Q(x) = 0$, we get the conditions $a + b + c + 2 = 0$ and $a - b + c - 2 = 0$, from which we get $b = -2$ and $a + c = 0$.

When these conditions are satisfied, $P(x)$ and $Q(x)$ can be written as

$$P(x) = x^4 + ax^3 - 2x^2 - ax + 1 = (x^2 - 1)(x^2 + ax - 1),$$

$$Q(x) = x^4 - ax^3 - 2x^2 + ax + 1 = (x^2 - 1)(x^2 - ax - 1).$$

Thus, the roots of $P(x) = 0$ are

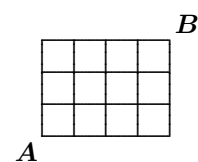
$$-1, \quad 1, \quad \frac{-a - \sqrt{a^2 + 4}}{2}, \quad \text{and} \quad \frac{-a + \sqrt{a^2 + 4}}{2},$$

and the roots of $Q(x) = 0$ are

$$-1, \quad 1, \quad \frac{a - \sqrt{a^2 + 4}}{2}, \quad \text{and} \quad \frac{a + \sqrt{a^2 + 4}}{2}.$$

2. The figure shows a street plan of twelve square blocks.

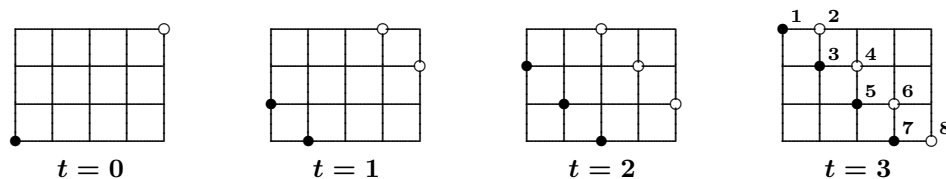
A person P goes from point A to point B , and a second person Q goes from B to A . Both of them (P and Q) leave at the same time with the same speed, following shortest paths on the grid. At each corner they choose among the possible streets with equal probability. What is the probability that P meets Q ?



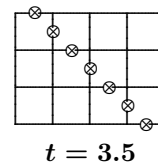
Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornshtein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give the solution by Bilinski.

We understand “following shortest paths on the grid” to mean that P can only go up or to the right and Q can only go down or to the left, so that each finishes the trip after travelling 7 blocks in total (4 horizontal and 3 vertical).

Let us equate time with the number of blocks covered. We illustrate the possible positions for both P (black circle) and Q (white circle) between times $t = 0$ (the start) and $t = 3$. The eight vertices containing the white or black circles at time $t = 3$ have been numbered for future reference.



All 7 potential meetings take place at time $t = 3.5$, at positions shown in the diagram to the right by crossed white circles. For $1 \leq i \leq 8$, let P_i denote the probability that P is at vertex i at time $t = 3$, and define Q_i similarly. There is only one way P could travel to reach vertex 1 or vertex 7, and there are three possible ways P could reach vertex 3 or vertex 5. Thus, we see that $P_1 = P_7 = \frac{1}{8}$ and $P_3 = P_5 = \frac{3}{8}$. Similarly, $Q_2 = Q_8 = \frac{1}{8}$ and $Q_4 = Q_6 = \frac{3}{8}$.



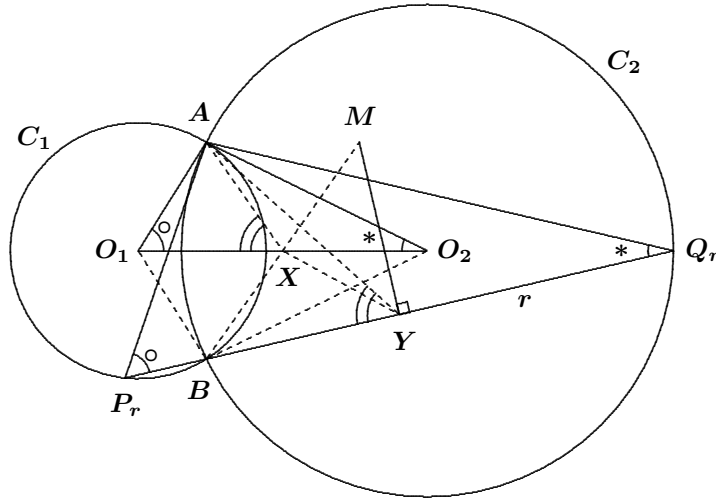
For $1 \leq i \leq 8$ and $1 \leq j \leq 8$, let P_{ij} denote the probability that P moves from vertex i at time $t = 3$ to vertex j at time $t = 4$, and define Q_{ij} similarly. At vertices 3, 5, and 7, the probabilities of P going up or across are the same, whereas P can only go across from vertex 1. It follows that $P_{12} = \frac{1}{8}$, $P_{32} = P_{34} = P_{54} = P_{56} = \frac{1}{2} \cdot \frac{3}{8} = \frac{3}{16}$, and $P_{76} = P_{78} = \frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$. By a similar argument, we obtain $Q_{87} = \frac{1}{8}$, $Q_{67} = Q_{65} = Q_{45} = Q_{43} = \frac{3}{16}$, and $Q_{23} = Q_{21} = \frac{1}{16}$.

Then the probability that P meets Q is:

$$\begin{aligned} p &= P_{12} \cdot Q_{21} + P_{32} \cdot Q_{23} + P_{34} \cdot Q_{43} + \cdots + P_{78} \cdot Q_{87} \\ &= \frac{1}{8} \cdot \frac{1}{16} + \frac{3}{16} \cdot \frac{1}{16} + 3 \cdot \frac{3}{16} \cdot \frac{3}{16} + \frac{1}{16} \cdot \frac{3}{16} + \frac{1}{16} \cdot \frac{1}{8} \\ &= \frac{1}{128} + \frac{3}{256} + 3 \cdot \frac{9}{256} + \frac{3}{256} + \frac{1}{128} = \frac{37}{256} \approx 14.45\%. \end{aligned}$$

3. Circles C_1 and C_2 intersect at points A and B . A line r through B intersects C_1 and C_2 again at points P_r and Q_r , respectively. Prove that there is a point M , which depends only on C_1 and C_2 , such that the perpendicular bisector of $P_r Q_r$ passes through M .

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; Toshio Seimiya, Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's solution.



Let O_1 and O_2 be the centres of C_1 and C_2 , respectively. Then O_1O_2 is the perpendicular bisector of AB . We have $\triangle AO_1O_2 \sim \triangle AP_rQ_r$, since

$$\begin{aligned} \angle AO_1O_2 &= \frac{1}{2} \angle AO_1B = \angle AP_rB = \angle AP_rQ_r \\ \text{and } \angle AO_2O_1 &= \frac{1}{2} \angle AO_2B = \angle AQ_rB = \angle AQ_rP_r. \end{aligned}$$

Let X and Y be the mid-points of O_1O_2 and P_rQ_r , respectively. Since O_1O_2 and P_rQ_r are corresponding sides in the similar triangles $\triangle AO_1O_2$ and $\triangle AP_rQ_r$, we get $\triangle AXO_1 \sim \triangle AYP_r$, and hence, $\angle AXO_1 = \angle AYP_r$.

Thus, $\angle AYB = \angle AYP_r = \angle AXO_1 = \frac{1}{2}\angle AXB$. Since $AX = BX$ and $\angle AXB = 2\angle AYB$, we see that X is the circumcentre of $\triangle AYB$. Then $AX = BX = YX$.

Let M be the reflection of B with respect to X . Then X is the midpoint of BM . Since $MX = BX = YX$, we have $\angle BYM = 90^\circ$. Thus, the perpendicular bisector of P_rQ_r passes through the fixed point M .

4. For any integer x , let $\lfloor x \rfloor$ denote the integer part of x . Find the largest integer N satisfying the following conditions:

- (a) $\lfloor \frac{N}{3} \rfloor$ has three identical digits, and
 (b) $\lfloor \frac{N}{3} \rfloor$ is the sum of n consecutive positive integers starting at 1; that is, there is a positive integer n such that

$$\left\lfloor \frac{N}{3} \right\rfloor = 1 + 2 + \cdots + (n-1) + n.$$

Solved by Pierre Bornsztejn, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

The answer is $N = 2000$. Let $N = 3k + r$ where $k, r \in \mathbb{N} \cup \{0\}$ and $0 \leq r \leq 2$. From condition (a) we have $k = \lfloor \frac{N}{3} \rfloor = a \cdot 111$, where a is a decimal digit different from 0. From condition (b) we get $k = n(n+1)/2$ for some $n \in \mathbb{N}$. Since $k \leq 999$, we have $n^2 < n(n+1) = 2k \leq 1998$, which implies that $n < \sqrt{1998} \approx 44.69$. Hence, $n \leq 44$.

Direct computations show that when $n = 44, 43, 42, 41, 40, 39, 38$, and 37 , the corresponding values of k are 990, 946, 903, 861, 820, 780, 741, and 703, none of which has three identical digits, while for $n = 36$ we have $k = 666$. Therefore, it follows that $N = 3 \times 666 + 2 = 2000$.

5. Four points are placed in a square of side 1. Show that the distance between some two of them is less than or equal to 1.

Solution by Pierre Bornsztejn, Maisons-Laffitte, France.

First note that the maximum distance between two points in a square (in its interior or on its boundary) is $\sqrt{2}$.

Let A, B, C , and D be four points placed in a square of side 1. Let \mathcal{C} be the convex hull of $\{A, B, C, D\}$. Clearly, \mathcal{C} is contained in the square.

Case 1. \mathcal{C} is a line segment.

Without loss of generality, we may assume that A, B, C , and D are collinear in that order. Thus, $AB + BC + CD = AD \leq \sqrt{2}$, so that $\min\{AB, BC, CD\} \leq \frac{\sqrt{2}}{3} < 1$, and we are done with this case.

Case 2. \mathcal{C} is a triangle.

Without loss of generality, we may assume that D is inside or on the boundary of ABC , and that $\theta = \angle CDA = \max\{\angle ADB, \angle BDC, \angle CDA\}$.

Since $\angle ADB + \angle BDC + \angle CDA = 2\pi$, we see that $\theta \geq \frac{2\pi}{3}$ and $\cos \theta \leq -\frac{1}{2}$. If AD and DC are greater than 1, then, from the Law of Cosines,

$$AC^2 = AD^2 + DC^2 - 2AD \cdot DC \cos \theta \geq 1 + 1 + AD \cdot DC > 3,$$

which leads to $AC > \sqrt{3} > \sqrt{2}$, a contradiction.

Then $AD \leq 1$ or $DC \leq 1$, and we are done again. In fact, the above argument can easily be strengthened to show that $AD < 1$ or $CD < 1$.

Case 3. C is a (convex) quadrilateral, say $ABCD$.

Since the convex quadrilateral $ABCD$ is contained in the square, it follows that its perimeter is not greater than the perimeter of the square; that is, $AB + BC + CD + DA \leq 4$. Therefore, $\min\{AB, BC, CD, DA\} \leq 1$, and we are done.

We note that equality occurs if and only if $ABCD$ is the square itself.

6. Show that there is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(f(n)) = n + 1$.

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Maisons-Laffitte, France; Skotidas Sotirios, Karditso, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

Suppose for the purpose of contradiction that such a function exists. Let m be the natural number defined by $f(0) = m$. Then $f(k) = m + k$ for $k = 0$. Let $k \geq 0$ be any natural number such that $f(k) = m + k$. Then $f(m + k) = f(f(k)) = k + 1$. Thus, $f(k + 1) = f(f(m + k)) = m + k + 1$. It follows by induction that $f(k) = m + k$ for all $k \in \mathbb{N}$. However, this yields $f(m) = 2m$, while $f(0) = m$ yields $f(m) = f(f(0)) = 1$, a contradiction.

Note: Bornshtein points out that the well-known problem #4 of the 1987 IMO was to prove that there is no function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(f(n)) = n + 1987$. Any solution of this problem contains a remark that the result holds just because 1987 is odd.

Next we give a solution to one of the problems of the Taiwan (ROC) Mathematical Olympiad 2000 given [2004 : 203].

2. In an acute triangle ABC with $|AC| > |BC|$, let M be the mid-point of AB . Let AP be the altitude from A and BQ be the altitude from B . These altitudes meet at H , and the lines AB and PQ meet at R . Prove that the two lines RH and CM are perpendicular.

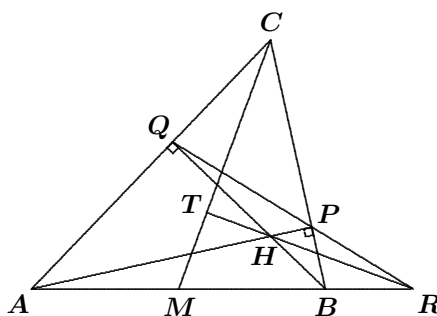
Solved by Christopher J. Bradley, Bristol, UK; and D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution by Bradley.

We will use vectors with the circumcentre O as origin and $\mathbf{x} = \overrightarrow{OA}$, $\mathbf{y} = \overrightarrow{OB}$, $\mathbf{z} = \overrightarrow{OC}$. It is known that QP meets AB at R , where

$$\overrightarrow{OR} = \frac{-(a \cos B)x + (b \cos A)y}{b \cos A - a \cos B}$$

and $\overrightarrow{OH} = x + y + z$. We also have $\overrightarrow{CM} = \frac{1}{2}(x + y) - z$. Hence,

$$\begin{aligned} \overrightarrow{RH}(b \cos A - a \cos B) \\ = (b \cos A)x - (a \cos B)x \\ + (b \cos A - a \cos B)z. \end{aligned}$$



Thus, if ρ is the radius of the circumcircle of $\triangle ABC$, we get

$$\begin{aligned} 2(b \cos A - a \cos B)\overrightarrow{RH} \cdot \overrightarrow{CM} \\ = (-b \cos A + a \cos B)\rho^2 + (b \cos A - a \cos B)x \cdot y \\ + (b \cos A + a \cos B)(y - x) \cdot z \\ = \rho^2(a \cos B - b \cos A)(1 - \cos 2C) + c(\cos 2A - \cos 2B) \\ = 2\rho^2 c(a \cos B - b \cos A) \sin C((\sin A \cos B - \sin B \cos A) \\ + \sin(B - A)) \\ = 0. \end{aligned}$$

Hence, $RH \perp CM$.

Next we look at solutions to problems of the 2000 Hungarian National Olympiad given [2004 : 204].

First Round

1. Let x , y , and z denote positive real numbers, each less than 4. Prove that at least one of the numbers $\frac{1}{x} + \frac{1}{4-y}$, $\frac{1}{y} + \frac{1}{4-z}$, and $\frac{1}{z} + \frac{1}{4-x}$ is greater than or equal to 1.

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's version.

From the HM-AM Inequality, we have

$$\frac{1}{2} \left(\frac{1}{x} + \frac{1}{4-x} \right) \geq \frac{2}{x+4-x} = \frac{1}{2};$$

whence, $\frac{1}{x} + \frac{1}{4-x} \geq 1$. Now, if the three numbers

$$\frac{1}{x} + \frac{1}{4-y}, \quad \frac{1}{y} + \frac{1}{4-z}, \quad \text{and} \quad \frac{1}{z} + \frac{1}{4-x}$$

were all less than 1, their sum S would be less than 3. However,

$$S = \left(\frac{1}{x} + \frac{1}{4-x}\right) + \left(\frac{1}{y} + \frac{1}{4-y}\right) + \left(\frac{1}{z} + \frac{1}{4-z}\right) \geq 3.$$

This contradiction proves the requested result.

2. Find the integer solutions of $5x^2 - 14y^2 = 11z^2$.

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

Clearly, $x = y = z = 0$ is a solution. We show that this is the only solution.

If $x = 0$, then $y = z = 0$. Next note that x and z must have the same parity. Let us now work modulo 8. If x and z are both odd, then $x^2 \equiv z^2 \equiv 1$, which implies that $5x^2 - 11z^2 \equiv 2$. Since $14y^2 \equiv 14 \equiv 6$ if y is odd and $14y^2 \equiv 0$ if y is even, we have a contradiction. Thus, x and z are both even.

Suppose there are solutions in which $x \neq 0$. Let x_0 denote the least positive integer for which there exist $y, z \in \mathbb{Z}$ such that (x_0, y, z) is a solution. Setting $x_0 = 2x_1, z = 2z_1$, we get $20x_1^2 - 14y^2 = 44z_1^2$, or $10x_1^2 - 7y^2 = 22z_1^2$, which implies that y is even. Setting $y = 2y_1$, we then have $10x_1^2 - 28y_1^2 = 22z_1^2$, or $5x_1^2 - 14y_1^2 = 11z_1^2$, showing that (x_1, y_1, z_1) is also a solution. This is a contradiction, since $0 < x_1 < x_0$.

4. If $1 \leq m \leq n$, prove that m is a divisor of

$$n \left(\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^{m-1} \binom{n}{m-1} \right).$$

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

Substituting $\binom{n-1}{0}$ for $\binom{n}{0}$ and using systematically the law of formation of Pascal's Triangle, we easily see that

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^{m-1} \binom{n}{m-1} = (-1)^{m-1} \binom{n-1}{m-1}.$$

Thus,

$$\begin{aligned} n \left(\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^{m-1} \binom{n}{m-1} \right) \\ = (-1)^{m-1} n \binom{n-1}{m-1} = (-1)^{m-1} m \binom{n}{m}. \end{aligned}$$

The result follows.

Final Round

1. Let c denote a positive integer, and let c_1 , c_3 , c_7 , and c_9 be the number of divisors of c which have last digit 1, 3, 7, and 9, respectively (in the decimal system). Prove that $c_3 + c_7 \leq c_1 + c_9$.

Solution by Pierre Bornsstein, Maisons-Laffitte, France.

The result is clear for $c = 1$. Thus, we assume that $c \geq 2$. Let

$$c = p_1^{\alpha_1} \cdots p_k^{\alpha_k} \quad (1)$$

be the prime factorization of c . Since we are interested only in the divisors of n which are odd and not divisible by 5, we may assume that $\gcd(c, 10) = 1$.

Let A be the set of divisors of c which have last digit 3 or 7, and let B be the set of divisors of c which have last digit 1 or 9. Thus, $|A| = c_3 + c_7$ and $|B| = c_1 + c_9$.

Let a be the number of primes which appear in (1), counted with their multiplicities, and which belong to A , and let b be the number of such primes (with multiplicity) belonging to B .

It is easy to verify that:

- The product of any two elements from A belongs to B .
- The product of any two elements from B belongs to B .
- The product of an element from A and an element from B belongs to A .

It follows that $d \in A$ if and only if the prime decomposition of d contains an odd number of primes which belong to A ; that is, $d = D_A \times D_B$, where D_A is the product of an odd number of primes which belong to A (so that there are exactly $(a+1)/2$ choices for D_A) and D_B is the product of an arbitrary number of primes which belong to B (so that there are exactly $b+1$ choices for D_B). Therefore, $|A| = (b+1) \lfloor \frac{a+1}{2} \rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part.

In the same way, we have $|B| = (b+1) \lfloor \frac{a+2}{2} \rfloor$. Then $|A| \leq |B|$ and we are done.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.