EDITORIAL

Jim Totten

Over the years, ***Crux Mathematicorum*** and ***CRUX with MAYHEM*** have been extremely fortunate to have found many people who have made and continue to make regular and significant contributions. Everyone is aware of the contributions of the editorial board—their names appear on the inside cover of each issue. However, there is another set of individuals without whose participation this publication would not be viable. I am speaking, of course, of our problem posers, problem solvers, and the writers of our expository articles.

We have decided that recognition of these contributors is long overdue. We are inaugurating a "Contributor Profiles" section. It will appear on occasion starting with this issue. In each profile we will identify one long-time regular contributor and present a short biography of that individual, together with a photograph, if possible. That way, all our readers will be able to find out more about those whose problems, solutions, or articles have captured our imagination over the years.

There is already quite an impressive list of contributors from whom to choose. We have gathered biographical information on some contributors we wish to feature. Next comes the difficult task of choosing the order in which they should appear. There is no right way to do this. We have simply made an editorial decision to feature K.R.S. Sastry in the first profile.

I hope you will enjoy finding out more about Sastry and about others whose names are familiar to readers of ***CRUX with MAYHEM***.

On a separate matter entirely, those who access ***CRUX with MAYHEM*** on the web will notice that, as of this issue, the full Mayhem section will be available to the public, instead of just the Mayhem Problems. Please let your colleagues who do not frequent our website know about this. We are hoping that by making Mayhem more readily available, we will entice more solvers, particularly high school students and their teachers, to submit solutions to the Mayhem problems.

Mayhem readers may have noticed that the column *Pólya's Paragon* has gradually moved away from its original mandate, which was to explore problem solving. We have decided to reverse this trend. As a result, there will be no *Pólya's Paragon* in this issue, but it will return in March. It will be authored by guest columnists and will likely not appear in every issue. If you have an interesting solution to a problem which you feel will add to others' repertoire of solution strategies, please forward your idea for a column in *Polya's Paragon* to the Mayhem Editor.

We welcome your feedback on all the changes we are implementing with this issue of ***CRUX with MAYHEM***.

Contributor Profiles:
K.R.S. Sastry

Shankaranarayana Sastry grew up in a joint family environment in Dodballapur in the state of Karnataka, India. His uncle wanted him to become an astrologer partly because their ancestors were members of the *Panchanga* committee. This membership was by invitation of the palace of the (then) Maharaja of Mysore. The duties of members included the preparation of *Panchanga*, the Hindu calendar of sacred events, which contains the daily descriptions of positions of the heavenly objects for the following year. In addition they were expected to predict their likely effects on India in general and the (then) Kingdom of Mysore in particular, and to suggest possible precautions and remedies.

His father, a teacher of languages, wanted him to become a Sanskrit Scholar, which (incidentally) is a necessary condition to be a worthy successor to his forefathers. However, Sastry was not interested in either of these professions. Intuitively, he felt that his interests lay elsewhere. He went on to earn his B.Sc. (Hons.) and M.Sc. degrees in mathematics from the University of Mysore.

From his school days, Sastry was interested in mathematics. The reason for this interest, strangely enough, was not to acquire more mathematical knowledge for himself, but rather to assist his weaker classmates (without compensation—a family tradition).

He taught mathematics in India for some time, and thereafter mostly in Ethiopia. If a student approached him with a mathematical problem, he used to "force" its solution from the student's mind. The Ethiopian government selected him teacher of the year for 1971, and that provided him with an opportunity to meet with the late Emperor Haile Selassie. Whenever he identified a mathematically talented student, he encouraged the student to solve a problem in more than one way. He believes that $n + 1$ solutions of a problem are better than n solutions, because each solution illuminates a different aspect of the problem. He has demonstrated this truth on a number of occasions—in recent years, by presenting a number of descriptions of Heron triangles and Brahmagupta quadrilaterals.

On a lighter note, a former Editor-in-Chief of ***Crux Mathematicorum***, Bill Sands, tells how the Ethiopians used to relate Sastry's mother tongue *Kannada* to CRUX's motherland *Canada* [1995 : 305].

Sastry is now retired and resides in a home for seniors in Bangalore, India. In this issue, Sastry has supplied us with a new problem in the Problems Section, #3101, and a new Mayhem problem, M228. In adition, solutions to two of his proposals to Mayhem are solved in this issue.

SKOLIAD No. 91

Robert Bilinski

Please send your solutions to the problems in this edition by ***1 August, 2006***. A copy of MATHEMATICAL MAYHEM Vol. 2 will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our problems come from a contest organized by Collège Montmorency for the secondary school boards in the Laval region of Quebec.

Je voudrais remercier André Labelle du Collège Montmorency pour nous avoir fourni gracieusement une copie de ce concours.

Montmorency Contest 2003–04

Grade 11, November 2003

1. A magician says he can quickly calculate the square of any number between 50 and 59 inclusive.

You tell him to calculate 57^2.

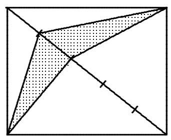
He answers: "Abracadabra! $5^2 = 25$ and $25 + 7 = 32$".

He continues with: "Abracadabra! $7^2 = 49$" (Hint: if the second number had been a 2, then we would have $2^2 = 04$).

He ends with: "The square you seek is 3249!".

He is right! Prove it algebraically.

2. Consider a rectangle with width 8 and length 10. Cutting one of its diagonals in five equal parts, calculate the area of the shaded region.

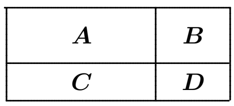


3. (a) Show that, for any pair of real numbers $a > 0$ and $b > 0$, we have $\frac{a}{b} + \frac{b}{a} \geq 2$.

(b) Deduce, from the result of part (a), that for $x > 0$, $y > 0$, and $z > 0$, we always have $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) > 8$.

4. Given a rectangular plot having area 4000 m^2. Using two straight line cuts parallel to the sides of the plot, we want to cut it into four small rectangular plots A, B, C, and D.

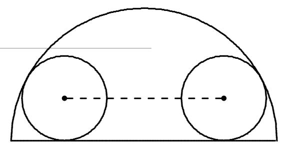
Is it possible to do it in such a way that the area of $A = 2000$ m^2, the area of $B = 1000$ m^2, the area of $C = 600$ m^2, and the area of $D = 400$ m^2?



5. During a power outage, two candles of the same length are lit at 6:00 pm. The first candle takes 6 hours to burn completely, and the second takes 8 hours. At a certain time both candles were extinguished and it was observed that the first one was exactly half as long as the second. What was the exact time when this happened?

6. Two circles of radius 8 are placed inside a semicircle of radius 25. The two circles are each tangent to the diameter and to the semicircle.

What is the distance between the centres of the two circles?

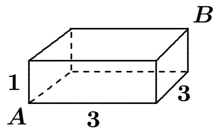


7. Evaluate the very long product that follows:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2003^2}\right).$$

(Hint: $\left(1-\frac{1}{n^2}\right)=\frac{n^2-1}{n^2}=\cdots$)

8. A rectangular box has a base of 3×3 and a height of 1. Find the minimal length of the path a spider could follow, along the surface, to get from corner A to the opposite corner B.



Concours Montmorency 2003-04

Sec V, novembre 2003

1. Un magicien vous propose de calculer le carré de n'importe quel nombre entre 50 et 59 inclusivement.

Vous lui proposez de calculer 57^2.

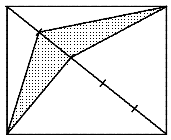
Il répond : «Abracadabra : $5^2 = 25$ et $25 + 7 = 32$».

Il continue : «Abracadabra : $7^2 = 49$». (Rem : si le deuxième chiffre avait été 2, il aurait écrit $2^2 = 04$).

Il ajoute finalement : «Le carré est 3249».

Il a raison ! Justifiez-le algébriquement.

2. Considérons un rectangle de largeur 8 et de longueur 10. En séparant la diagonale en cinq parties égales, calculer l'aire de la zone hachurée.



3. (a) Montrer que, pour toute paire de nombres réels $a > 0$ et $b > 0$, on a : $\frac{a}{b}+\frac{b}{a}\geq 2$.

(b) Déduire, grâce au résultat obtenu en (a), que pour $x > 0$, $y > 0$ et $z > 0$, on a toujours $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) > 8$.

4. On veut partager un terrain rectangulaire de 4000 m^2 à l'aide de deux lignes droites parallèles aux côtés, en quatre petits terrains rectangulaires A, B, C et D.

Est-il possible de le faire de telle manière que l'aire de $A = 2000$ m^2, l'aire de $B = 1000$ m^2, l'aire de $C = 600$ m^2 et l'aire de $D = 400$ m^2 ?

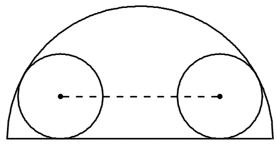
A	B
C	D

5. Durant une panne d'électricité, deux chandelles de même longueur sont allumées à 18 :00 h. La première chandelle se consume en 6 heures et la seconde, en 8 heures. À une certaine heure, on éteint les deux chandelles et on observe que la première est exactement deux fois plus courte que la seconde.

À quelle heure exactement, a-t-on éteint les deux chandelles ?

6. Deux cercles de rayon 8 sont à l'intérieur d'un demi-cercle de rayon 25. Ces deux cercles sont à la fois tangents au diamètre et à la circonférence du demi-cercle.

Quelle est la distance entre les deux centres des cercles intérieurs ?

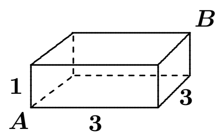


7. Évaluer le très long produit suivant :

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{2003^2}\right).$$

(Indice : $\left(1-\frac{1}{n^2}\right) = \frac{n^2-1}{n^2} = \cdots$)

8. Un parallélépipède rectangle a une base 3×3 et une hauteur 1. Trouver la longueur minimale d'un chemin qu'une araignée pourrait suivre, le long de la surface, pour se rendre du sommet A au sommet opposé B.



Next we give the solutions to the 2005 BC Junior High School Mathematics Contests (Preliminary and Final rounds). [2005 : 261–270].

BC Colleges High School Mathematics Contest 2005
Junior Preliminary Round
Wednesday, March 2, 2005

1. A wire is cut into two parts in the ratio $3 : 2$. Each part is bent to form a square. The ratio of the perimeter of the larger square to the perimeter of the smaller square is :

(A) $3 : 2$ (B) $9 : 4$ (C) $5 : 3$ (D) $5 : 2$ (E) $12 : 5$

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Si le rapport de deux longueurs est de $3 : 2$, alors le rapport du périmètre des carrés formés à partir de ces longueurs sera également de $3 : 2$. La réponse est A.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

2. Given the following

I. even II. odd III. a perfect square IV. a multiple of 5

then it is true that the product $21 \times 35 \times 15$ is:

(A) II & IV (B) I & IV (C) II & III (D) III & I (E) II, III, & IV

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

We have $21 \times 35 \times 15 = 7 \times 3 \times 7 \times 5 \times 5 \times 3 = 7^2 \times 5^2 \times 3^2$. From this, we can conclude that the number is odd, the number is a multiple of 5, and the number is a perfect square. Hence, the answer is E.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

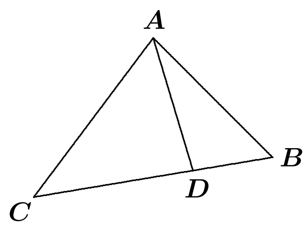
3. The radius of the largest sphere that can fit entirely inside a rectangular box with dimensions 5 cm $\times$7 cm $\times$11 cm is:

(A) 2 cm (B) $\frac{5}{2}$ cm (C) 3 cm (D) $\frac{23}{6}$ cm (E) $\frac{9}{2}$ cm

Identical solutions by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON; and Jean-François Désilets, student, Collège Montmorency, Laval, QC.

Since the smallest dimension of the box is 5 cm, the diameter of the sphere can only be a maximum of 5 cm. Therefore, the radius is 2.5 cm. The answer is B.

4. In the diagram, the area of the triangle ABC is 60. If DB is one third of CB, then the area of triangle ACD is:



(A) 20 (B) 30 (C) 40

(D) 45 (E) 50

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

L'aire d'un triangle est égal à $\frac{1}{2}b \cdot h$. Dans un produit, si on réduit un facteur d'un tiers, le produit est réduit d'un tiers. Ainsi, on a que l'aire du triangle ADB est $\frac{1}{3} \cdot 60 = 20$. La réponse est A.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

5. If $\dfrac{1}{n+5} = 4$, then $\dfrac{1}{n+6}$ equals:

(A) 3 (B) $\frac{1}{5}$ (C) $\frac{5}{4}$ (D) $\frac{4}{5}$ (E) None of these

Identical solutions by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON; and Jean-François Désilets, student, Collège Montmorency, Laval, QC.

From the given data, $n + 5 = \dfrac{1}{4}$; thus, $n + 6 = \dfrac{5}{4}$. Hence, $\dfrac{1}{n+6} = \dfrac{4}{5}$. The answer is D.

6. A standard 6-sided die is tossed twice. The probability of obtaining a sum of 5 is:

(A) $\frac{1}{12}$ (B) $\frac{1}{9}$ (C) $\frac{5}{36}$ (D) $\frac{1}{6}$ (E) $\frac{2}{9}$

I. *Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.*

Au premier lancer, il faut avoir un chiffre entre 1 et 4 pour que l'addition soit possible. Au deuxième lancer, il faut obtenir le chiffre qui additionné à notre premier chiffre donnera 5. Donc, $P = \frac{4}{6} \cdot \frac{1}{6} = \frac{4}{36} = \frac{1}{9}$. La réponse est B.

II. *Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.*

Since the 6-sided die is tossed twice, there are 36 possible outcomes. The only combinations which add up to 5 are $(1,4)$, $(2,3)$, $(3,2)$, $(4,1)$. Therefore, the probability of obtaining a sum of 5 is $\frac{4}{36} = \frac{1}{9}$. The answer is B.

7. A rectangle has dimensions 20 cm $\times$ 50 cm. If the length is increased by 20% and the width is decreased by 20%, then the change in the area is:

(A) an 8% increase (B) a 4% increase (C) a 0% increase

(D) a 4% decrease (E) an 8% decrease

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

Let us denote the length of the original rectangle by ℓ and the width by w. The area of the rectangle is ℓw. Increasing ℓ by 20% and decreasing w by 20%, we get 1.2ℓ and $0.8w$ for the new rectangle. Hence, the area of the new rectangle is $0.96\ell w$. Therefore, the area has decreased by $0.04\ell w$ or 4%. The answer is D.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

8. The greatest prime factor of 21831 is:

(A) 435 (B) 57 (C) 783 (D) 383 (E) 10917

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON, modified by the editor.

Let us take all the answers given and see if they are prime or not. The answer 435 is not prime because its last digit is 5, which means that it is divisible by 5. Answers 57, 783, and 10917 are not prime because $5+7=12$, $7+8+3=18$, and $1+0+9+1+7=18$, which means they are each divisible by 3. The only number left in the list is 383. The answer is D.

[*Ed:* While this argument shows that 383 is the only feasible answer, for completeness one should still verfiy that 383 is prime and is a divisor of 21831, both of which can be checked easily.]

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC.

9. The number of houses sold in Kamloops in 2004 is exactly 40% more than the number sold in 2003. Assuming that at least one house was sold in 2004, the smallest possible number of houses sold in Kamloops in 2004 is:

(A) 5 (B) 7 (C) 14 (D) 70 (E) 140

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Puisqu'on ne peut pas vendre de demi-maison, le plus petit nombre n pour lequel $0,4n$ est entier est 5. Pour savoir le nombre de maisons vendues en 2004, il suffit de faire $1,4\times 5=7$. La réponse est B.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

10. The minimum number of students that must be in a room to ensure that at least 10 are boys or at least 10 are girls is:

(A) 10 (B) 11 (C) 18 (D) 19 (E) 20

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC.

Il peut y avoir 9 filles et 9 garçons dans 18 étudiants. Le $19^{\text{ième}}$ fait qu'il doit y avoir au moins 10 garçons ou 10 filles. La réponse est D.

Une solution incorrecte a été soumise.

11. The number of integers that satisfy the inequality $\frac{3}{7}<\frac{n}{14}<\frac{2}{3}$ is:

(A) 0 (B) 2 (C) 3 (D) 4 (E) 5

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

The inequality can be written as $\frac{6}{14} < \frac{n}{14} < \frac{28}{42}$. Therefore we can easily conclude that n starts at 7. To find the maximum value for n, let us consider $n = 9$. We get $\frac{9}{14} = \frac{27}{42}$. Therefore, $n = 9$ is possible. For $n = 10$, we have $\frac{10}{14} = \frac{30}{42} > \frac{28}{42}$. Hence, $n = 10$ is not possible. Therefore, n can be 7, 8, or 9. The answer is C.

An incorrect solution was submitted.

12. Given that $20! = 20 \times 19 \times 18 \times \cdots \times 2 \times 1$ and 2^n is a factor of $20!$, then the largest possible value of n is:

(A) 10 (B) 12 (C) 18 (D) 20 (E) 24

Une solution identique soumise par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC ; et Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

Pour trouver n, il suffit de regarder combien de fois 2 entre dans chaque facteur de 20!. Nous placerons dans le tableau les facteurs pairs de 20! (les seuls contenant des 2) et le nombre de deux qu'ils contiennent.

Facteur	2	4	6	8	10	12	14	16	18	20	Total
Nombre de deux	1	2	1	3	1	2	1	4	1	2	18

La réponse est C.

13. Terry has \$28.00 in nickels, dimes, and quarters. The value of the dimes is twice the value of the quarters, and it is half the value of the nickels. The total number of coins that Terry has is:

(A) 72 (B) 264 (C) 416 (D) 560 (E) 632

Official solution.

Let Q be the value of the quarters; then the value of the dimes is $2Q$, and, since this is half the value of the nickels, the value of the nickels is $4Q$. Hence, $Q + 2Q + 4Q = 2800$, which means that $Q = 400$. Thus, there must be \$4 in quarters, \$8 in dimes, and \$16 in nickels.

Finally, since there are 4 quarters per dollar, 10 dimes per dollar, and 20 nickels per dollar, the number of coins is $4 \times 4 + 10 \times 8 + 20 \times 16 = 416$. The answer is C.

Also solved by Jean-François Désilets, student, Collège Montmorency, Laval, QC; and Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

14. The number 2005 can be written in the form $a^2 - b^2$, where a and b are integers that are greater than one, in exactly one way. The value of $a^2 + b^2$ is:

(A) 160825 (B) 160801 (C) 80418 (D) 80413 (E) 80406

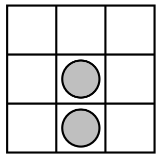
Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

We can write $a^2 - b^2 = (a+b)(a-b)$, and we can factor 2005 either as 2005×1 or as 401×5.

Suppose we let $a + b = 2005$ and $a - b = 1$. Solving these two equations, we get $a = 1003$ and $b = 1002$. However, this solution produces a value for $a^2 + b^2$ which is exceeds every possible solution in the list.

Now let $a+b = 401$ and $a-b = 5$. Then we get $a = 203$ and $b = 198$. Hence, $a^2 = 41209$ and $b^2 = 39204$. Thus, $a^2 + b^2 = 80413$. The answer is D.

15. The game of Solitaire JumpIt is played on a 3×3 grid with two identical game discs. If the two discs are adjacent horizontally, vertically, or diagonally, one disc can jump the other by moving onto the open space opposite the other disc. The disc that is jumped is removed. (See the diagram). The number of ways to place two identical game discs on the grid so that no jump is possible is:



(A) 16 (B) 20 (C) 24 (D) 32 (E) 40

Solution par Jean-François Désilets, étudiant, Collège Montmorency, Laval, QC, adapté par le rédacteur.

Nous allons identifier toutes les solutions possibles avec méthode. Nous évaluons toutes les possibilités pour chaque carré ayant un O dedans. Les carrés ayant des X représentent les possibilités pour le second jeton. De plus, les carrés ayant déjà contenu des O ne pourront plus être utilisés par la suite pour éviter le double comptage d'une solution déjà comptée.

O		X
		X
X	X	X

	X	X
O		X
	X	X

	X	X
		X
O		X

	X	X
		X
	O	

	X	X
	O	

	X	
		O

Il y a vingt X dans les tableaux, donc 20 solutions possibles. La réponse est B.

Solutioné aussi par Karthik Natarajan, étudiant, Edgewater Park Public School, Thunder Bay, ON.

That brings us to the end of another issue. This month's winners of a past Volume of *Mathematical Mayhem* are Jean-François Désilets and Khartik Natarajan. Congratulations, Jean-François and Khartik! Please continue to send in your contests and solutions.

MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

Mayhem Problems

Please send your solutions to the problems in this edition by **1 June 2006**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M226. *Proposed by John Ciriani, Kamloops, BC.*

Antonino has a drawer full of identical black socks and identical white socks. If he were to select two socks at random from his drawer, the probability that they match would be $\frac{1}{2}$. How many of each colour of sock does Antonino have? (There is more than one answer.)

M227. *Proposed by Kenneth S. Williams, Carleton University, Ottawa, ON.*

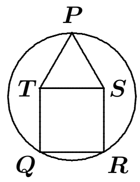
Let N be a positive integer such that N leaves a remainder of 2 or 4 when divided by 6 and there are integers x and y such that $N = x^2 + 27y^2$. Prove that there exist integers a and b with $N = a^2 + 3b^2$ where b is not divisible by 3.

M228. *Proposed by K.R.S. Sastry, Bangalore, India.*

(a) The zeros of the polynomial $P(x) = x^2 - 5x + 2$ are precisely the dimensions of a rectangle in centimetres. Determine the perimeter and the area of the rectangle.

(b) The zeros of the polynomial $P(x) = x^3 - 70x^2 + 1629x - 12600$ are precisely the inner dimensions of a rectangular room in metres. Find the total surface area and the volume of the interior of the room (when doors and windows are closed).

M229. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*



An equilateral triangle sits atop a square as in the diagram. All sides have length 1. A circle passes through points P, Q and R. What is the radius of the circle?

M230. *Proposed by the Mayhem Staff.*

Al, Betty, Cecil, Dora, and Eugene are going to divide n coins among themselves knowing that:

1. Everyone receives at least one coin.
2. Al gets fewer coins than Betty, who gets fewer than Cecil, who gets fewer than Dora, who gets fewer than Eugene.
3. Each person knows only the total n and how many coins he or she got.

What is the smallest possible value of n such that nobody can deduce the number of coins received by each of the others without more information?

M231. *Proposed by Edward J. Barbeau, University of Toronto, Toronto, ON.*

Cordelia and Kent play the following game. Cordelia goes first and they take alternate turns. Each selects a number from 1 to 6 inclusive that has not already been selected; the game ends in six moves. At the end of each move, the player making the move takes the sum of all the numbers selected by either player up to that point and claims all of its positive divisors. When the game is over, the score of each player is the highest number k for which the player has claimed all the consecutive numbers $1, 2, 3, \ldots, k$ from 1 to k inclusive. The winner is the player with the highest score; if both have the same score, neither wins and the game is a draw. For example, suppose the six moves are as follows: C:2; K:4; C:1; K:3; C:5; K:6. The respective claims by C are 1, 2; 1, 7; 1, 3, 5, 15; and by K are 1, 2, 3, 6; 1, 2, 5, 10; 1, 3, 7, 21. Cordelia and Kent have the same score, 3, and the game is a draw. The example does not demonstrate very good play. Is there any way that Cordelia can be prevented from winning assuming she is playing as an expert?

. . . .

M226. *Proposé par John Ciriani, Kamloops, BC.*

Antonino a un tiroir plein de chaussettes noires identiques et de chaussettes blanches identiques. S'il devait choisir deux chaussettes au hasard, la probabilité qu'elles soient de même couleur serait $\frac{1}{2}$. Combien de chaussettes de chaque couleur Antonino possède-t-il ? (Il y a plus d'une réponse.)

M227. *Proposé par Kenneth S. Williams, Université Carleton, Ottawa, ON.*

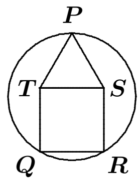
Soit N un nombre entier positif tel que N donne un reste de 2 ou 4 lorsque divisé par 6 et il y a des entiers x et y tels que $N = x^2 + 27y^2$. Montrer qu'il existe des entiers a et b avec $N = a^2 + 3b^2$ où b n'est pas divisible par 3.

M228. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

(a) Les zéros du polynôme $P(x) = x^2 - 5x + 2$ donnent en centimètres les dimensions d'un rectangle. Déterminer le périmètre et l'aire du rectangle.

(b) Les zéros du polynôme $P(x) = x^3 - 70x^2 + 1629x - 12600$ donnent en mètres les dimensions intérieures d'une chambre rectangulaire. Trouver la surface totale et le volume de l'intérieur de la chambre (les portes et fenêtres étant fermées).

M229. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*



On dessine un triangle équilatéral à partir du côté supérieur d'un carré suivant la figure ci-contre. Tous les côtés sont de longueur 1. Si un cercle passe par les points P, Q et R, quel est son rayon ?

M230. *Proposé par l'Equipe de Mayhem.*

Alex, Berthe, Carole, Denise et Eugène se partagent n jetons de sorte que :

1. Chacun reçoît au moins un jeton.
2. Par ordre alphabétique, chacun en reçoît strictement moins que le suivant.
3. À part le total des n jetons, chaque personne ne connaît que le nombre de jetons qu'elle a reçus.

Quelle est la plus petite valeur possible de n de sorte que personne ne puisse en déduire, sans information supplémentaire, combien les autres ont reçu de jetons ?

M231. *Proposé par Edward J. Barbeau, Université de Toronto, Toronto, ON.*

Catherine et Patrick jouent le jeu suivant. On joue tour à tour et c'est Catherine qui commence. Chacun choisit un nombre, de 1 à 6 inclusivement, qui n'a pas encore été sélectionné ; le jeu s'arrête après six tours. À la fin de chaque tour, le joueur à qui c'est le tour fait la somme de tous les nombres choisis jusqu'ici par chacun des joueurs et marque tous les diviseurs positifs de celle-ci. À la fin du jeu, la marque de chaque jouer est le plus grand nombre k pour lequel le joueur a marqué tous les nombres consécutifs 1, 2, 3, $\dots$, k de 1 à k inclusivement. Le gagnant est le joueur avec la plus haute marque ; il y a un match nul si les deux joueurs obtiennent la même marque. Par exemple, supposons que les six tours donnent : C :2 ; P :4 ; C :1 ; P :3 ; C :5 ; P :6. Les marques respectives de C sont 1, 2 ; 1, 7 ; 1, 3, 5, 15 ; et celles de P sont1, 2, 3, 6 ; 1, 2, 5, 10 ; et 1, 3, 7, 21. Catherine et Patrick ont la même marque, 3, et c'est match nul. Cet exemple ne représente pas la meilleure manière de jouer. Y a-t-il une manière d'empêcher Catherine de gagner en supposant qu'elle joue comme un expert ?

Mayhem Solutions

M163. Corrected. *Proposed by the Mayhem Staff.*

Show that it is possible to put non-negative integer values on the faces of two dice (not necessarily the same on both dice) so that, when the dice are tossed, the outcomes $1, 2, 3, \ldots, 12$ are equally probable.

Solution by Geoffrey Siu, London Central Secondary School, London, ON.

There are many ways in which this can be done. For example,

Die 1	Die 2
0 0 0 6 6 6	1 2 3 4 5 6
0 0 0 1 1 1	1 3 5 7 9 11
1 1 1 2 2 2	0 2 4 6 8 10
1 1 1 7 7 7	0 1 2 3 4 5
0 6 13 14 15 16	1 2 3 4 5 6
6 6 6 6 6 6	7 7 7 7 7 7

It can be verified that the first four examples above yield pairs where $1, 2, 3, \ldots, 12$ all occur with probability $\frac{1}{12}$. In the fifth example above, $1, 2, 3, \ldots, 12$ all occur with probability $\frac{1}{36}$; while in the sixth example, they occur with probability 0, which fits the requirements of the problem.

Also solved by the Austrian IMO team; Roger He, Prince of Wales Collegiate, St. Johns, NL; Titu Zvonaru, Comanesti, Romania.

M164. *Proposed by the Mayhem Staff.*

Consider the following procedure for dividing the three-digit number 375 by 8. Write down the number formed by the first two digits, namely, 37. Multiply this by 2 to get 74. Add to this the units digit of 375 (the original number), obtaining $74 + 5 = 79$. Then divide by 8 to get 9 with a remainder of 7. Add this result (9, remainder 7) to the number 37 (the first two digits of 375) to get your answer: 46, remainder 7. Thus, 375 divided by 8 equals 46 with a remainder of 7.

Does this method always work for three-digit numbers? Why, or why not?

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

The answer to the question is yes.

Let $100a + 10b + c$ be a three-digit number (a, b, and c are the digits). Then we have

$$\frac{100a + 10b + c}{8} = \frac{80a + 8b}{8} + \frac{20a + 2b + c}{8} = 10a + b + \frac{2(10a + b) + c}{8}.$$

The calculation on the right side is exactly the "method" described in the problem.

Notice that this algorithm can be extended quite naturally to an n–digit number, since

$$\frac{\sum\limits_{i=0}^{n} a_i 10^i}{8} = \frac{a_0 + \sum\limits_{i=1}^{n}(8+2)a_i 10^{i-1}}{8} = \sum_{i=1}^{n} a_i 10^{i-1} + \frac{a_0 + 2\sum\limits_{i=1}^{n} a_i 10^{i-1}}{8}\,.$$

M165. *Proposed by Babis Stergiou, Chalkida, Greece.*

If a, $b > 0$, prove that

(a) $\sqrt{ab} \;\geq\; \dfrac{2}{1/a+1/b}$.

(b) $a^6 + b^6 + 8a^3 + 8b^3 + 2a^3b^3 + 16 \;\geq\; 36ab$.

Solution by Mark Yang, St. Paul's Junior High School, St. John's, NL.

(a) By the AM–GM Inequality, we know that

$$\frac{a^3b + a^2b^2 + a^2b^2 + ab^3}{4} \;\geq\; \sqrt[4]{a^8b^8}\,;$$

that is, $ab\big(a^2+2ab+b^2\big) \geq 4a^2b^2$. Since a, $b > 0$, we rearrange and take the square root of both sides to get

$$\sqrt{ab} \;\geq\; \frac{2ab}{a+b} \;=\; \frac{2}{1/a+1/b}\,.$$

(b) By the AM–GM Inequality, we know that

$$\frac{a^6+b^6+a^3+b^3+\cdots+a^3+b^3+a^3b^3+a^3b^3+1+\cdots+1}{36} \;\geq\; ab\,,$$

where the numerator on the left side contains 8 terms each of a^3 and b^3, and 16 terms of 1. The result follows immediately.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC; and Vedula N. Murty, Dover, PA, USA.

M166. *Proposed by the Mayhem Staff.*

(a) Simplify

$$(3n)^2 + (4n-1)^2 - (5n-1)^2\,, \qquad (3n+2)^2 + (4n)^2 - (5n+1)^2\,.$$

(b) Using (a) or otherwise, prove that all positive integers can be represented in the form $a^2+b^2-c^2$ where a, b, c are integers and $0 < a < b < c$.

Solution by James T. Bruening, Southeast Missouri State University, Cape Girardeau, MO, USA.

(a) Note that

$$(3n)^2 + (4n-1)^2 - (5n-1)^2$$
$$= \; 9n^2 + 16n^2 - 8n + 1 - 25n^2 + 10n - 1 \; = \; 2n\,.$$

Similarly,

$$(3n+2)^2 + (4n)^2 - (5n+1)^2$$
$$= \; 9n^2 + 12n + 4 + 16n^2 - 25n^2 - 10n - 1 \; = \; 2n+3\,.$$

(b) Let m be a positive integer. Assume first that m is odd. For the first several odd integers, observe that $1 = 4^2 + 7^2 - 8^2$, $3 = 4^2 + 6^2 - 7^2$, $5 = 4^2 + 5^2 - 6^2$, and $7 = 10^2 + 14^2 - 17^2$. Now let $m = 2n + 3$ with $n > 2$. Then $m = (3n+2)^2 + (4n)^2 - (5n+1)^2$, and since $n > 2$, we have $0 < 3n + 2 < 4n < 5n + 1$.

Next, assume m is even. We note first that $2 = 5^2 + 11^2 - 12^2$. Now, for $n > 1$, let $m = 2n$. Then $m = 2n = (3n)^2 + (4n-1)^2 - (5n-1)^2$, and since $n > 1$, we have $0 < 3n < 4n - 1 < 5n - 1$.

Also solved by Robert Bilinski, Collège Montmorency, Laval, QC.

M167. *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Solve the following inequality:

$$(2+\cos x)(1+\sin x-\cos x) \;\geq\; \cos x(1+2\sin x-\cos x)\,.$$

Solution by Vedula N. Murty, Dover, PA, USA, modified by the editor.

The inequality we are to solve is equivalent to

$$2(1-\cos x) + \sin x(2-\cos x) \;\geq\; 0\,. \qquad (1)$$

Since the inequality is periodic with period 2π, it will suffice to solve it in an interval of length 2π.

Consider any $x \in (-\pi, \pi)$. Then $-\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}$. Let $t = \tan\left(\frac{x}{2}\right)$ (or equivalently, $\frac{x}{2} = \arctan t$). Using the formulas

$$\cos x \;=\; \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) \;=\; \frac{1-t^2}{1+t^2}$$

$$\text{and} \qquad \sin x \;=\; 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) \;=\; \frac{2t}{1+t^2}\,,$$

the inequality (1) can be stated equivalently as

$$2t(t+1)(2t^2+t+1) \;\geq\; 0\,,$$

which holds for all t such that $t \leq -1$ or $t \geq 0$.

Therefore, the original inequality is valid for all $x \in (-\pi, \pi)$ such that $\tan\left(\frac{x}{2}\right) \leq -1$ or $\tan\left(\frac{x}{2}\right) \geq 0$; that is, for all x such that $-\pi < x \leq -\frac{\pi}{2}$ or

$0 \le x < \pi$. By continuity (or by direct calculation), the inequality must also hold when $x = \pm\pi$.

Our results can be stated more simply if we use the interval $[0, 2\pi)$ in place of $[-\pi, \pi)$. In $[0, 2\pi)$, the inequality holds for $0 \le x \le \frac{3\pi}{2}$. The complete solution set is

$$\left\{x : 2k\pi \le x \le 2k\pi + \tfrac{3\pi}{2} \quad \text{for some integer } k\right\} .$$

One incorrect solution was received.

M168. *Proposed by Neven Jurič, Zagreb, Croatia.*

How many different 3×3 arrays of non-negative integers is it possible to construct so that each of the three horizontal sums and each of the three vertical sums is equal to 7, the first diagonal sum is equal to 10, and the second diagonal sum is equal to 9? (Two arrays which may be transformed into one another by rotations and/or reflections are not considered to be different.)

Here is an example of such an array:
$$\begin{matrix} 2 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 0 & 4 \end{matrix}$$

Solution by Geneviève Lalonde, Massey, ON.

Let the entries in the array be $\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$. Then the conditions may be written as a system of equations:

$$\begin{aligned}
a_1 + a_2 + a_3 &= 7\,, && (1)\\
a_4 + a_5 + a_6 &= 7\,, && (2)\\
a_7 + a_8 + a_9 &= 7\,, && (3)\\
a_1 + a_4 + a_7 &= 7\,, && (4)\\
a_2 + a_5 + a_8 &= 7\,, && (5)\\
a_3 + a_6 + a_9 &= 7\,, && (6)\\
a_1 + a_5 + a_9 &= 10\,, && (7)\\
a_3 + a_5 + a_7 &= 9\,. && (8)
\end{aligned}$$

(There is no loss of generality here, since the array may be reflected or rotated.) By adding equations (1), (2), and (3), we get

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 \;=\; 21\,. \qquad (9)$$

On the other hand, if we add (2), (5), (7), and (8), we obtain

$$a_1 + a_2 + a_3 + a_4 + 4a_5 + a_6 + a_7 + a_8 + a_9 \;=\; 33\,. \qquad (10)$$

Subtracting (9) from (10) yields $3a_5 = 12$, from which we get $a_5 = 4$. We may assume, without loss of generality, that $a_1 \ge a_9$ and $a_3 \ge a_7$. From (7) we see that $a_1 \geq 3$, and from (8) we see that $a_3 \geq 3$. Then, from (1) we see that there are only three possibilities for a_1 and a_3, namely $a_1 = a_3 = 3$, $a_1 = 3$ and $a_3 = 4$, or $a_1 = 4$ and $a_3 = 3$. Each of these lead to a unique array satisfying (1) to (8) (the details are left to the reader):

$$\begin{bmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 2 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 0 & 4 \\ 3 & 4 & 0 \\ 1 & 3 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 2 \\ 2 & 3 & 2 \end{bmatrix}.$$

The last array above is equivalent to the one given as an example in the problem statement.

M169. *Proposed by the Mayhem Staff.*

Prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2003}-\frac{1}{2004} = \frac{1}{1003}+\frac{1}{1004}+\frac{1}{1005}+\cdots+\frac{1}{2003}+\frac{1}{2004}.$$

Solution by Geneviève Lalonde, Massey, ON.

We claim that, for all positive integers n,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2n-1}-\frac{1}{2n} = \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}.$$

(The problem posed is simply the case where $n = 1002$.)

To prove our claim, we apply Mathematical Induction. For $n = 1$, the statement is $1 - \frac{1}{2} = \frac{1}{2}$, which is clearly true. Assume that, for some positive integer k, the statement is true; that is,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k} = \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}.$$

Let us now examine $n = k + 1$. Starting with the left side, and using the above assumption, we have

$$\begin{aligned} 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}&+\cdots+\frac{1}{2k-1}-\frac{1}{2k}+\frac{1}{2k+1}-\frac{1}{2k+2} \\ &= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right) \\ &= \left(\frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}\right)+\left(\frac{1}{2k+1}-\frac{1}{2k+2}\right) \\ &= \frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\left(\frac{2}{2k+2}-\frac{1}{2k+2}\right) \\ &= \frac{1}{k+2}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2k+2}, \end{aligned}$$

which means that the the result is true for $n = k + 1$. Thus, it is true for $n = k + 1$ whenever it is true for $n = k$. Therefore, it is true for all n.

Also solved by Mihály Bencze, Brasov, Romania. Bencze noted that the identity proved above is due to Catalan.

M170. *Proposed by the Mayhem Staff.*

Evaluate $\cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 90^\circ$.

Solution by Robert Bilinski, Collège Montmorency, Laval, QC.

Let S denote the given sum. Since $\cos 90^\circ = 0$, we have

$$S = \cos^2 1^\circ + \cos^2 2^\circ + \cos^2 3^\circ + \cdots + \cos^2 89^\circ .$$

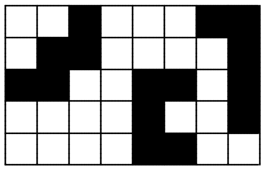
Then, since $\cos x = \sin(90^\circ - x)$ and $\sin^2 x = 1 - \cos^2 x$, we get

$$\begin{aligned} S &= \sin^2(90^\circ - 1^\circ) + \sin^2(90^\circ - 2^\circ) + \cdots + \sin^2(90^\circ - 89^\circ) \\ &= \sin^2 89^\circ + \sin^2 88^\circ + \cdots + \sin^2 1^\circ \\ &= (1 - \cos^2 89^\circ) + (1 - \cos^2 88^\circ) + \cdots + (1 - \cos^2 1^\circ) \\ &= 89 - S . \end{aligned}$$

Thus, $2S = 89$, which implies that $S = 44.5$.

Also solved by Mihály Bencze, Brasov, Romania.

M171. *Proposed by Neven Jurič, Zagreb, Croatia.*



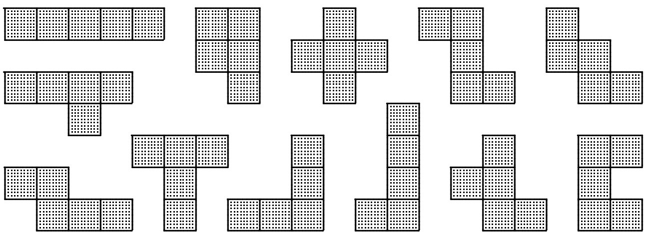
There are 12 distinct (non-congruent) pentominoes, 3 of which are shown to the right. Each pentomino covers an area of 5 square units. (Note: *Pentominoes* is a registered trademark of Solomon W. Golomb.)

1. Find the remaining 9 pentominoes.
2. Arrange all 12 pentominoes on the 60 numbered cells in the diagram to the right, so that each pentomino covers numbers whose sum is 10.

2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

Solution by Roger He, Prince of Wales Collegiate, St. John's, NL.

First, we list the 12 possible pentominoes.



Next we need to fit them in and have their blocks sum to 10. Note that only the numbers 1, 2, 3, 4, 5 are used in the given grid. Looking at all the ways to use five positive integers to add to 10 we get: $5+2+1+1+1$, $4+3+1+1+1$, $4+2+2+1+1$, $3+3+2+1+1$, $3+2+2+2+1$, and

$2+2+2+2+2$. We quickly note that there is no grouping in the given grid that can make the last sum; so we need only consider the others. Counting the number of occurrences of each number in the puzzle, we find that there are 5 fives, 5 fours, 8 threes, 9 twos, and 33 ones. Since 5 only appears in the first pattern above, we can block off around the 5s the cells that could possibly go with it (one region is marked below left).

Piecing together the conditions for the other numbers, we come up with the solution below right.

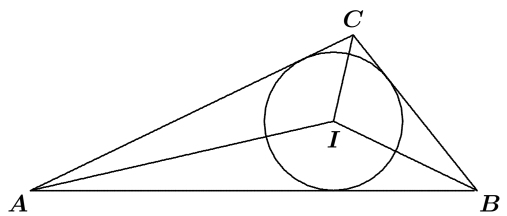
2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

2	1	1	3	1	1		
3	1	1	3	5	2		
3	1	1	2	1	1	4	1
2	1	5	2	1	2	2	1
1	1	5	4	3	1	4	1
3	4	1	5	1	1	1	1
1	4	1	1	2	1	1	3
3	1	1	1	1	1	2	5

M172. *Proposed by Mihály Bencze, Brasov, Romania.*

Let I denote the centre of the inscribed circle in triangle ABC. Prove that if one of the triangles AIB, BIC, or CIA is similar to triangle ABC, then the angles of triangle ABC are in geometric progression.

Solution by the proposer.



Without loss of generality, we may suppose that $\triangle BIC$ is the triangle which is similar to $\triangle ABC$. Then the angles of $\triangle BIC$ will match up with the angles in $\triangle ABC$ in some order. The angles in $\triangle BIC$ are $\frac{1}{2}B$, $\frac{1}{2}C$, and $\frac{\pi}{2}+\frac{1}{2}A$.

If $\frac{\pi}{2}+\frac{1}{2}A = A$, then $A=\pi$, which is impossible. If $\frac{1}{2}B = B$, then $B=0$, which is impossible. We can similarly rule out the case where $\frac{1}{2}C = C$.

Thus, we have two possibilities:

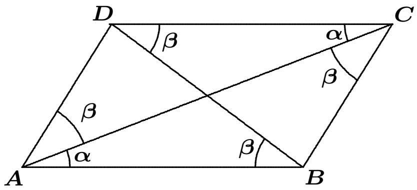
$$A = \tfrac{1}{2}B\,,\quad B = \tfrac{1}{2}C\,,\quad \text{and}\quad C = \tfrac{\pi}{2}+\tfrac{1}{2}A$$

$$\text{or}\quad A = \tfrac{1}{2}C\,,\quad B = \tfrac{\pi}{2}+\tfrac{1}{2}A\,,\quad \text{and}\quad C = \tfrac{1}{2}B\,,$$

both of which lead to $\triangle ABC$ having angles $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$, which are in geometric progression.

M173. *Proposed by K.R.S. Sastry, Bangalore, India.*

Suppose that the diagonals AC and BD of a parallelogram $ABCD$ determine angles α and β as shown in the diagram below.



1. Prove that such an arrangement of angles is possible if and only if the diagonals are proportional to the sides.
2. Use trigonometry to express β in terms of α.

Combination of solutions by Bruce Crofoot, Thompson Rivers University, Kamloops, BC; and the proposer.

1. Let E denote the intersection point of the diagonals AC and BD. The given arrangement of angles occurs if and only if $\triangle ABE \sim \triangle ACB$ (or equivalently, $\triangle CDE \sim \triangle CAD$). We will prove that this is true if and only if the diagonals are proportional to the sides (meaning that there is some $k > 0$ such that $AC = k\,AB$ and $BD = k\,BC$).

First suppose that $\triangle ABE \sim \triangle ACB$. Then

$$\frac{AE}{AB} = \frac{AB}{AC} \quad \text{and} \quad \frac{BE}{AE} = \frac{BC}{AB}\,.$$

Since $AE = \frac{1}{2}AC$ and $BE = \frac{1}{2}BD$, we have

$$\frac{\frac{1}{2}AC}{AB} = \frac{AB}{AC} \quad \text{and} \quad \frac{BD}{AC} = \frac{BC}{AB}\,.$$

The first equation implies that $AC = \sqrt{2}\,AB$. Using this result in the second equation, we obtain $BD = \sqrt{2}\,BC$. Thus, the diagonals are proportional to the sides with proportionality constant $\sqrt{2}$.

Conversely, suppose that $AC = k\,AB$ and $BD = k\,BC$ for some $k > 0$. We apply the Parallelogram Law (which holds for any parallelogram):

$$AC^2 + BD^2 = 2AB^2 + 2BC^2\,.$$

Substituting $AC = k\,AB$ and $BD = k\,BC$ on the left, we find that $k = \sqrt{2}$. Then $AC^2 = 2AB^2$, or equivalently, $\frac{AE}{AB} = \frac{AB}{AC}$. This implies that $\triangle ABE \sim \triangle ACB$.

2. Applying the Law of Sines to $\triangle ABC$, we have

$$\frac{a}{\sin\beta} = \frac{b}{\sin\alpha} = \frac{\sqrt{2}a}{\sin(\alpha+\beta)}\,.$$

From the first and third expressions above, we have $\sin(\alpha+\beta)=\sqrt{2}\sin\beta$, from which we deduce that

$$\tan\beta \;=\; \frac{\sin\alpha}{\sqrt{2}-\cos\alpha}\,.$$

M174. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let x denote the measure of an angle of a non-degenerate triangle. Determine x, given that

$$\frac{1}{\sin x} \;=\; \frac{1}{\sin 2x}+\frac{1}{\sin 3x}\,.$$

Solution by Miguel Marañón, IES Sagasta, Logroño, Spain.

First we rewrite the given condition successively as

$$\begin{aligned}\frac{1}{\sin x} &= \frac{\sin 2x+\sin 3x}{\sin 2x\sin 3x}\,,\\ \sin 2x\sin 3x &= \sin x(\sin 2x+\sin 3x)\,,\\ 2\sin x\cos x\sin 3x &= \sin x(\sin 2x+\sin 3x)\,.\end{aligned}$$

We now divide by $\sin x$ (since the given condition makes sense only for $\sin x \neq 0$) to get

$$2\cos x\sin 3x \;=\; \sin 2x+\sin 3x\,.$$

Using the trigonometric identity

$$\sin A+\sin B \;=\; 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

with $A=4x$ and $B=2x$, we get $\sin 4x+\sin 2x = 2\sin 3x\cos x$. Then we can rewrite our equation above as

$$\sin 4x+\sin 2x \;=\; \sin 2x+\sin 3x\,,$$

or $\sin 4x = \sin 3x$. This has solutions of the form $4x = 3x+2k\pi$ or $4x = \pi+2k\pi-3x$ for $k\in\mathbb{Z}$; that is, $x=2k\pi$ or $x=(2k+1)\frac{\pi}{7}$. The former implies that $\sin x = 0$, which we have already ruled out. For the same reason, we can rule out the case where $2k+1$ is a multiple of 7 in the latter case.

Therefore, the solutions of the proposed equation are $x=(2k+1)\frac{\pi}{7}$ where k is an integer and $k\not\equiv 3 \pmod 7$.

Also solved by Mihály Bencze, Brasov, Romania.

Problem of the Month

Ian VanderBurgh, University of Waterloo

A couple of years ago, while fiddling with some problems in preparation for our annual June Mathematics Contest Seminar here in Waterloo, I came across the following problem:

Solve the following system of equations:

$$\begin{aligned} x^2 + xy &= 12\,, \\ 2xy + 3y^2 &= -5\,. \end{aligned}$$

Luckily, my good friend (and former Mayhem columnist) Paul Ottaway happened by. We set to work. With our combined wisdom, we managed to remember that the usual method for solving a system of equations was to eliminate one of the variables.

Following this idea, we solved the first equation for y to get $y = x - \dfrac{12}{x}$, which we then substituted into the second equation, obtaining

$$2x\left(x - \frac{12}{x}\right) + 3\left(x - \frac{12}{x}\right)^2 \;=\; -5\,.$$

After expanding and clearing out denominators, we ended up with a quartic equation, which was hardly appetizing. It was also apparent that solving the second equation for x and substituting into the first equation was not going to be any better.

Thus, the standard technique of eliminating one of the variables was not working. Then one of us had the clever idea to try eliminating the constants instead. (Since the idea was clever, odds are it was Paul's idea, not mine!)

We multiplied the first equation by 5 (obtaining $5x^2 + 5xy = 60$) and the second equation by 12 (obtaining $24xy + 36y^2 = -60$) and added the equations to obtain

$$5x^2 + 29xy + 36y^2 \;=\; 0\,,$$

which we were then able to factor to give

$$(x + 4y)(5x + 9y) \;=\; 0\,,$$

yielding $x = -4y$ or $x = -\frac{9}{5}y$.

Substituting $x = -4y$ into $x^2 + xy = 12$ gives $16y^2 - 4y^2 = 12$; that is, $y^2 = 1$. Hence, $y = \pm 1$, and $(x, y) = (-4, 1)$ or $(x, y) = (4, -1)$.

Substituting $x = -\frac{9}{5}y$ into $x^2 + xy = 12$ gives $\frac{81}{25}y^2 - \frac{9}{5}y^2 = 12$; that is, $\frac{36}{25}y^2 = 12$. Hence, $y = \pm\frac{5}{\sqrt{3}}$, and $(x, y) = \left(-\frac{9}{\sqrt{3}}, \frac{5}{\sqrt{3}}\right)$ or $(x, y) = \left(\frac{9}{\sqrt{3}}, -\frac{5}{\sqrt{3}}\right)$.

We found a neat way to tackle certain systems of equations. Much to my delight, as I was flipping through a recent journal looking for problems, I found the following problem from a European competition of which I had never heard:

Problem. (2005 Mathematical Duel)
Determine all integer solutions of the system of equations

$$\begin{aligned} x^2z + y^2z + 4xy &= 40\,, \\ x^2 + y^2 + xyz &= 20\,. \end{aligned}$$

I attempted this problem for a few minutes by the usual method of eliminating one of the variables. This met with little success. Then I remembered Paul's trick!

Solution: If we subtract 2 times the second equation from the first, the constants are eliminated, and we obtain

$$x^2z + y^2z + 4xy - 2x^2 - 2y^2 - 2xyz \;=\; 0\,.$$

Where to now? Our only real hope is to try to factor this equation somehow. If we rearrange the terms to get

$$x^2z - 2x^2 + y^2z - 2y^2 - 2xyz + 4xy \;=\; 0\,,$$

the light at the end of the tunnel starts to appear. We can now do some factoring to get

$$x^2(z-2) + y^2(z-2) - 2xy(z-2) \;=\; 0\,.$$

Thus, there is a common factor of $z-2$, which we can factor out, giving

$$(z-2)(x^2 - 2xy + y^2) \;=\; 0\,,$$

or

$$(z-2)(x-y)^2 \;=\; 0\,.$$

Therefore, $z = 2$ or $x = y$.

If $z = 2$, then the second equation gives $x^2 + 2xy + y^2 = 20$, or $(x+y)^2 = 20$. But we are looking for integer solutions (unlike the first problem). There are no solutions here, since 20 is not a perfect square.

If $x = y$, the second equation gives $2x^2 + zx^2 = 20$, or $x^2(z+2) = 20$. (The first equation reduces to this same equation as well.) Since x and z are integers and the only perfect squares which are divisors of 20 are 1 and 4, the possibilities for x are 1, -1, 2, and -2, giving values for z of 18, 18, 3, and 3, respectively.

Therefore, the integer solutions for (x,y,z) are $(1,1,18)$, $(-1,-1,18)$, $(2,2,3)$, $(-2,-2,3)$.

We have discovered a new technique for solving old problems—always a satisfying feeling. Problems involving systems of equations occur frequently on contests and in real life, and it is important to have a wide variety of different techniques in our toolbox.

THE OLYMPIAD CORNER

No. 251

R.E. Woodrow

Here it is—the start of another year and a new volume of ***CRUX with MAYHEM***. It is appropriate to look back over the 2005 numbers of the *Corner* and thank all those who provided us with problems, comments, and solutions:

Mohammed Aassila	Bob Serkey
Arkady Alt	Achilleas Sinefakopoulos
Miguel Amengual Covas	Skotidas Sotirios
Michelle Bataille	Chris Small
Robert Bilinski	D.J. Smeenk
Pierre Bornsztein	Mike Spivey
Christopher J. Bradley	The Samford University Problem Solving Group
Yuming Chen	Stan Wagon
José Luis Díaz-Barrero	Tracy Walker
George Evagelopoulos	Edward T.H. Wang
Geoffrey A. Kandall	Kaiming Zhao
Andy Liu	Yufei Zhao
Bill Sands	Titu Zvonaru
Toshio Seimiya	

I also want to thank Joanne Canape (neé Longworth) who continues to work miracles with my scribbles, turning them into high-quality LaTeX files.

As a first set of problems to warm up your problem solving capacity for 2006 we give the problems of the 2003 Vietnamese Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to the IMO in Japan, for collecting the problems for our use.

2003 VIETNAMESE MATHEMATICAL OLYMPIAD

1. Let ABC be a triangle inscribed in a circle with centre O, and let M and N be two points on the line AC such that $\overline{MN} = \overline{AC}$. Let D be the orthogonal projection of M onto the line BC, and E that of N onto AB.

(i) Prove that the orthocentre H of triangle ABC lies on the circumcircle with centre O' of triangle BED.

(ii) Prove that the mid-point of segment AN is symmetric to B with respect to the mid-point of segment OO'.

2. Let Γ_1 and Γ_2 be two circles in the plane tangent to each other at M. Let O_1 and O_2 be their respective centres, and let R_1 and R_2 be their respective radii. Suppose that $R_2 > R_1$. Let A be a point on Γ_2 which is not on the line O_1O_2. Let AB and AC be the tangent lines to Γ_1 where B and C are the points of tangency. The lines MB and MC meet Γ_2 again at E and F, respectively. Let D be the point of intersection of the line EF and the tangent to Γ_2 at A. Prove that D moves on a fixed line as A moves along Γ_2 as long as the three points O_1, O_2, and A are not collinear.

3. Find all polynomials $P(x)$ with real coefficients, satisfying the relation

$$(x^3 + 3x^2 + 3x + 2)P(x - 1) \;=\; (x^3 - 3x^2 + 3x - 2)P(x)$$

for every real number x.

4. Let $P(x) = 4x^3 - 2x^2 - 15x + 9$ and $Q(x) = 12x^3 + 6x^2 - 7x + 1$.

(i) Prove that each of these polynomials has three distinct real roots.

(ii) Let α and β be the greatest roots of $P(x)$ and $Q(x)$, respectively. Prove that $\alpha^2 + 3\beta^2 = 4$.

5. Find the greatest positive integer n such that the following system of equations has an integral solution $(x, y_1, y_2, \ldots, y_n)$:

$$\begin{aligned}(x+1)^2 + y_1^2 \;&=\; (x+2)^2 + y_2^2 \;=\; \cdots \\ &=\; (x+k)^2 + y_k^2 \;=\; \cdots \;=\; (x+n)^2 + y_n^2\,.\end{aligned}$$

6. Let f be a function defined on the set of real numbers $\mathbb{R}$, taking values in $\mathbb{R}$, and satisfying the condition $f(\cot x) = \sin 2x + \cos 2x$ for every x belonging to the open interval $(0, \pi)$. Find the least and the greatest values of the function $g(x) = f(x) \cdot f(1 - x)$ on the closed interval $[-1, 1]$.

7. Let α be a real number, $\alpha \neq 0$. Consider the sequence of real numbers $\{x_n\}$, $n = 1, 2, 3, \ldots$, defined by $x_1 = 0$ and $x_{n+1}(x_n + \alpha) = \alpha + 1$ for $n = 1, 2, 3, \ldots$.

(i) Find the general term of the sequence $\{x_n\}$.

(ii) Prove that the sequence $\{x_n\}$ has a finite limit when $n \to +\infty$. Find this limit.

8. Let $\mathbb{R}^+$ denote the set of all positive real numbers, and let F be the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the condition

$$f(3x) \;\geq\; f\big(f(2x)\big) + x$$

for every $x \in \mathbb{R}^+$. Find the greatest real number α such that $f(x) \geq \alpha x$ for all $f \in F$ and $x \in \mathbb{R}^+$.

9. Consider an integer $n > 1$. Colour all natural numbers red and blue so that the following conditions are simultaneously satisfied:

(i) Every number is coloured red or blue, and there are infinitely many numbers coloured red and infinitely many coloured blue.

(ii) The sum of n distinct red numbers is coloured red and the sum of n distinct blue numbers is coloured blue.

Is it possible to colour in such a manner when (a) $n = 2002$? (b) $n = 2003$?

10. For each integer $n > 1$, denote by s_n the number of permutations $(a_1, a_2, \ldots, a_n)$ of the first n positive integers such that each permutation satisfies the condition $1 \le |a_k - k| \le 2$ for $k = 1, 2, \ldots, n$. Prove that $1.75 \cdot s_{n-1} < s_n < 2 \cdot s_{n-1}$ for all integers $n > 6$.

Next we give problems of the XXIX Russian Mathematical Olympiad V (Final) Round. Thanks again go to Andy Liu for collecting the set.

XXIX RUSSIAN MATHEMATICAL OLYMPIAD

V (Final) Round — 10^{th} Form

1. (N. Agakhanov) Let M be a set containing 2003 different positive real numbers, such that for any 3 different elements a, b, c from M the number $a^2 + bc$ is rational. Prove that it is possible to choose a natural number n such that for each a from M the number $a\sqrt{n}$ is rational.

2. (S. Berlov) Diagonals of the inscribed quadrilateral $ABCD$ intersect at point O. Let S_1 and S_2 be the circumcircles of triangles ABO and CDO, respectively, and let K be the second point of intersection of S_1 and S_2. Straight lines passing through O parallel to AB and CD intersect S_1 and S_2 again at points L and M, respectively. Points P and Q are chosen on segments OL and OM, respectively, so that $OP : PL = MQ : QO$. Prove that points O, K, P, and Q lie on the same circle.

3. (V. Dolnikov) A tree is given on $n \ge 2$ vertices (that is, a graph on n vertices and $n-1$ edges in which it is possible to pass from one vertex to any other vertex by edges, and there is no cyclical path passing through edges). Numbers $x_1, x_2, \ldots, x_n$ are placed on the tree's vertices and the product of numbers at the ends of an edge is written on that edge. Let S be the sum of numbers on all the edges. Prove that

$$\sqrt{n-1}(x_1^2 + x_2^2 + \cdots + x_n^2) \ \ge\ 2S\,.$$

4. (V. Dolnikov, R. Karasev) Let X be a finite set of points in the plane, and let T be an equilateral triangle in the same plane. Suppose that, if X' is any subset of X consisting of no more than 9 points, then X' can be covered by two translations of the triangle T. Prove that the entire set X can be covered by two translations of T.

5. (O. Podlipsky) There are N cities in a country. Between any two cities there is either a highway or a railroad. A tourist wants to travel around the country, visiting each city exactly once, and to return to the city where he started his journey. Prove that the tourist can choose a city with which to begin his journey, as well as the path, in such a way that he will have to change the type of transportation no more than once.

6. (A. Khrabrov) Starting with some natural number a_0, the sequence of natural numbers a_n is created in the following way: $a_{n+1} = a_n/5$, if a_n is divisible by 5, and $a_{n+1} = \lfloor\sqrt{5}a_n\rfloor$ if a_n is not divisible by 5. Prove that, for some number N, the sequence $\{a_n\}_{n\geq N}$ is increasing.

7. (P. Kozhevnikov) In triangle ABC let O and I be the circumcentre and incentre, respectively. Let ω_a be the excircle which touches the extensions of sides AB and AC at points K and M, respectively, and touches side BC at point N. If the mid-point P of the segment KM lies on the circumcircle of triangle ABC, prove that O, N, and I are collinear.

8. (D. Khramtsov) Find the greatest natural number N such that, for any arrangement of the natural numbers from 1 to 400 in the cells of a square table of size 20×20, there can be found two numbers located in the same row or in the same column, the difference of which is not less than N.

11th Form

1. (N. Agakhanov, A. Golovanov, V. Senderov) Let α, β, γ, and τ be positive numbers such that, for all x,

$$\sin\alpha x + \sin\beta x = \sin\gamma x + \sin\tau x .$$

Prove that $\alpha = \gamma$ or $\alpha = \tau$.

2. [*Ed:* Same problem as #2 in the previous set.]

3. (A. Khrabrov) Let $f(x)$ and $g(x)$ be polynomials with non-negative integer coefficients, and let m be the greatest coefficient of $f(x)$. Suppose that there are natural numbers $a < b$ such that $f(a) = g(a)$ and $f(b) = g(b)$. Prove that if $b > m$, then $f(x)$ and $g(x)$ are the same polynomial.

4. (E. Cherepanov) Originally, Anna and Boris each had a long sheet of paper. The letter A was written on one sheet of paper, and the letter B on the other. Every minute either Anna or Boris, not necessarily taking turns, adds (on the right or the left) to the word on his or her sheet of paper a word from the other sheet of paper. Prove that in 24 hours the word from Anna's sheet of paper could be cut into two parts which could be switched in such a way as to obtain the same word written backwards.

5. (N. Agakhanov) The lengths of the sides of a triangle are roots of a cubic equation with rational coefficients. Prove that the lengths of the altitudes of the triangle are roots of an equation of degree 5 with rational coefficients.

6. (S. Berlov) Is it possible to arrange natural numbers in the cells of an infinite checkerboard in such a way that for any natural numbers $m > 100$ and $n > 100$ the sum of the numbers in any $m \times n$ rectangle is divisible by $m + n$?

7. (I. Ivanov) There are 100 cities in a country; some pairs of cities are connected by roads. For each set of four cities there are at least two roads between them. Suppose that there is no path that passes through each city exactly once. Prove that one could choose two cities in such a way that each of the remaining cities would be connected by a road with at least one of the two chosen cities.

8. (F. Bakharev) A sphere inscribed in the tetrahedron $ABCD$ touches its four faces ABC, ABD, ACD, and BCD at points D_1, C_1, B_1, and A_1, respectively. Consider the plane equidistant from point A and from the plane $B_1C_1D_1$, and the three planes analogous to it. Prove that the tetrahedron formed by these four planes has the same centre for the circumscribed sphere as the tetrahedron $ABCD$.

Next we turn to solutions from our readers to problems of the XXXVI Spanish Mathematical Olympiad National Round given [2004 : 202–203].

1. Let $P(x) = x^4 + ax^3 + bx^2 + cx + 1$ and $Q(x) = x^4 + cx^3 + bx^2 + ax + 1$, with a, b, c real numbers and $a \neq c$. Find conditions on a, b, and c so that $P(x)$ and $Q(x)$ have two common roots. In this case, solve the equations $P(x) = 0$, $Q(x) = 0$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Robert Bilinski, Collège Montmorency, Laval, QC; Christopher J. Bradley, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Skotidas Sotirios, Karditso, Greece. We give the write-up by Díaz-Barrero.

The common roots of $P(x)$ and $Q(x)$ are among the roots of the polynomial

$$P(x) - Q(x) = (a - c)x(x^2 - 1) .$$

Thus, they are among -1, 0, and 1. Since $P(0) = Q(0) = 1$, the common roots must be -1 and 1. Substituting these values of x into the equations $P(x) = 0$ and $Q(x) = 0$, we get the conditions $a + b + c + 2 = 0$ and $a - b + c - 2 = 0$, from which we get $b = -2$ and $a + c = 0$.

When these conditions are satisfied, $P(x)$ and $Q(x)$ can be written as

$$\begin{aligned} P(x) &= x^4 + ax^3 - 2x^2 - ax + 1 = (x^2-1)(x^2+ax-1)\,, \\ Q(x) &= x^4 - ax^3 - 2x^2 + ax + 1 = (x^2-1)(x^2-ax-1)\,. \end{aligned}$$

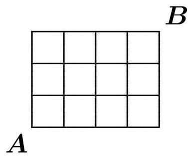
Thus, the roots of $P(x) = 0$ are

$$-1\,,\quad 1\,,\quad \frac{-a-\sqrt{a^2+4}}{2}\,,\quad \text{and}\quad \frac{-a+\sqrt{a^2+4}}{2}\,,$$

and the roots of $Q(x) = 0$ are

$$-1\,,\quad 1\,,\quad \frac{a-\sqrt{a^2+4}}{2}\,,\quad \text{and}\quad \frac{a+\sqrt{a^2+4}}{2}\,.$$

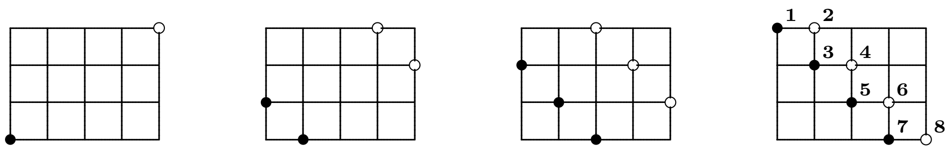
2. The figure shows a street plan of twelve square blocks. A person P goes from point A to point B, and a second person Q goes from B to A. Both of them (P and Q) leave at the same time with the same speed, following shortest paths on the grid. At each corner they choose among the possible streets with equal probability. What is the probability that P meets Q?



Solved by Robert Bilinski, Collège Montmorency, Laval, QC; Pierre Bornsztein, Maisons-Laffitte, France; and Christopher J. Bradley, Bristol, UK. We give the solution by Bilinski.

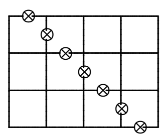
We understand "following shortest paths on the grid" to mean that P can only go up or to the right and Q can only go down or to the left, so that each finishes the trip after travelling 7 blocks in total (4 horizontal and 3 vertical).

Let us equate time with the number of blocks covered. We illustrate the possible positions for both P (black circle) and Q (white circle) between times $t = 0$ (the start) and $t = 3$. The eight vertices containing the white or black circles at time $t = 3$ have been numbered for future reference.



$t = 0$ $t = 1$ $t = 2$ $t = 3$

All 7 potential meetings take place at time $t = 3.5$, at positions shown in the diagram to the right by crossed white circles. For $1 \le i \le 8$, let P_i denote the probability that P is at vertex i at time $t = 3$, and define Q_i similarly. There is only one way P could travel to reach vertex 1 or vertex 7, and there are three possible ways P could reach vertex 3 or vertex 5. Thus, we see that $P_1 = P_7 = \frac{1}{8}$ and $P_3 = P_5 = \frac{3}{8}$. Similarly, $Q_2 = Q_8 = \frac{1}{8}$ and $Q_4 = Q_6 = \frac{3}{8}$.



$t = 3.5$

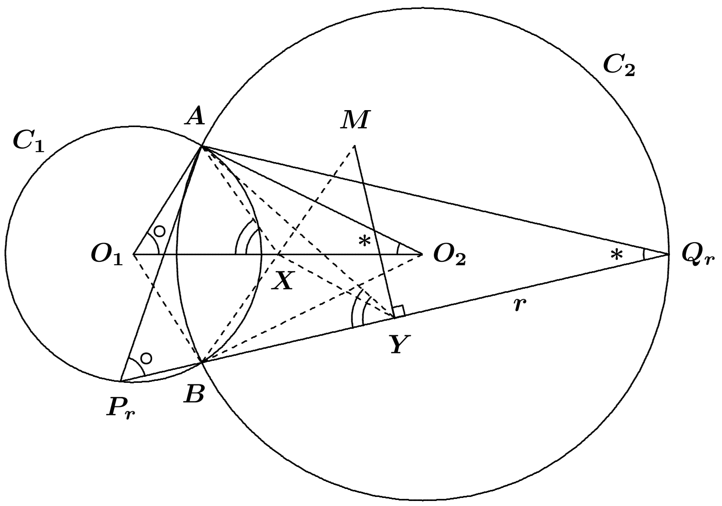
For $1 \le i \le 8$ and $1 \le j \le 8$, let P_{ij} denote the probability that P moves from vertex i at time $t = 3$ to vertex j at time $t = 4$, and define Q_{ij} similarly. At vertices 3, 5, and 7, the probabilities of P going up or across are the same, whereas P can only go across from vertex 1. It follows that $P_{12} = \frac{1}{8}$, $P_{32} = P_{34} = P_{54} = P_{56} = \frac{1}{2}\cdot\frac{3}{8} = \frac{3}{16}$, and $P_{76} = P_{78} = \frac{1}{2}\cdot\frac{1}{8} = \frac{1}{16}$. By a similar argument, we obtain $Q_{87} = \frac{1}{8}$, $Q_{67} = Q_{65} = Q_{45} = Q_{43} = \frac{3}{16}$, and $Q_{23} = Q_{21} = \frac{1}{16}$.

Then the probability that P meets Q is:

$$\begin{aligned} p &= P_{12}\cdot Q_{21} + P_{32}\cdot Q_{23} + P_{34}\cdot Q_{43} + \cdots + P_{78}\cdot Q_{87} \\ &= \tfrac{1}{8}\cdot\tfrac{1}{16} + \tfrac{3}{16}\cdot\tfrac{1}{16} + 3\cdot\tfrac{3}{16}\cdot\tfrac{3}{16} + \tfrac{1}{16}\cdot\tfrac{3}{16} + \tfrac{1}{16}\cdot\tfrac{1}{8} \\ &= \tfrac{1}{128} + \tfrac{3}{256} + 3\cdot\tfrac{9}{256} + \tfrac{3}{256} + \tfrac{1}{128} \;=\; \tfrac{37}{256} \approx 14.45\% \,. \end{aligned}$$

3. Circles C_1 and C_2 intersect at points A and B. A line r through B intersects C_1 and C_2 again at points P_r and Q_r, respectively. Prove that there is a point M, which depends only on C_1 and C_2, such that the perpendicular bisector of P_rQ_r passes through M.

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Bristol, UK; Toshio Seimiya, Kawasaki, Japan; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Seimiya's solution.



Let O_1 and O_2 be the centres of C_1 and C_2, respectively. Then O_1O_2 is the perpendicular bisector of AB. We have $\triangle AO_1O_2 \sim \triangle AP_rQ_r$, since

$$\begin{aligned} \angle AO_1O_2 &= \tfrac{1}{2}\angle AO_1B \;=\; \angle AP_rB \;=\; \angle AP_rQ_r \\ \text{and} \quad \angle AO_2O_1 &= \tfrac{1}{2}\angle AO_2B \;=\; \angle AQ_rB \;=\; \angle AQ_rP_r \,. \end{aligned}$$

Let X and Y be the mid-points of O_1O_2 and P_rQ_r, respectively. Since O_1O_2 and P_rQ_r are corresponding sides in the similar triangles AO_1O_2 and AP_rQ_r, we get $\triangle AXO_1 \sim \triangle AYP_r$, and hence, $\angle AXO_1 = \angle AYP_r$.

Thus, $\angle AYB = \angle AYP_r = \angle AXO_1 = \frac{1}{2}\angle AXB$. Since $AX = BX$ and $\angle AXB = 2\angle AYB$, we see that X is the circumcentre of $\triangle AYB$. Then $AX = BX = YX$.

Let M be the reflection of B with respect to X. Then X is the mid-point of BM. Since $MX = BX = YX$, we have $\angle BYM = 90^\circ$. Thus, the perpendicular bisector of P_rQ_r passes through the fixed point M.

4. For any integer x, let $\lfloor x \rfloor$ denote the integer part of x. Find the largest integer N satisfying the following conditions:

(a) $\lfloor \frac{N}{3} \rfloor$ has three identical digits, and

(b) $\lfloor \frac{N}{3} \rfloor$ is the sum of n consecutive positive integers starting at 1; that is, there is a positive integer n such that

$$\left\lfloor \frac{N}{3} \right\rfloor = 1 + 2 + \cdots + (n-1) + n\,.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Christopher J. Bradley, Bristol, UK; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

The answer is $N = 2000$. Let $N = 3k + r$ where k, $r \in \mathbb{N} \cup \{0\}$ and $0 \le r \le 2$. From condition (a) we have $k = \left\lfloor \frac{N}{3} \right\rfloor = a \cdot 111$, where a is a decimal digit different from 0. From condition (b) we get $k = n(n+1)/2$ for some $n \in \mathbb{N}$. Since $k \le 999$, we have $n^2 < n(n+1) = 2k \le 1998$, which implies that $n < \sqrt{1998} \approx 44.69$. Hence, $n \le 44$.

Direct computations show that when $n = 44$, 43, 42, 41, 40, 39, 38, and 37, the corresponding values of k are 990, 946, 903, 861, 820, 780, 741, and 703, none of which has three identical digits, while for $n = 36$ we have $k = 666$. Therefore, it follows that $N = 3 \times 666 + 2 = 2000$.

5. Four points are placed in a square of side 1. Show that the distance between some two of them is less than or equal to 1.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

First note that the maximum distance between two points in a square (in its interior or on its boundary) is $\sqrt{2}$.

Let A, B, C, and D be four points placed in a square of side 1. Let $\mathcal{C}$ be the convex hull of $\{A, B, C, D\}$. Clearly, $\mathcal{C}$ is contained in the square.

Case 1. $\mathcal{C}$ is a line segment.

Without loss of generality, we may assume that A, B, C, and D are collinear in that order. Thus, $AB + BC + CD = AD \le \sqrt{2}$, so that $\min\{AB, BC, CD\} \le \frac{\sqrt{2}}{3} < 1$, and we are done with this case.

Case 2. $\mathcal{C}$ is a triangle.

Without loss of generality, we may assume that D is inside or on the boundary of ABC, and that $\theta = \angle CDA = \max\{\angle ADB, \angle BDC, \angle CDA\}$.

Since $\angle ADB + \angle BDC + \angle CDA = 2\pi$, we see that $\theta \geq \frac{2\pi}{3}$ and $\cos\theta \leq -\frac{1}{2}$. If AD and DC are greater than 1, then, from the Law of Cosines,

$$AC^2 \;=\; AD^2 + DC^2 - 2AD \cdot DC\cos\theta \;\geq\; 1 + 1 + AD \cdot DC \;>\; 3\,,$$

which leads to $AC > \sqrt{3} > \sqrt{2}$, a contradiction.

Then $AD \leq 1$ or $DC \leq 1$, and we are done again. In fact, the above argument can easily be strengthened to show that $AD < 1$ or $CD < 1$.

Case 3. $\mathcal{C}$ is a (convex) quadrilateral, say $ABCD$.

Since the convex quadrilateral $ABCD$ is contained in the square, it follows that its perimeter is not greater than the perimeter of the square; that is, $AB + BC + CD + DA \leq 4$. Therefore, $\min\{AB,\, BC,\, CD,\, DA\} \leq 1$, and we are done.

We note that equality occurs if and only if $ABCD$ is the square itself.

6. Show that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f\big(f(n)\big) = n + 1$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Skotidas Sotirios, Karditso, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

Suppose for the purpose of contradiction that such a function exists. Let m be the natural number defined by $f(0) = m$. Then $f(k) = m + k$ for $k = 0$. Let $k \geq 0$ be any natural number such that $f(k) = m + k$. Then $f(m+k) = f(f(k)) = k+1$. Thus, $f(k+1) = f(f(m+k)) = m+k+1$. It follows by induction that $f(k) = m+k$ for all $k \in \mathbb{N}$. However, this yields $f(m) = 2m$, while $f(0) = m$ yields $f(m) = f(f(0)) = 1$, a contradiction.

Note: Bornsztein points out that the well-known problem #4 of the 1987 IMO was to prove that there is no function $f : \mathbb{N} \to \mathbb{N}$ such that $f\big(f(n)\big) = n + 1987$. Any solution of this problem contains a remark that the result holds just because 1987 is odd.

Next we give a solution to one of the problems of the Taiwan (ROC) Mathematical Olympiad 2000 given [2004 : 203].

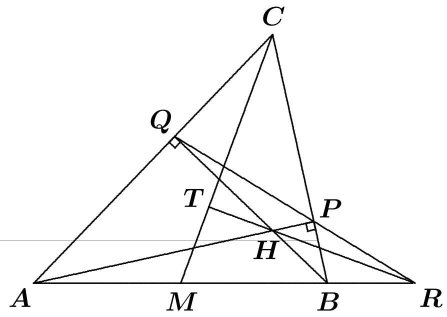
2. In an acute triangle ABC with $|AC| > |BC|$, let M be the mid-point of AB. Let AP be the altitude from A and BQ be the altitude from B. These altitudes meet at H, and the lines AB and PQ meet at R. Prove that the two lines RH and CM are perpendicular.

Solved by Christopher J. Bradley, Bristol, UK; and D.J. Smeenk, Zaltbommel, the Netherlands. We give the solution by Bradley.

We will use vectors with the circumcentre O as origin and $\mathrm{x} = \overrightarrow{OA}$, $\mathrm{y} = \overrightarrow{OB}$, $\mathrm{z} = \overrightarrow{OC}$. It is known that QP meets AB at R, where

$$\overrightarrow{OR} = \frac{-(a\cos B)\mathrm{x} + (b\cos A)\mathrm{y}}{b\cos A - a\cos B}$$

and $\overrightarrow{OH} = \mathrm{x} + \mathrm{y} + \mathrm{z}$. We also have $\overrightarrow{CM} = \frac{1}{2}(\mathrm{x} + \mathrm{y}) - \mathrm{z}$. Hence,



$$\begin{aligned}\overrightarrow{RH}&(b\cos A - a\cos B)\\ &= (b\cos A)\mathrm{x} - (a\cos B)\mathrm{x}\\ &\quad + (b\cos A - a\cos B)\mathrm{z}\,.\end{aligned}$$

Thus, if ρ is the radius of the circumcircle of $\triangle ABC$, we get

$$\begin{aligned}2(b\cos A &- a\cos B)\overrightarrow{RH}\cdot\overrightarrow{CM}\\ &= (-b\cos A + a\cos B)\rho^2 + (b\cos A - a\cos B)\mathrm{x}\cdot\mathrm{y}\\ &\qquad + (b\cos A + a\cos B)(\mathrm{y} - \mathrm{x})\cdot\mathrm{z}\\ &= \rho^2(a\cos B - b\cos A)(1 - \cos 2C) + c(\cos 2A - \cos 2B)\\ &= 2\rho^2 c(a\cos B - b\cos A)\sin C\big((\sin A\cos B - \sin B\cos A)\\ &\qquad + \sin(B - A)\big)\\ &= 0\,.\end{aligned}$$

Hence, $RH \perp CM$.

Next we look at solutions to problems of the 2000 Hungarian National Olympiad given [2004 : 204].

First Round

1. Let x, y, and z denote positive real numbers, each less than 4. Prove that at least one of the numbers $\frac{1}{x} + \frac{1}{4-y}$, $\frac{1}{y} + \frac{1}{4-z}$, and $\frac{1}{z} + \frac{1}{4-x}$ is greater than or equal to 1.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's version.

From the HM–AM Inequality, we have

$$\frac{1}{2}\left(\frac{1}{x} + \frac{1}{4-x}\right) \geq \frac{2}{x + 4 - x} = \frac{1}{2}\,;$$

whence, $\frac{1}{x} + \frac{1}{4-x} \geq 1$. Now, if the three numbers

$$\frac{1}{x} + \frac{1}{4-y}\,,\quad \frac{1}{y} + \frac{1}{4-z}\,,\quad \text{and}\quad \frac{1}{z} + \frac{1}{4-x}$$

were all less than 1, their sum S would be less than 3. However,

$$S \;=\; \left(\frac{1}{x}+\frac{1}{4-x}\right)+\left(\frac{1}{y}+\frac{1}{4-y}\right)+\left(\frac{1}{z}+\frac{1}{4-z}\right) \;\geq\; 3\,.$$

This contradiction proves the requested result.

2. Find the integer solutions of $5x^2 - 14y^2 = 11z^2$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.

Clearly, $x = y = z = 0$ is a solution. We show that this is the only solution.

If $x = 0$, then $y = z = 0$. Next note that x and z must have the same parity. Let us now work modulo 8. If x and z are both odd, then $x^2 \equiv z^2 \equiv 1$, which implies that $5x^2 - 11z^2 \equiv 2$. Since $14y^2 \equiv 14 \equiv 6$ if y is odd and $14y^2 \equiv 0$ if y is even, we have a contradiction. Thus, x and z are both even.

Suppose there are solutions in which $x \neq 0$. Let x_0 denote the least positive integer for which there exist $y,\ z \in \mathbb{Z}$ such that (x_0, y, z) is a solution. Setting $x_0 = 2x_1$, $z = 2z_1$, we get $20x_1^2 - 14y^2 = 44z_1^2$, or $10x_1^2 - 7y^2 = 22z_1^2$, which implies that y is even. Setting $y = 2y_1$, we then have $10x_1^2 - 28y_1^2 = 22z_1^2$, or $5x_1^2 - 14y_1^2 = 11z_1^2$, showing that (x_1, y_1, z_1) is also a solution. This is a contradiction, since $0 < x_1 < x_0$.

4. If $1 \leq m \leq n$, prove that m is a divisor of

$$n\left(\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{m-1}\binom{n}{m-1}\right)\,.$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

Substituting $\binom{n-1}{0}$ for $\binom{n}{0}$ and using systematically the law of formation of Pascal's Triangle, we easily see that

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{m-1}\binom{n}{m-1} \;=\; (-1)^{m-1}\binom{n-1}{m-1}\,.$$

Thus,

$$\begin{aligned} & n\left(\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^{m-1}\binom{n}{m-1}\right) \\ &\qquad = \;(-1)^{m-1}n\binom{n-1}{m-1} \;=\; (-1)^{m-1}m\binom{n}{m}\,. \end{aligned}$$

The result follows.

Final Round

1. Let c denote a positive integer, and let c_1, c_3, c_7, and c_9 be the number of divisors of c which have last digit 1, 3, 7, and 9, respectively (in the decimal system). Prove that $c_3 + c_7 \leq c_1 + c_9$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

The result is clear for $c = 1$. Thus, we assume that $c \geq 2$. Let

$$c \;=\; p_1^{\alpha_1} \cdots p_k^{\alpha_k} \tag{1}$$

be the prime factorization of c. Since we are interested only in the divisors of n which are odd and not divisible by 5, we may assume that $\gcd(c, 10) = 1$.

Let A be the set of divisors of c which have last digit 3 or 7, and let B be the set of divisors of c which have last digit 1 or 9. Thus, $|A| = c_3 + c_7$ and $|B| = c_1 + c_9$.

Let a be the number of primes which appear in (1), counted with their multiplicities, and which belong to A, and let b be the number of such primes (with multiplicity) belonging to B.

It is easy to verify that:

- The product of any two elements from A belongs to B.
- The product of any two elements from B belongs to B.
- The product of an element from A and an element from B belongs to A.

It follows that $d \in A$ if and only if the prime decomposition of d contains an odd number of primes which belong to A; that is, $d = D_A \times D_B$, where D_A is the product of an odd number of primes which belong to A (so that there are exactly $(a+1)/2$ choices for D_A) and D_B is the product of an arbitrary number of primes which belong to B (so that there are exactly $b+1$ choices for D_B). Therefore, $|A| = (b+1)\left\lfloor \dfrac{a+1}{2} \right\rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part.

In the same way, we have $|B| = (b+1)\left\lfloor \dfrac{a+2}{2} \right\rfloor$. Then $|A| \leq |B|$ and we are done.

That completes the *Corner* for this issue. Send me your nice solutions and generalizations.

BOOK REVIEWS

John Grant McLoughlin

Hungary–Israel Mathematics Competition: The First Twelve Years
By S. Gueron, AMT Publishing, Canberra, 2004
ISBN 1-87642-015-4, paperback, 181 pages, AUD$35.20.
Reviewed by ***Stan Wagon****, Macalester College, St. Paul, MN, USA.*

This book, like any collection of problems at the Olympiad level, is excellent for students preparing for Olympiad-type contests. But the book should also appeal to many other readers. There is excellent variety in the problems, and there is a section with hints, as well as the complete solutions.

I found that a good number of the problems were interesting to me. For example, Problem 1993–1 is very nice: find all rationals b/a so that the number one gets when one writes the digits of a followed by the digits of b, but with a decimal point between them, exactly equals the given rational. It seems somewhat surprising that there is exactly one rational with this property, and the reader is encouraged to find it. Of course, this problem is screaming out for an investigation into other bases. All I found was the rational $18/4$ which works in base 6, because 18 is $30_{(6)}$ and $4.30_{(6)}$ equals the rational $9/2$. But this lacks elegance in that the rational is not in lowest terms. Michael Schweitzer (Berlin) was able to resolve this completely, obtaining the following nice result: A base B admits a rational b/a that is in lowest terms and equals $a.b$ if and only if B has the form $n(n^2+1)$ with $n \geq 2$; and, for such bases, there is exactly one such rational.

One unusual feature of this event is that, for 9 years, there has been a team competition that takes place on a separate day from the individual contest. Here the questions are all on a preannounced theme, thus allowing deeper exploration than in a traditional contest. This book includes those nine years of problems and solutions as well.

I found the theme in 1991 appealing. It concerns polynomials $p(x)$ which are quadratic with integer coefficients. They include the following:

1. Show that a non-square, monic $p(x)$ cannot take on infinitely many square values (for x an integer).

2. Show that, for any positive integer N, there is a non-square, monic $p(x)$ for which there are N square values $p(n)$.

3. Show that there is a non-square $p(x)$ that takes on infinitely many square values.

4. Find a non-square and monic $p(x)$ that is square for four consecutive integer values of x.

The book has a fair number of typographical errors, but most are not significant. There are no references or sources given for any of the problems. Presumably most are original.

Mathematical Adventures for Students and Amateurs
Edited by David F. Hayes and Tatiana Shubin, published by the Mathematical Association of America (Spectrum Series), 2004
ISBN 0-88385-548-8, paperbound, 291+xi pages, US$38.50.
Reviewed by ***David G. Poole****, Trent University, Peterborough, ON.*

The Bay Area Mathematical Adventures (BAMA) is a series of mathematical talks for bright middle and high school students hosted alternately by Santa Clara University and San Jose State University. In existence since 1998, the program has featured six talks per year by many well-known mathematicians—Ronald Graham, Carl Pomerance, Karl Rubin, Joseph Gallian, Jean Pedersen, Sherman Stein, and Robin Wilson, to name a few. This book is a collection of nineteen of the talks presented at BAMA. The essays are not transcripts of the talks but rather presentations of the mathematics contained therein. For a list of all the talks that have been presented as part of BAMA, see the program's website:

`http://www.mathcs.sjsu.edu/faculty/dfhayes/bama.html`

The range of topics is diverse and accessible. The essays are organized under five headings: general, number theory, combinatorics and probability, geometry and topology, applications and history. Graham discusses the mathematics of juggling; Rubin introduces elliptic curves; Gallian describes mathematical detective work needed to decode drivers' licenses; Pedersen addresses the question of how many bounded and unbounded regions in space result when the planes of the Platonic solids are extended in space; Stein revisits Archimedes' analysis of the equilibrium of a floating paraboloid; Wilson presents an account of the mathematics of Charles Lutwidge Dodgson (Lewis Carroll) in the form of a play inspired by Alice in Wonderland. In addition there are talks on communicating with extraterrestrials, the space shuttle Challenger disaster, the mathematics of map-making, collaboration between mathematicians and computers, and many more.

Since the BAMA talks are aimed at talented students, the level of the mathematical exposition is higher than one might initially expect. The required level of mathematical maturity varies from talk to talk, but all of the talks are self-contained and will be accessible to a bright high school student. Although "user-friendly", the mathematics found here is very rigorous, with precisely stated definitions, theorems and conjectures. In addition, most of the articles provide references for further reading or links to related websites. As a result, this book is suitable for a wide variety of audiences. High school students and teachers, college and university students, mathematics educators and mathematicians will all find something of interest here.

This book is an excellent example of how to present stimulating topics in mathematics that will inspire both students and teachers. Good students will be able to read the articles independently; teachers may wish to dip into the book to find enrichment material for their classes. As the book's title suggests, the theme is exploration. The articles do a very good job of conveying the sense of wonder and excitement one has when plunged into a rich mathematical world.

Some Inversion Formulas for Sums of Quotients

Natalio H. Guersenzvaig and Michael Z. Spivey

In this note we establish some formulas for certain sums of quotients of a positive integer n, which are closely related to an identity established by Préville-Ratelle in Problem M40 of the April 2003 issue of this journal [1]. We also establish some elementary facts that are not well known about quotients and remainders. Our main result is the following theorem.

Theorem 1. Let n and k be any positive integers with $k \le n$. Then

$$\sum_{d=1}^{k} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=\lfloor \frac{n}{k} \rfloor + 1}^{n} \left\lfloor \frac{n}{d} \right\rfloor = k \left\lfloor \frac{n}{k} \right\rfloor . \qquad (F_k)$$

The first sum is clearly the sum of the quotients of n from $\lfloor n/1 \rfloor$ through $\lfloor n/k \rfloor$. We show below that the second sum is the sum of the quotients of n that are equal to one of $\{1, \ldots, k-1\}$. The "inversion" aspect of the formula is that the quotient sums are being taken in opposite directions.

As an illustration, here are the quotients for $n = 15$:

divisor d	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
quotient $\lfloor n/d \rfloor$	15	7	5	3	3	2	2	1	1	1	1	1	1	1	1

Theorem 1 is trivial for $k = 1$. When $k = 2$, for example, the formula says that $15 + 7 - (1+1+1+1+1+1+1+1) = 2\lfloor \frac{15}{2} \rfloor$; whereas for $k = 4$, the formula gives $15+7+5+3-(1+1+1+1+1+1+1+1+2+2+3+3) = 4\lfloor \frac{15}{4} \rfloor$.

In order to prove Theorem 1, we use the following result.

Lemma 1. If d and i are positive integers not exceeding n, then

$$\left\lfloor \frac{n}{i} \right\rfloor = d \qquad \Longleftrightarrow \qquad \left\lfloor \frac{n}{d+1} \right\rfloor < i \le \left\lfloor \frac{n}{d} \right\rfloor .$$

Proof: We first suppose that $\lfloor n/i \rfloor = d$. By definition of the floor function, d is the unique integer such that $d \le n/i < d+1$. Inverting the inequality yields $n/(d+1) < i \le n/d$. Certainly, $\lfloor n/(d+1) \rfloor \le n/(d+1)$. On the other hand, since i is an integer, we get $i \le \lfloor n/d \rfloor$ from $i \le n/d$. Hence, $\lfloor n/(d+1) \rfloor < i \le \lfloor n/d \rfloor$. Since these steps are reversible the proof is complete. ∎

Proof of Theorem 1: From Lemma 1, we get at once

$$\sum_{d=\lfloor \frac{n}{k+1} \rfloor+1}^{\lfloor \frac{n}{k} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor = k \left(\left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n}{k+1} \right\rfloor \right) .$$

Letting $Q(n,k) = \lfloor n/k \rfloor - \lfloor n/(k+1) \rfloor$, we have

$$\sum_{d=\lfloor \frac{n}{k} \rfloor+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{j=1}^{k-1} \sum_{d=\lfloor \frac{n}{j+1} \rfloor+1}^{\lfloor \frac{n}{j} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{j=1}^{k-1} jQ(n,j) . \qquad (1)$$

On the other hand, it is easy to verify that, for $d = 2, 3, \ldots, k$, we have

$$(d-1)Q(n,d-1) + d \left\lfloor \frac{n}{d} \right\rfloor - (d-1) \left\lfloor \frac{n}{d-1} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor .$$

Part of the left side of this equation telescopes when we sum d from 2 to k. Therefore, using (1), we obtain

$$\sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=1}^{k-1} dQ(n,d) + k \left\lfloor \frac{n}{k} \right\rfloor - n = \sum_{d=\lfloor \frac{n}{k} \rfloor+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor + k \left\lfloor \frac{n}{k} \right\rfloor - n .$$

Adding n to each side of this expression completes the proof. ■

Remark. Expression (1) and Lemma 1 show that the second sum in the statement of Theorem 1 is, in fact, the sum of the quotients of n that are equal to one of $\{1, 2, \ldots, k-1\}$.

Next we establish some facts about quotients that are a direct consequence of Lemma 1.

Corollary 1. Let d and i be positive integers not exceeding n.

(a) $d \leq \left\lfloor \dfrac{n}{\lfloor n/d \rfloor} \right\rfloor$.

(b) $\left\lfloor \dfrac{n}{\lfloor n/\lfloor n/d \rfloor \rfloor} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor$.

(c) $\left\lfloor \dfrac{n}{\lfloor n/i \rfloor} \right\rfloor = \left\lfloor \dfrac{n}{\lfloor n/d \rfloor} \right\rfloor$ if and only if $\left\lfloor \dfrac{n}{i} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor$.

(d) $\left\lfloor \dfrac{n}{i} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor$ if and only if $\left\lfloor \dfrac{n}{\lfloor n/d \rfloor + 1} \right\rfloor < i \leq \left\lfloor \dfrac{n}{\lfloor n/d \rfloor} \right\rfloor$.

(e) $\left\lfloor \dfrac{n}{\lfloor n/i \rfloor} \right\rfloor = d$ if and only if $\left\lfloor \dfrac{n}{i} \right\rfloor = \left\lfloor \dfrac{n}{d} \right\rfloor > \left\lfloor \dfrac{n}{d+1} \right\rfloor$.

Proof: We use freely the fact that $\lfloor n/d \rfloor \geq \lfloor n/(d+1) \rfloor$ for any $d \in \mathbb{N}$.

(a) Let $k = \lfloor n/\lfloor n/d \rfloor \rfloor$. By Lemma 1, we see that $\lfloor n/(k+1) \rfloor < \lfloor n/d \rfloor$. Thus, $k+1 > d$; that is, $k \geq d$.

(b) We use (a). Replacing d by $\lfloor n/d \rfloor$, we get $\lfloor n/\lfloor n/\lfloor n/d \rfloor \rfloor \rfloor \geq \lfloor n/d \rfloor$. On the other hand, we have $\lfloor n/\lfloor n/\lfloor n/d \rfloor \rfloor \rfloor \leq \lfloor n/d \rfloor$, because $\lfloor n/d \rfloor$ is a decreasing function of d.

(c) This follows at once from (b).

(d) This follows immediately from Lemma 1 by replacing d with $\lfloor n/d \rfloor$.

(e) Case $i = d$ follows from Lemma 1 by replacing i with $\lfloor n/d \rfloor$. Now we prove the general case. Suppose that $\lfloor n/\lfloor n/i \rfloor \rfloor = d$. Thus,

$$\left\lfloor \frac{n}{\lfloor n/\lfloor n/i \rfloor \rfloor} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor .$$

From (b) we have $\lfloor n/i \rfloor = \lfloor n/d \rfloor$, which implies that $\lfloor n/\lfloor n/d \rfloor \rfloor = d$. Hence, from the case $i = d$, we get $\lfloor n/d \rfloor > \lfloor n/(d+1) \rfloor$. Now suppose that $\lfloor n/i \rfloor = \lfloor n/d \rfloor > \lfloor n/(d+1) \rfloor$. Then $\lfloor n/\lfloor n/i \rfloor \rfloor = \lfloor n/\lfloor n/d \rfloor \rfloor$. Using the case $i = d$ again, we obtain $\lfloor n/\lfloor n/d \rfloor \rfloor = d$. Hence, $\lfloor n/\lfloor n/i \rfloor \rfloor = d$. ∎

We can also reformulate Corollary 1 in terms of $n \bmod d = n - d\lfloor n/d \rfloor$, the remainder on division of n by d. For example, reformulation of (a) and case $i = d$ of (e) yields the following result:

Corollary 2. Let d and n be positive integers with $d \leq n$. Then

(a) $n \bmod \lfloor n/d \rfloor \leq n \bmod d$.

(b) $n \bmod \lfloor n/d \rfloor < n \bmod d$ if and only if $\lfloor n/d \rfloor = \lfloor n/(d+1) \rfloor$.

Furthermore, from Theorem 1 and Lemma 1, we get some unusual expressions for $n \bmod k$ and, hence, a criterion for divisibility of n by k.

Corollary 3. Let d and n be positive integers with $d \leq n$. Then

(a) $$n \bmod k = \sum_{d=k+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{\lfloor \frac{n}{k} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=\lfloor \frac{n}{k} \rfloor + 1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor.$$

(b) $n \bmod k = \frac{1}{2}(F(k) + F(\lfloor n/k \rfloor))$, where

$$F(k) = \sum_{d=k+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor .$$

Moreover, $n \bmod k = F(k)$ if and only if $\lfloor n/(k+1) \rfloor < k = \lfloor n/k \rfloor$.

(c) $k \mid n$ if and only if $$\frac{n}{k} = \sum_{d=\lfloor \frac{n}{k} \rfloor + 1}^{n} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=2}^{k-1} \left\lfloor \frac{n}{d} \right\rfloor.$$

Proof: (a) We get the second identity by partitioning the sum $\sum_{d=2}^{n} \left\lfloor \frac{n}{d} \right\rfloor$ in two obvious ways. To prove the first identity, we add and subtract $\sum_{d=1}^{\lfloor \frac{n}{k} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor$ on the left side of (F_k) to obtain the following equivalent formula:

$$\sum_{d=1}^{\lfloor n/k \rfloor} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=k+1}^{n} \left\lfloor \frac{n}{d} \right\rfloor = k \left\lfloor \frac{n}{k} \right\rfloor .$$

Thus, replacing $k\lfloor n/k \rfloor$ by $n - n \bmod k$, cancelling n, and multiplying both sides of the equation by -1, we complete the proof of (a).

(b) The formula for $n \bmod k$ holds because the right member is the arithmetic mean of the second and third member of (a). From this, we have $n \bmod k = F(k)$ if and only if $F(k) = F(\lfloor n/k \rfloor)$. Partitioning the sum $\sum_{d=2}^{n} \left\lfloor \frac{n}{d} \right\rfloor$ as was done in (a), we get $\sum_{d=2}^{k} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=2}^{\lfloor \frac{n}{k} \rfloor} \left\lfloor \frac{n}{d} \right\rfloor$. Since each quotient is positive, we have $\lfloor n/k \rfloor = k$. Since these steps are reversible we have proved that $n \bmod k = F(k)$ if and only if $\lfloor n/k \rfloor = k$. Then, from Lemma 1, after we replace i and d by k, the proof of (b) is complete.

(c) This follows at once from (a) using the rightmost expression for $n \bmod k$. ∎

Next we establish a generalization of Theorem 1 that clearly shows the process of inversion of the sums involved.

Corollary 4. Let n, j, and k be positive integers with $j \le k \le n$. Then

$$\sum_{d=j+1}^{k} \left\lfloor \frac{n}{d} \right\rfloor - \sum_{d=\lfloor n/k \rfloor + 1}^{\lfloor n/j \rfloor} \left\lfloor \frac{n}{d} \right\rfloor = k \left\lfloor \frac{n}{k} \right\rfloor - j \left\lfloor \frac{n}{j} \right\rfloor . \qquad (F_{j,k})$$

Proof: This follows at once from Theorem 1 by subtracting (F_j) from (F_k). ∎

Remark. Theorem 1 and Corollary 4 are logically equivalent, because (F_k) follows from $(F_{1,k})$.

We have now generalized Préville-Ratelle's identity, since it is precisely Corollary 4 for the case when j and k both divide n.

Corollary 5. Let n, j, and k be positive integers with $j \le k \le n$. Then

$$\sum_{d=j+1}^{k} \left\lfloor \frac{n}{d} \right\rfloor = \sum_{d=\lfloor n/k \rfloor + 1}^{\lfloor n/j \rfloor} \left\lfloor \frac{n}{d} \right\rfloor \iff n \bmod j = n \bmod k .$$

Proof: If one replaces $k\lfloor n/k \rfloor - j\lfloor n/j \rfloor$ with $(n \bmod j) - (n \bmod k)$ in $(F_{j,k})$, then the result follows at once from Corollary 4. ∎

Concluding remark. Préville-Ratelle's solution to Problem M40 gives a nice graphical interpretation of $(F_{j,k})$ for the case when j and k are divisors of n. Can the reader generalize that graphical approach to prove $(F_{j,k})$ for arbitrary j and k with $j \leq k$?

References

[1] Louis-François Préville-Ratelle, Solution to Problem M40, *Crux Mathematicorum with Mathematical Mayhem* 29:3 (2003), pp. 140–141.

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PROBLEMS

Solutions to problems in this issue should arrive no later than **1 September 2006**. *An asterisk* (⋆) *after a number indicates that a problem was proposed without a solution.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English. In the solutions section, the problem will be stated in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

3056. Correction. *Proposed by Paul Bracken, University of Texas, Edinburg, TX, USA.*

If $f(x)$ is a non-negative, continuous, concave function on the closed interval $[0,1]$ such that $f(0)=1$, show that

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \;\le\; \left[\int_0^1 f(x)\,dx\right]^2 .$$

3101. *Proposed by K.R.S. Sastry, Bangalore, India.*

The two distinct cevians AP and AQ of $\triangle ABC$ satisfy the equation $AQ^2 = AP^2 + |AC - AB|^2$.

(a) If $BP = CQ$, show that AP bisects $\angle BAC$.

(b)★ If AP bisects $\angle BAC$, prove or disprove that $BP = CQ$.

3102. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let D be the mid-point of the side BC of $\triangle ABC$. Let E and F be the projections of B onto AC and C onto AB, respectively. Let P be the point of intersection of AD and EF. Show that, if $AD = \frac{\sqrt{3}}{2}BC$, then P is the mid-point of AD.

3103. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be an acute-angled triangle with circumcentre O. Let the lines AO, BO, and CO meet the circles BCO, CAO, and ABO for the second time at A', B', and C', respectively. Let $|XYZ|$ denote the perimeter and $[XYZ]$ the area of the triangle XYZ. Prove that

(a) $\dfrac{BC}{|BCA'|} + \dfrac{CA}{|CAB'|} + \dfrac{AB}{|ABC'|} \;=\; 1$;

(b) $[BCO]\cdot[BCA'] + [CAO]\cdot[CAB'] + [ABO]\cdot[ABC'] \;=\; [ABC]^2$.

3104. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$, let D, E, and F be the mid-points of the sides BC, CA, and AB, respectively. Show that, if $AD = \frac{\sqrt{3}}{2}BC$, then $\angle BEC = \angle AFC$.

3105. *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a, b, c, d be positive real numbers.

(a) Prove that the following inequality holds for $0 \le x \le (5-\sqrt{17})/2$ and also for $x = 1$:

$$\sum_{\text{cyclic}} \frac{a}{a+(3-x)b+xc} \;\ge\; 1\,.$$

(b)★ Prove the above inequality for $0 \le x \le 1$.

3106. *Proposed by Mihály Bencze, Brasov, Romania.*

Prove the following identities:

(a) $$\sum_{k=1}^{n}\sum_{i=1}^{k+1} \frac{\binom{k}{i-1}^2\binom{2k}{k}}{2^{2k}\binom{2k}{2i-2}(2i-1)} \;=\; \frac{2^{2n+1}}{\binom{2n+1}{n}} - 2.$$

(b) $$\sum_{k=1}^{n}\sum_{i=1}^{k+1} \frac{\binom{k}{i-1}^2\binom{2k}{k}}{2^{2k}\binom{2k}{2i-2}i} \;=\; \frac{(2n+3)\binom{2n+2}{n+1}}{2^{2n+1}} - 3.$$

3107. *Proposed by Victor Oxman, Western Galilee College, Israel.*

Let $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles with $A_1C_1 = A_2C_2$. Suppose that the interior angle bisectors A_1D_1 and A_2D_2 are equal.

(a) If the altitudes B_1H_1 and B_2H_2 are equal, show that the triangles are congruent.

(b) If the interior angle bisectors B_1E_1 and B_2E_2 are equal, show that the triangles are congruent.

3108. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle in which angles B and C are both acute. Let H be the point on side BC such that $AH \perp BC$. Let r, r_1, and r_2 be the incircles of triangles ABC, ABH, and AHC, respectively. Show that $r + r_1 + r_2 - AH$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

3109. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle in which angles B and C are both acute, and let a, b, c be the lengths of the sides opposite the vertices A, B, C, respectively. If h_a is the altitude from A to BC, prove that $\frac{1}{h_a^2} - \left(\frac{1}{b^2} + \frac{1}{c^2}\right)$ is positive, negative, or zero according as $\angle A$ is obtuse, acute, or right-angled.

3110. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let m_b be the length of the median to side b in $\triangle ABC$, and define m_c similarly. Prove that $4a^4 + 9b^2c^2 - 16m_b^2m_c^2$ is positive, negative, or zero according as angle A is acute, obtuse, or right-angled.

3111. *Proposed by Mihály Bencze, Brasov, Romania.*

Let a_k, b_k, and c_k be the lengths of the sides opposite the vertices A_k, B_k, and C_k, respectively, in triangle $A_kB_kC_k$, for $k = 1, 2, \ldots, n$. If r_k is the inradius of triangle $A_kB_kC_k$ and if R_k is its circumradius, prove that

$$\begin{aligned} 6\sqrt{3}\left(\prod_{k=1}^{n} r_k\right)^{\frac{1}{n}} &\le \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} c_k\right)^{\frac{1}{n}} \\ &\le 3\sqrt{3}\left(\prod_{k=1}^{n} R_k\right)^{\frac{1}{n}}. \end{aligned}$$

3112★. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let $MABC$ be a tetrahedron, and let M' be any point in the interior of $\triangle ABC$. Denote the area of $\triangle XYZ$ by $[XYZ]$. Prove that

$$\begin{aligned} (MM')^2 &= MA^2\,\frac{[BM'C]}{[ABC]} + MB^2\,\frac{[CM'A]}{[ABC]} + MC^2\,\frac{[AM'B]}{[ABC]} \\ &\quad - \left(AB^2\,\frac{[BM'C][CM'A]}{[ABC]^2} + BC^2\,\frac{[CM'A][AM'B]}{[ABC]^2}\right. \\ &\qquad \left. +\; CA^2\,\frac{[AM'B][BM'C]}{[ABC]^2}\right). \end{aligned}$$

Comment: This result for a tetrahedron is "similar" to Stewart's Theorem for a triangle. If $M' = G$, the centroid of $\triangle ABC$, then the relation becomes

$$MG^2 = \tfrac{1}{3}(MA^2 + MB^2 + MC^2) - \tfrac{1}{9}(AB^2 + BC^2 + CA^2),$$

which is well known.

3113. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle and let a be the length of the side opposite the vertex A. If m_a is the length of the median from A to BC, and if R is the circumradius of $\triangle ABC$, prove that $m_a - R$ is positive, negative, or zero, according as $\angle A$ is obtuse, acute, or right-angled.

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3056. Correction. *Proposé par Paul Bracken, Université du Texas, Edinburg, TX, USA.*

Si $f(x)$ est une fonction continue, concave et non négative sur l'intervalle fermé $[0,1]$ telle que $f(0) = 1$, montrer que

$$2\int_0^1 x^2 f(x)\,dx + \frac{1}{12} \;\le\; \left[\int_0^1 f(x)\,dx\right]^2 .$$

3101. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Les deux céviennes distinctes AP et AQ d'un triangle ABC satisfont l'équation $AQ^2 = AP^2 + |AC - AB|^2$.

(a) Si $BP = CQ$, montrer que AP est une bissectrice de l'angle BAC.

(b)★ Si AP est une bissectrice de l'angle BAC, démontrer ou réfuter l'égalité $BP = CQ$.

3102. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit D le point milieu du côté BC du triangle ABC. Soit respectivement E et F les projections de B sur AC et de C sur AB. Soit P le point d'intersection de AD et EF. Si $AD = \frac{\sqrt{3}}{2}BC$, montrer que P est le point milieu de AD.

3103. *Proposé par Michel Bataille, Rouen, France.*

Soit ABC un triangle acutangle et O le centre du cercle circonscrit. Désignons respectivement par A', B' et C' les points où les droites AO, BO et CO coupent les cercles BCO, CAO et ABO pour la seconde fois. Si $|XYZ|$ dénote le périmètre du triangle XYZ et $[XYZ]$ son aire, montrer que

(a) $\dfrac{BC}{|BCA'|} + \dfrac{CA}{|CAB'|} + \dfrac{AB}{|ABC'|} \;=\; 1$;

(b) $[BCO]\cdot[BCA'] + [CAO]\cdot[CAB'] + [ABO]\cdot[ABC'] \;=\; [ABC]^2$.

3104. *Proposé par D.J. Smeenk, Zaltbommel, Pays-Bas.*

Soit respectivement D, E et F les points milieu des côtés BC, CA et AB du triangle ABC. Montrer que, si $AD = \frac{\sqrt{3}}{2}BC$, alors $\angle BEC = \angle AFC$.

3105. *Proposé par Vasile Cîrtoaje, Université de Ploiesti, Roumanie.*

Soit a, b, c et d quatre nombres réels positifs.

(a) Montrer que, pour $0 \leq x \leq (5-\sqrt{17})/2$ et pour $x = 1$, on a l'inégalité suivante :

$$\sum_{\text{cyclique}} \frac{a}{a+(3-x)b+xc} \geq 1 \,.$$

(b)★ Montrer l'inégalité ci-dessus pour $0 \leq x \leq 1$.

3106. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Établir les identités suivantes :

(a) $$\sum_{k=1}^{n}\sum_{i=1}^{k+1} \frac{\binom{k}{i-1}^2\binom{2k}{k}}{2^{2k}\binom{2k}{2i-2}(2i-1)} = \frac{2^{2n+1}}{\binom{2n+1}{n}} - 2.$$

(b) $$\sum_{k=1}^{n}\sum_{i=1}^{k+1} \frac{\binom{k}{i-1}^2\binom{2k}{k}}{2^{2k}\binom{2k}{2i-2}i} = \frac{(2n+3)\binom{2n+2}{n+1}}{2^{2n+1}} - 3.$$

3107. *Proposé par Victor Oxman, Western Galilee College, Israël.*

Soit $A_1B_1C_1$ et $A_2B_2C_2$ deux triangles tels que $A_1C_1 = A_2C_2$. Supposons que les bissectrices intérieures A_1D_1 et A_2D_2 sont égales.

(a) Si les hauteurs B_1H_1 et B_2H_2 sont égales, montrer que les triangles sont congruents.

(b) Si les bissectrices intérieures B_1E_1 et B_2E_2 sont égales, montrer que les triangles sont congruents.

3108. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit ABC un triangle dont les deux angles B et C sont aigus. Soit H le point du côté BC tel que $AH \perp BC$. Désignons par r, r_1 et r_2 les rayons respectifs des cercles inscrits des triangles ABC, ABH et AHC. Montrer que $r + r_1 + r_2 - AH$ est positif, négatif ou nul suivant que $\angle A$ est obtus, aigu ou droit.

3109. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit ABC un triangle dont les deux angles B et C sont aigus, et soit respectivement a, b et c les longueurs des côtés opposés aux sommets A, B et C. Si h_a est la hauteur de A à BC, montrer que $\frac{1}{h_a^2} - \left(\frac{1}{b^2} + \frac{1}{c^2}\right)$ est positif, négatif ou nul suivant que $\angle A$ est obtus, aigu ou droit.

3110. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit m_b la longueur de la médiane aboutissant sur le côté b du triangle ABC. On définit de même m_c. Montrer que $4a^4 + 9b^2c^2 - 16m_b^2m_c^2$ est positif, négatif ou nul suivant que l'angle A est aigu, obtus ou droit.

3111. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Soit a_k, b_k et c_k les longueurs respectives des côtés opposés aux sommets A_k, B_k et C_k du triangle $A_kB_kC_k$, pour $k = 1, 2, \ldots, n$. Si r_k est le rayon du cercle inscrit du triangle $A_kB_kC_k$ et si R_k est le rayon de son cercle circonscrit, montrer que

$$\begin{aligned} 6\sqrt{3}\left(\prod_{k=1}^{n} r_k\right)^{\frac{1}{n}} &\le \left(\prod_{k=1}^{n} a_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} b_k\right)^{\frac{1}{n}} + \left(\prod_{k=1}^{n} c_k\right)^{\frac{1}{n}} \\ &\le 3\sqrt{3}\left(\prod_{k=1}^{n} R_k\right)^{\frac{1}{n}}. \end{aligned}$$

3112★. *Proposé par Mohammed Aassila, Strasbourg, France.*

Soit $MABC$ un tétraèdre et soit M' un point intérieur quelconque du triangle ABC. Désigner l'aire du triangle XYZ par $[XYZ]$. Montrer que

$$\begin{aligned} (MM')^2 &= MA^2\,\frac{[BM'C]}{[ABC]} + MB^2\,\frac{[CM'A]}{[ABC]} + MC^2\,\frac{[AM'B]}{[ABC]} \\ &\quad - \left(AB^2\,\frac{[BM'C][CM'A]}{[ABC]^2} + BC^2\,\frac{[CM'A][AM'B]}{[ABC]^2}\right. \\ &\quad \left. + \; CA^2\,\frac{[AM'B][BM'C]}{[ABC]^2}\right). \end{aligned}$$

Commentaire : Ce résultat concernant un tétraèdre est “semblable” au théorème de Stewart pour un triangle. Si $M' = G$, le centre de gravité du triangle ABC, alors la relation devient

$$MG^2 = \frac{1}{3}(MA^2 + MB^2 + MC^2) - \frac{1}{9}(AB^2 + BC^2 + CA^2),$$

qui est bien connue.

3113. *Proposé par Juan-Bosco Romero Márquez, Université de Valladolid, Valladolid, Espagne.*

Soit a la longueur du côté opposé au sommet A d'un triangle ABC. Si m_a est la longueur de la médiane de A à BC, et si R est le rayon du cercle circonscrit du triangle ABC, montrer que $m_a - R$ est positif, négatif ou nul suivant que $\angle A$ est obtus, aigu ou droit.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2923. [2004 : 108, 111; 2005 : 124–125] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that $x,\ y \geq 0$ ($x,\ y \in \mathbb{R}$) and $x^2 + y^3 \geq x^3 + y^4$. Prove that $x^3 + y^3 \leq 2$.

Generalization by Li Zhou, Polk Community College, Winter Haven, FL, USA.

The ingenious Solution I by Chip Curtis [2005 : 124–125] can be easily modified to prove an infinite chain of inequalities.

Theorem. Suppose that x_i, a_i, and b_i are real numbers and that $x_i > 0$ for all i, $1 \leq i \leq n$. Let $c_i \in \{a_i,\ b_i\}$. If $x_1^{a_1} + \cdots + x_n^{a_n} \geq x_1^{b_1} + \cdots + x_n^{b_n}$, then, for $k = 0,\ 1,\ 2,\ \ldots$, we have

$$x_1^{c_1+k(a_1-b_1)} + x_2^{c_n+k(a_n-b_n)} \leq x_1^{c_1+(k+1)(a_1-b_1)} + \cdots + x_n^{c_n+(k+1)(a_n-b_n)}\,.$$

Proof : For each $k = 0,\ 1,\ 2,\ \ldots$, set

$$S_k \;=\; x_1^{c_1+k(a_1-b_1)} + \cdots + x_n^{c_n+k(a_n-b_n)}\,.$$

Then, by the Cauchy–Schwarz Inequality, the assumption, and the AM–GM Inequality, we have

$$\begin{aligned} S_0 &= \left\langle x_1^{\frac{b_1}{2}}, \ldots, x_n^{\frac{b_n}{2}} \right\rangle \cdot \left\langle x_1^{c_1-\frac{b_1}{2}}, \ldots, x_n^{c_n-\frac{b_n}{2}} \right\rangle \\ &\leq \sqrt{x_1^{b_1} + \cdots + x_n^{b_n}}\sqrt{x_1^{2c_1-b_1} + \cdots + x_n^{2c_n-b_n}} \\ &\leq \sqrt{x_1^{a_1} + \cdots + x_n^{a_n}}\sqrt{x_1^{2c_1-b_1} + \cdots + x_n^{2c_n-b_n}} \\ &\leq \tfrac{1}{2}\big(x_1^{a_1} + \cdots + x_n^{a_n} + x_1^{2c_1-b_1} + \cdots + x_n^{2c_n-b_n}\big)\,. \end{aligned}$$

Note that if $c_i = a_i$, then $2c_i - b_i = c_i + (a_i - b_i)$; and if $c_i = b_i$, then $2c_i - b_i = c_i$ and $a_i = c_i + (a_i - b_i)$. In either case,

$$x_i^{a_i} + x_i^{2c_i-b_i} \;=\; x_i^{c_i} + x_i^{c_i+(a_i-b_i)}\,.$$

Hence, $S_0 \leq \frac{1}{2}(S_0 + S_1)$; that is, $S_0 \leq S_1$.

As an induction hypothesis, assume that $S_{k-1} \le S_k$ for some $k \ge 1$. Then, by the Cauchy–Schwarz Inequality again, we have

$$\begin{aligned} S_k &= \left\langle x_1^{\frac{c_1+(k-1)(a_1-b_1)}{2}}, \ldots, x_n^{\frac{c_n+(k-1)(a_n-b_n)}{2}} \right\rangle \\ &\qquad \cdot \left\langle x_1^{\frac{c_1+(k+1)(a_1-b_1)}{2}}, \ldots, x_n^{\frac{c_n+(k+1)(a_n-b_n)}{2}} \right\rangle \\ &\le \sqrt{S_{k-1}}\sqrt{S_{k+1}} \le \sqrt{S_k}\sqrt{S_{k+1}}\,. \end{aligned}$$

Hence, $S_k \le S_{k+1}$, and the induction is complete. ∎

The following related inequalities may also be of interest to the reader.

Theorem. Suppose that u_i, v_i, and t_i are real numbers and that $u_i > 0$ and $v_i > 0$ for all i, $1 \le i \le n$. Let $w_i \in \{u_i, v_i\}$. If

$$u_1^{t_1} + \cdots + u_n^{t_n} \ge v_1^{t_1} + \cdots + v_n^{t_n}\,,$$

then, for $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned} &\left[w_1\left(\frac{u_1}{v_1}\right)^k\right]^{t_1} + \cdots + \left[w_n\left(\frac{u_n}{v_n}\right)^k\right]^{t_n} \\ &\qquad \le \left[w_1\left(\frac{u_1}{v_1}\right)^{k+1}\right]^{t_1} + \cdots + \left[w_n\left(\frac{u_n}{v_n}\right)^{k+1}\right]^{t_n}. \end{aligned}$$

Proof : Let $x_i = e^{t_i}$, $a_i = \ln u_i$, $b_i = \ln v_i$, and $c_i = \ln w_i$. It follows that $u_i^{t_i} = x_i^{a_i}$, $v_i^{t_i} = x_i^{b_i}$, $\left[w_i\left(\frac{u_i}{v_i}\right)^k\right]^{t_i} = x_i^{c_i+k(a_i-b_i)}$, and all the inequalities are transformed into those of the previous theorem. ∎

2984. [2004 : 431, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij(i+j)} = 2\sum_{n=1}^{\infty}\frac{1}{n^3}\,.$$

Ed : At the end of the solution to 2984 [2005 : 480], we included a conjecture by the proposer : If

$$P(k) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{i_1 i_2 \cdots i_k (i_1 + i_2 + \cdots + i_k)}\,,$$

for $k = 1, 2, 3, \ldots$, then $P(k) = k\zeta(k+1)$, where $\zeta(k+1)$ is the Riemann Zeta function evaluated at $k+1$. [The conjecture, as it originally appeared, was that $\zeta(k) = k\zeta(k+1)$, which was obviously incorrect.]

David Bradley, University of Maine, Orono, ME, USA; Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA, have each determined that the conjecture is false, but can be easily fixed, namely $P(k) = k!\zeta(k+1)$.

Bradley states that Mordell proved this in the 1950s. See equation (5) of L.J. Mordell, "On the evaluation of some multiple series", *J. London Math Soc.* (2) 33 (1958), 368–371. His equation (10) is the case $k = 2$, which is the published version of the problem. Mordell actually proved the generalization

$$\sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{i_1 i_2 \cdots i_k (i_1 + i_2 + \cdots + i_k + a)} = k! \sum_{i=0}^{\infty} \frac{(-1)^i \binom{a-1}{i}}{(i+1)^{k+1}},$$

which reduces to the above formula for $P(k)$ when $a = 0$.

Furdui and Zhou both provided a proof of the corrected conjecture. Their combined proof is given below.

Proof of corrected conjecture.

We first notice that

$$\begin{aligned} P(k) &= \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{i_1 i_2 \cdots i_k} \int_0^1 x^{i_1+i_2+\cdots+i_k-1}\, dx \\ &= \int_0^1 \frac{1}{x} \left(\sum_{i_1=1}^{\infty} \frac{x^{i_1}}{i_1} \right) \cdots \left(\sum_{i_k=1}^{\infty} \frac{x^{i_k}}{i_k} \right) dx \\ &= \int_0^1 (-1)^k \frac{\left(\ln(1-x)\right)^k}{x}\, dx\,. \end{aligned}$$

We now use the substitution $t = 1 - x$ in the above integral to get

$$\begin{aligned} P(k) &= (-1)^k \int_0^1 \frac{(\ln t)^k}{1-t}\, dt = (-1)^k \int_0^1 (\ln t)^k \left(\sum_{n=0}^{\infty} t^n \right) dt \\ &= (-1)^k \sum_{n=0}^{\infty} \int_0^1 t^n (\ln t)^k\, dt\,. \end{aligned}$$

In view of the integration by parts formulas, we see that

$$\int_0^1 t^n (\ln t)^k\, dt = (-1)^k \frac{k!}{(n+1)^{k+1}}\,.$$

Therefore,

$$\begin{aligned} P(k) &= (-1)^k \sum_{n=0}^{\infty} (-1)^k \frac{k!}{(n+1)^{k+1}} \\ &= k! \sum_{n=0}^{\infty} \frac{1}{(n+1)^{k+1}} = k!\zeta(k+1)\,. \end{aligned}$$

3001. [2005 : 43, 46] *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Given a, b, c, d, $e > 0$ such that $a^2+b^2+c^2+d^2+e^2 \geq 1$, prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c} \;\geq\; \frac{\sqrt{5}}{3}\,.$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON, modified by the editor.

We first show that the above inequality holds for a, b, c, d, $e > 0$ such that $a^2+b^2+c^2+d^2+e^2 = 1$.

By the Cauchy-Schwarz Inequality we have

$$\begin{aligned} 1 \;=\; \Big(\sum_{\text{cyclic}} a^2\Big)^2 \;&=\; \Big(\sum_{\text{cyclic}} \frac{a}{\sqrt{b+c+d}}\left(a\sqrt{b+c+d}\right)\Big)^2 \\ &\leq\; \Big(\sum_{\text{cyclic}} \frac{a^2}{b+c+d}\Big)\Big(\sum_{\text{cyclic}} a^2(b+c+d)\Big)\,. \end{aligned}$$

Thus, it suffices to show that

$$\sum_{\text{cyclic}} a^2(b+c+d) \;\leq\; \frac{3}{\sqrt{5}}\,. \qquad (1)$$

Again, by the Cauchy-Schwarz Inequality, we have

$$\begin{aligned} \Big(\sum_{\text{cyclic}} a^2(b+c+d)\Big)^2 \;&\leq\; \Big(\sum_{\text{cyclic}} a^2\Big)\Big(\sum_{\text{cyclic}} a^2(b+c+d)^2\Big) \\ &=\; \sum_{\text{cyclic}} a^2(b+c+d)^2\,. \end{aligned}$$

Hence, (1) follows if we can show that

$$\sum_{\text{cyclic}} a^2(b+c+d)^2 \;\leq\; \frac{9}{5}\,.$$

The Cauchy-Schwarz Inequality yields the following two inequalities :

$$(b+c+d)^2 \;\leq\; (1^2+1^2+1^2)(b^2+c^2+d^2) \;=\; 3(b^2+c^2+d^2)$$

and

$$\Big(\sum_{\text{cyclic}} (a^2+e^2)\Big)^2 \;\leq\; (1^2+1^2+1^2+1^2+1^2)\sum_{\text{cyclic}} (a^2+e^2)^2 = 5\sum_{\text{cyclic}} (a^2+e^2)^2.$$

These, together with the fact that $2\sum_{\text{cyclic}} (a^4+a^2e^2) = \sum_{\text{cyclic}} (a^2+e^2)^2$, imply that

$$\begin{aligned}
\sum_{\text{cyclic}} a^2(b+c+d)^2 &\le 3\sum_{\text{cyclic}} a^2(b^2+c^2+d^2) \;=\; 3\sum_{\text{cyclic}} a^2(1-a^2-e^2) \\
&= 3\sum_{\text{cyclic}} a^2 \;-\; 3\sum_{\text{cyclic}}(a^4+a^2e^2) \\
&= 3-\frac{3}{2}\sum_{\text{cyclic}}(a^2+e^2)^2 \\
&\le 3-\frac{3}{10}\Bigg(\sum_{\text{cyclic}}(a^2+e^2)\Bigg)^2 \;=\; 3-\frac{3}{10}(2)^2 \;=\; \frac{9}{5}\,.
\end{aligned}$$

The original inequality follows from the proven inequality if we replace (a,b,c,d,e) with $\left(\frac{a}{\sqrt{r}},\frac{b}{\sqrt{r}},\frac{c}{\sqrt{r}},\frac{d}{\sqrt{r}},\frac{e}{\sqrt{r}}\right)$ where $r=a^2+b^2+c^2+d^2+e^2$.

Also solved by KEE-WAI LAU, Hong Kong, China ; MARIAN TETIVA, Bîrlad, Romania ; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria ; and the proposer.

Janous has proposed the following generalization : Determine the set of all positive exponents α such that

$$\sum_{\text{cyclic}} \frac{a^\alpha}{b+c+d} \;\ge\; \frac{5^{(1-\gamma)}}{3}\left(a^\beta+b^\beta+c^\beta+d^\beta+e^\beta\right)^\gamma ,$$

where $\beta=\beta(\alpha)$ and $\gamma=\gamma(\alpha)$ are suitable.

3002. [2005 : 43, 46] *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let $r,\ s\in\mathbb{R}$ with $0<r<s$, and let $a,\ b,\ c\in(r,s)$. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \;\le\; \frac{3}{2}+\frac{(r-s)^2}{2r(r+s)} ,$$

and determine when equality occurs.

Similar solutions by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON ; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let

$$F(a,b,c) \;=\; \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

with domain $[r,s]$. Then $\frac{\partial^2 f}{\partial a^2}=\frac{2b}{(c+a)^3}+\frac{2c}{(a+b)^3}>0$. By symmetry, we conclude that F is a strictly convex function in each one of its three variables. Then F attains its maximum at the vertices of the box $[r,s]^3$. By symmetry, we need only check four of the vertices :

$$F(r,r,r) \;=\; \frac{3}{2} \;=\; F(s,s,s)$$

$$\text{and}\qquad F(r,s,s) \;=\; \frac{3}{2}+\frac{(r-s)^2}{2s(r+s)} \;<\; \frac{3}{2}+\frac{(r-s)^2}{3r(r+s)} \;=\; F(r,r,s)\,.$$

Thus, $F(a,b,c) \leq \frac{3}{2} + \frac{(r-s)^2}{2r(r+s)}$. Equality occurs if and only if two of the three numbers are equal to r while the third is equal to s.

Also solved by the AUSTRIAN IMO-TEAM 2005; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; MARIAN TETIVA, Bîrlad, Romania; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer. There was also one incorrect solution.

3003. [2005 : 43, 46] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let ABC be a triangle with $AB \neq AC$. Let AD be the altitude from A to BC and let BE and CF be the internal angle bisectors of $\angle B$ and $\angle C$, respectively, with E on AC and F on AB. Let B' and C' be the points of intersection of AD with BE and CF, respectively, and let A' be the point where BE intersects CF.

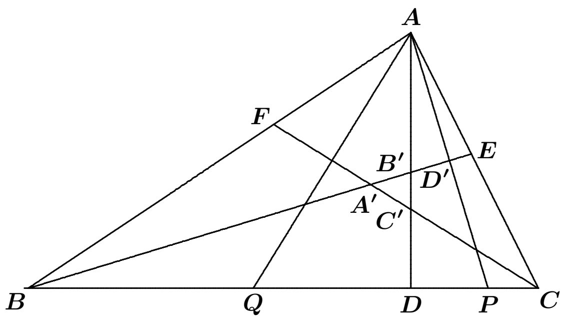
Construct the point Q on BC on the same side of C as B such that $QC = AC$, and construct the point P on BC on the same side of B as C such that $PB = AB$.

Prove that $\triangle A'B'C'$ is similar to $\triangle AQP$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let D' be the intersection point of AP and BE.

Since BE bisects $\angle ABP$ of the isosceles triangle ABP, we have $BE \perp AP$. Consequently, $PDB'D'$ is a cyclic quadrilateral, and we get $\angle APQ = \angle A'B'C'$. Similarly, $\angle AQP = \angle A'C'B'$, completing the proof.



Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; ROBERT BILINSKI, Collège Montmorency, Laval, QC; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; IMO-TEAM AUSTRIA 2005; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

Several solvers had similar proofs, but this editor felt that Zhou's solution was the neatest (in keeping with his love of pure geometric arguments).

3004. [2005 : 43, 46] *Proposed by Mihály Bencze, Brasov, Romania.*

Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Prove that

$$\frac{\left(\sqrt{a}-\sqrt{b}\right)^2+\left(\sqrt{b}-\sqrt{c}\right)^2+\left(\sqrt{c}-\sqrt{a}\right)^2}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2} \;\le\; \frac{4}{9}\left(\frac{R}{r}-2\right).$$

Solution and generalization by Gabriel Dospinescu, Paris, France, and Marian Tetiva, Bîrlad, Romania, adapted by the editor.

We shall establish the much stronger result that $D \le \frac{1}{16}\left(\frac{R}{r}-2\right)$, where D denotes the expression on left side of the inequality above.

Let $x = s-a$, $y = s-b$, and $z = s-c$, where $s = \frac{1}{2}(a+b+c)$ denotes the semiperimeter of $\triangle ABC$, and let F denote the area of $\triangle ABC$.

From the well-known formulas $R = \frac{abc}{4F}$ and $F = rs$, we have

$$\frac{R}{r} \;=\; \frac{abcs}{4F^2} \;=\; \frac{abcs}{4s(s-a)(s-b)(s-c)} \;=\; \frac{(x+y)(y+z)(z+x)}{4xyz}.$$

However,

$$\begin{aligned}\frac{(x+y)(y+z)(z+x)}{4xyz} &= \frac{(x+y)(y+z)}{4yz}+\frac{(x+y)(y+z)}{4xy}\\ &\ge \frac{x+y}{y+z}+\frac{y+z}{x+y} = \frac{c}{a}+\frac{a}{c},\end{aligned}$$

since $\frac{y+z}{4yz} \ge \frac{1}{y+z}$ and $\frac{x+y}{4xy} \ge \frac{1}{x+y}$. Thus, $\frac{R}{r} \ge \frac{c}{a}+\frac{a}{c}$, which implies that

$$\begin{aligned}\frac{R}{r}-2 &\ge \frac{c^2+a^2}{ac}-2 = \frac{(a-c)^2}{ac} = \frac{(\sqrt{a}+\sqrt{c})^2(\sqrt{a}-\sqrt{c})^2}{ac}\\ &\ge \frac{4\sqrt{ac}(\sqrt{a}-\sqrt{c})^2}{ac}. \qquad (1)\end{aligned}$$

Due to the symmetry in D, we may assume, without loss of generality, that $a \ge b \ge c$. Since

$$\begin{aligned}(\sqrt{a}-\sqrt{c})^2 &= \left((\sqrt{a}-\sqrt{b})+(\sqrt{b}-\sqrt{c})\right)^2\\ &\ge (\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2,\end{aligned}$$

and since $(\sqrt{a}+\sqrt{b}+\sqrt{c})^2 \ge (\sqrt{a}+2\sqrt{c})^2 \ge 8\sqrt{ac}$, we have

$$\begin{aligned}D &= \frac{(\sqrt{a}-\sqrt{b})^2+(\sqrt{b}-\sqrt{c})^2+(\sqrt{c}-\sqrt{a})^2}{(\sqrt{a}+\sqrt{b}+\sqrt{c})^2}\\ &\le \frac{2(\sqrt{a}-\sqrt{c})^2}{8\sqrt{ac}} = \frac{4\sqrt{ac}(\sqrt{a}-\sqrt{c})^2}{16ac}. \qquad (2)\end{aligned}$$

From (1) and (2), we obtain $\frac{R}{r}-2 \ge 16D$, and hence $D \le \frac{1}{16}\left(\frac{R}{r}-2\right)$.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; VEDULA N. MURTY, Dover, PA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposers.

Arslanagić pointed out that in the paper "Another Inequality Strengthening Euler's Inequality $R-2r \geq 0$, Octogon Mathematical Magazine, *Vol. 11, No. 2, October, 2003, p. 746, Janous had obtained the stronger result* $D \leq \frac{4}{27}\left(\frac{R}{r}-2\right)$, *which is of course weaker than the one featured above. Janous himself also submitted a solution of his result. Dospinescu and Tetiva asked the natural question : what is the largest constant* $k > 0$ *for which* $kD \leq \frac{R}{r} - 2$ *holds for all triangles?*

3005. [2005 : 44, 46] *Proposed by Pham Van Thuan, Hanoi City, Viet Nam.*

Let R and r be the circumradius and inradius, respectively, of $\triangle ABC$. Let h_a, h_b, h_c be the lengths of the altitudes of $\triangle ABC$ issuing from A, B, C, respectively, and let w_a, w_b, w_c be the lengths of the interior angle bisectors of A, B, C, respectively. Prove that

$$\frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \;\geq\; 1 + \frac{4r}{R}\,.$$

Solution by Michel Bataille, Rouen, France, modified slightly by the editor.

We will prove the stronger inequality

$$\frac{h_a^2}{w_a^2} + \frac{h_b^2}{w_b^2} + \frac{h_c^2}{w_c^2} \;\geq\; 1 + \frac{4r}{R}\,.$$

First we note that if AH is the altitude and AW the interior bisector of $\angle CAB$, then $\angle HAW = \frac{1}{2}\left|\angle B - \angle C\right|$, and hence, $\frac{h_a}{w_a} = \cos\frac{B-C}{2} \leq 1$. Similarly, $\frac{h_b}{w_b} = \cos\frac{C-A}{2} \leq 1$ and $\frac{h_c}{w_c} = \cos\frac{A-B}{2} \leq 1$. Thus,

$$\begin{aligned}
\frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} &\geq \frac{h_a^2}{w_a^2} + \frac{h_b^2}{w_b^2} + \frac{h_c^2}{w_c^2} \\
&= \cos^2\frac{B-C}{2} + \cos^2\frac{C-A}{2} + \cos^2\frac{A-B}{2} \\
&= \tfrac{3}{2} + \tfrac{1}{2}\left[\cos(B-C) + \cos(C-A) + \cos(A-B)\right]\,.
\end{aligned}$$

Using the inequality

$$\cos(A-B) + \cos(B-C) + \cos(C-A) \;\geq\; 8(\cos A + \cos B + \cos C) - 9$$

(***CRUX with MAYHEM*** 2760 [2002 : 396; 2003 : 342]), and the well-known identities

$$\cos A + \cos B + \cos C \;=\; 1 + 4\sin\left(\tfrac{1}{2}A\right)\sin\left(\tfrac{1}{2}B\right)\sin\left(\tfrac{1}{2}C\right) \;=\; 1 + \frac{r}{R}\,,$$

we obtain

$$\begin{aligned}
&\tfrac{3}{2}+\tfrac{1}{2}\left[\cos(B-C)+\cos(C-A)+\cos(A-B)\right] \\
&\qquad \geq\ \tfrac{3}{2}+\tfrac{1}{2}\left[8(\cos A+\cos B+\cos C)-9\right] \\
&\qquad =\ \tfrac{3}{2}+\tfrac{1}{2}\left[8\left(1+4\sin\left(\tfrac{1}{2}A\right)\sin\left(\tfrac{1}{2}B\right)\sin\left(\tfrac{1}{2}C\right)\right)-9\right] \\
&\qquad =\ \tfrac{3}{2}+4\left(1+r/R\right)-\tfrac{9}{2}\ =\ 1+4r/R\,,
\end{aligned}$$

which completes the proof. Equality holds if and only if the triangle is equilateral.

Also solved by SCOTT BROWN, Auburn University, Montgomery, AL, USA ; JOHN G. HEUVER, Grande Prairie, AB (2 solutions)*; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria ; VEDULA N. MURTY, Dover, PA, USA ; MARIAN TETIVA, Bîrlad, Romania ; PETER Y. WOO, Biola University, La Mirada, CA, USA ; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON ; LI ZHOU, Polk Community College, Winter Haven, FL, USA ; and the proposer.*

The stronger inequality from the above solution was also proved by Janous, Murty, Tetiva, and the proposer.

3006. [2005 : 44, 47] *Proposed by Luis V. Dieulefait, Centre de Recerca Matemática, Ballaterra, Spain.*

An old man willed that, upon his death, his three sons would receive the u^{th}, v^{th}, and w^{th} parts of his herd of camels respectively. He had N camels in the herd when he died, where $N+1$ is a common multiple of u, v, and w. Since the three sons could not divide N exactly into u, v, or w parts, they approached a distinguished ***CRUX*** problem solver for help. He rode over on his own camel, which he added to the herd. The herd was then divided up according to the old man's wishes. Our ***CRUX*** problem solver then took back the one camel that remained, which was, of course, his own.

(a) Find all solutions (u,v,w,N).

(b)$\star$ Solve the same problem if there are four sons.

(c)$\star$ Let there be k sons. Find an upper bound $f(k)$ on N for the problem to have a solution.

[*Ed* : This is a generalization of Problem 2226 [1997 : 166 ; 1998 : 186].]

Combination of solutions by Michel Bataille, Rouen, France ; and Chip Curtis, Missouri Southern State University, Joplin, MO, USA, for parts (a) *and* (b). *Only Curtis solved part* (c).

(a) In the case of three sons, we seek an integer $N\geq 3$ and positive integers u, v, and w such that u, v, and w each divide $N+1$ (but not N), and $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}\right)(N+1)=N$, or equivalently,

$$\frac{1}{u}+\frac{1}{v}+\frac{1}{w} \;=\; \frac{N}{N+1}\,.$$

Without loss of generality, we may assume that $u \leq v \leq w$. The assumption $u \geq 5$ would yield $\frac{N}{N+1} \leq \frac{3}{5}$, contradicting $N \geq 3$. Thus, $2 \leq u \leq 4$.

• If $u = 4$, then $\frac{1}{v}+\frac{1}{w}=\frac{3N-1}{4N+4}$ and, arguing as above, we see that $v \geq 5$ is impossible. Since $v \geq u$, we must have $v = 4$. Hence, $\frac{1}{w}=\frac{N-1}{2N+2}$ and $w = \frac{2N+2}{N-1} = 2+\frac{4}{N-1}$. It follows that $N-1$ divides 4. Therefore, $N = 3$ or $N = 5$. Now $N = 4$ leads to $w = 3 < v$, a contradiction. Thus, $N = 3$ and $w = 4$, which yields $(u,v,w,N) = (4,4,4,3)$. Conversely, this is obviously suitable (with $a = b = c = 1$).

• If $u = 3$, then a similar argument shows that we must have $v = 4$ or $v = 3$. As above, $v = 4$ leads to $w = \frac{12N+12}{5N-7}$, implying that $5N-7$ must divide $12N+12$. Then $5N-7$ must divide $5(12N+12)-12(5N-7) = 144$. Since $N+1$ is a multiple of both 3 and 4, we see that $N \geq 11$. It is easily seen that the only possibility is $5N-7 = 48$ and $N = 11$. But this gives $w = 3 < v$, a contradiction. Similarly, $v = 3$ leads to $N = 5$ or 11, and to the solutions $(3,3,6,5)$ and $(3,3,4,11)$.

• If $u = 2$, the same method calls for the examination of the cases $v = 6$, 5, 4, or 3 and leads to 9 more solutions.

In conclusion, there are 12 solutions which are displayed in the chart below.

N	(u,v,w)
3	$(4,4,4)$
5	$(2,6,6)$
5	$(3,3,6)$
7	$(2,4,8)$
9	$(2,5,5)$
11	$(2,3,12)$
11	$(2,4,6)$
11	$(3,3,4)$
17	$(2,3,9)$
19	$(2,4,5)$
23	$(2,3,8)$
41	$(2,3,7)$

(b) For the case of four sons, we seek an integer $N \geq 4$ and positive integers u, v, w, x, such that u, v, w, and x each divide $N+1$ (but none divide N), and $\left(\frac{1}{u}+\frac{1}{v}+\frac{1}{w}+\frac{1}{x}\right)(N+1)=N$, or equivalently,

$$\frac{1}{u}+\frac{1}{v}+\frac{1}{w}+\frac{1}{x} = \frac{N}{N+1}.$$

Without loss of generality, we may again assume that $u \leq v \leq w \leq x$. Arguing in a similar vein to part (a) we find the solutions in the chart below.

N	(u,v,w,x)	N	(u,v,w,x)	N	(u,v,w,x)
4	$(5,5,5,5)$	29	$(2,3,15,15)$	83	$(2,3,7,84)$
5	$(3,6,6,6)$	29	$(2,5,5,15)$	83	$(2,3,12,14)$
7	$(4,4,4,8)$	29	$(2,5,6,10)$	83	$(2,4,6,14)$
7	$(2,8,8,8)$	35	$(3,3,4,18)$	89	$(2,3,10,18)$
8	$(3,3,9,9)$	35	$(2,3,9,36)$	95	$(2,3,8,32)$
9	$(2,5,10,10)$	35	$(2,3,12,18)$	99	$(2,4,5,25)$
11	$(4,4,4,6)$	35	$(2,4,6,18)$	109	$(2,5,5,11)$
11	$(3,4,4,12)$	35	$(2,4,9,9)$	119	$(3,3,5,8)$
11	$(3,4,6,6)$	39	$(2,4,5,40)$	119	$(2,3,8,30)$
11	$(3,3,6,12)$	39	$(2,4,8,10)$	119	$(2,4,5,24)$
11	$(2,4,12,12)$	41	$(3,3,6,7)$	119	$(2,5,6,8)$
11	$(2,6,6,12)$	41	$(2,3,14,14)$	125	$(2,3,7,63)$
13	$(2,7,7,7)$	41	$(2,6,6,7)$	125	$(2,3,9,21)$
14	$(3,5,5,5)$	44	$(3,3,5,9)$	139	$(2,4,7,10)$
14	$(3,3,5,15)$	47	$(3,3,4,16)$	155	$(3,3,4,13)$
15	$(2,4,8,16)$	47	$(2,3,8,48)$	155	$(2,3,12,13)$
17	$(3,3,6,9)$	47	$(2,3,12,16)$	155	$(2,4,6,13)$
17	$(2,3,18,18)$	47	$(2,4,6,16)$	167	$(2,3,7,56)$
17	$(2,6,6,9)$	53	$(2,3,9,27)$	167	$(2,3,8,28)$
19	$(4,4,4,5)$	59	$(3,4,5,5)$	179	$(2,3,9,20)$
19	$(2,4,10,10)$	59	$(3,3,4,15)$	215	$(2,3,8,27)$
19	$(2,5,5,20)$	59	$(2,3,10,20)$	219	$(2,4,5,22)$
20	$(3,3,7,7)$	59	$(2,3,12,15)$	239	$(2,3,10,16)$
23	$(3,4,4,8)$	59	$(2,4,5,30)$	293	$(2,3,7,49)$
23	$(3,3,4,24)$	59	$(2,4,6,15)$	311	$(2,3,8,26)$
23	$(3,3,6,8)$	59	$(2,5,5,12)$	335	$(2,3,7,48)$
23	$(2,3,12,24)$	69	$(2,5,7,7)$	341	$(2,3,9,19)$
23	$(2,4,6,24)$	71	$(2,4,8,9)$	419	$(2,4,5,21)$
23	$(2,4,8,12)$	71	$(2,3,8,36)$	599	$(2,3,8,25)$
23	$(2,6,6,8)$	71	$(2,3,9,24)$	629	$(2,3,7,45)$
27	$(2,4,7,14)$	77	$(2,3,13,13)$	923	$(2,3,7,44)$
29	$(3,3,5,10)$	83	$(3,3,4,14)$	1805	$(2,3,7,43)$
29	$(2,3,10,30)$				

(c) For k sons, we claim that the maximal number of camels for which there is a solution is exactly given by

$$f(k) \;=\; a(k+1)-2\,,$$

where $a(k)$ is *Sylvester's sequence* (also called *Euclid numbers*), defined recursively (see sequence A000058 in [1]) by $a(0)=2$ and, for $k \geq 0$,

$$a(k+1) \;=\; a(k)^2 - a(k) + 1\,.$$

To see why, note first that N is maximal if and only if $\dfrac{N}{N+1}$ is as close to 1 as possible. But (again, see [1]) $\{a(k)\}$ is a "greedy sequence" ; that is, $a(k+1)$ is the smallest integer greater than $a(n)$ such that $\sum\limits_{j=1}^{k+1} \dfrac{1}{a(k)}$ does not exceed 1. Moreover,

$$\sum_{j=1}^{k} \frac{1}{a(k)} \;=\; \frac{a(k+1)-2}{a(k+1)-1}\,,$$

implying that the sum is of the form $\dfrac{N}{N+1}$. In fact, setting

$$\frac{a(k+1)-2}{a(k+1)-1} \;=\; \frac{N}{N+1}$$

gives $N = a(k+1)-2$, as claimed.

In particular, the formula gives the following :

# of sons	Maximum number of camels
1	1
2	5
3	41
4	1805
5	3263441
6	10650056950805
7	113423713055421844361000441
8	12864938683278671740537145998360961546653259485195805

Reference

[1] N.J.A. Sloane, "The On-line Encyclopedia of Integer Sequences",
http ://www.research.att.com/ njas/sequences.

Parts (a) *and* (b) *also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA ; and WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria.*

3007. [2005 : 44, 47 ; corrected 2005 : 173, 176] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be a triangle, and let $A_1 \in BC$, $B_1 \in CA$, $C_1 \in AB$ such that

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k > 0 .$$

1. Prove that the segments AA_1, BB_1, CC_1 are the sides of a triangle.

Let T_k denote this triangle. Let R_k and r_k be the circumradius and inradius of T_k. Prove that :

2. $P(T_k) < P(ABC)$, where $P(T)$ denotes the perimeter of triangle T ;
3. $[T_k] = \frac{k^2+k+1}{(k+1)^2}[ABC]$, where $[T]$ denotes the area of triangle T ;
4. $R_k \geq \frac{k\sqrt{k}\,P(ABC)}{(k+1)(k^2+k+1)}$;
5. $r_k > \frac{k^2+k+1}{(k+1)^2}\,r$, where r is the inradius of $\triangle ABC$.

[Editor : The problem was originally stated with equality in parts 4 and 5. This was the fault of the editor, not the proposer.]

Solution by Titu Zvonaru, Comăneşti, Romania.

As usual, let $a = BC$, $b = CA$, $c = AB$, and $s = \frac{1}{2}(a+b+c)$. Since

$$\frac{BA_1}{A_1C} = \frac{CB_1}{B_1A} = \frac{AC_1}{C_1B} = k ,$$

it follows that $BA_1 = \frac{ak}{k+1}$ and $A_1C = \frac{a}{k+1}$, with analogous expressions for CB_1, B_1A, AC_1, and C_1B. By the Law of Cosines, we obtain

$$\begin{aligned} AA_1^2 &= c^2 + \frac{a^2k^2}{(k+1)^2} - \frac{k}{k+1}\cdot 2ac\cos B \\ &= \frac{c^2(k+1)^2 + a^2k^2 - (k^2+k)(a^2+c^2-b^2)}{(k+1)^2} \\ &= \frac{b^2(k^2+k) + c^2(k+1) - a^2k}{(k+1)^2} . \end{aligned}$$

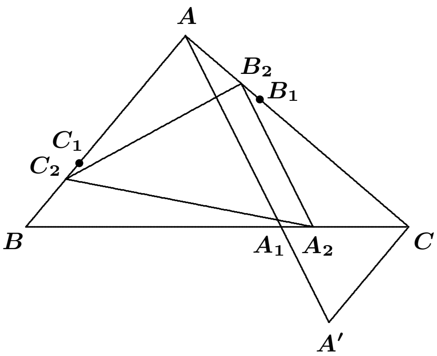
1. Let $A_2 \in BC$, $B_2 \in CA$, and $C_2 \in AB$ such that

$$\frac{BA_2}{A_2C} = \frac{CB_2}{B_2A} = \frac{AC_2}{C_2B} = k+1 .$$

Then $A_2C = \dfrac{a}{k+2}$ and $CB_2 = \dfrac{b(k+1)}{k+2}$. By the Law of Cosines,

$$\begin{aligned} A_2B_2^2 &= \frac{a^2}{(k+2)^2} + \frac{b^2(k+1)^2}{(k+2)^2} - \frac{k+1}{(k+2)^2}\cdot 2ab\cos C \\ &= \frac{a^2 + b^2(k+1)^2 - (k+1)(a^2+b^2-c^2)}{(k+2)^2} \\ &= \frac{b^2(k^2+k) + c^2(k+1) - a^2k}{(k+2)^2} = \frac{(k+1)^2}{(k+2)^2}\cdot AA_1^2\,. \end{aligned}$$

Hence, $AA_1 = \dfrac{k+2}{k+1}\cdot A_2B_2$. It follows that AA_1, BB_1, and CC_1 are the sides of a triangle similar to triangle $A_2B_2C_2$.



2. The line through C parallel to AB intersects AA_1 at a point A'. By similitude, $A'A_1 = AA_1/k$ and $CA' = c/k$. In $\triangle ACA'$, we have $AA' < AC + CA'$. Equivalently, $AA_1 + \dfrac{AA_1}{k} < b + \dfrac{c}{k}$, or $AA_1 < \dfrac{bk+c}{k+1}$. Hence,

$$P(T_k) = AA_1 + BB_1 + CC_1 < \frac{bk+c+ck+a+ak+b}{k+1} = P(ABC)\,.$$

We note an alternative way to prove the inequality $AA_1 < \dfrac{bk+c}{k+1}$. This inequality is equivalent to $b^2(k^2+k) + c^2(k+1) - a^2k < (bk+c)^2$, or (by simplification) $|b-c| < a$, which is the Triangle Inequality in $\triangle ABC$.

3. We have

$$[CA_2B_2] = \frac{A_2C\cdot B_2C\cdot\sin C}{2} = \frac{ab(k+1)\sin C}{2(k+2)^2} = \frac{k+1}{(k+2)^2}[ABC]\,.$$

Since the sides of $\triangle A_2B_2C_2$ are in the ratio $k+2 : k+1$ to the sides of T_k, we deduce that

$$\begin{aligned} [T_k] = \frac{(k+2)^2}{(k+1)^2}[A_2B_2C_2] &= \frac{(k+2)^2}{(k+1)^2}\left([ABC] - 3\frac{k+1}{(k+2)^2}[ABC]\right) \\ &= \frac{(k^2+k+1)}{(k+1)^2}[ABC]\,. \end{aligned}$$

4. We have

$$\begin{aligned} AA_1 &= \frac{\sqrt{b^2k^2 + c^2 + b^2k + c^2k - a^2k}}{k+1} \\ &\geq \frac{\sqrt{2bck + b^2k + c^2k - a^2k}}{k+1} = \frac{\sqrt{k}\sqrt{(b+c)^2 - a^2}}{k+1} \\ &= \frac{\sqrt{k}\sqrt{(b+c+a)(b+c-a)}}{k+1} = \frac{2\sqrt{k}\sqrt{s(s-a)}}{k+1}. \end{aligned}$$

Since $R_k = \dfrac{AA_1 \cdot BB_1 \cdot CC_1}{4[T_k]}$, it follows that

$$\begin{aligned} R_k &\geq \frac{8k\sqrt{k}\sqrt{s^3(s-a)(s-b)(s-c)}}{(k+1)^3 \cdot 4 \cdot \dfrac{k^2+k+1}{(k+1)^2}[ABC]} \\ &= \frac{2k\sqrt{k} \cdot s \cdot [ABC]}{(k+1)(k^2+k+1)[ABC]} = \frac{k\sqrt{k}\,P(ABC)}{(k+1)(k^2+k+1)}, \end{aligned}$$

where we have used Heron's Formula $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$. Equality holds if and only if $bk = c$, $ck = a$, and $ak = b$; that is, if and only if $abck^3 = abc$, which means that $k = 1$ and $a = b = c$.

5. Since $r_k = \dfrac{2[T_k]}{P(T_k)}$ and $r = \dfrac{2[ABC]}{P([ABC])}$, the inequality follows from parts 2 and 3.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain (parts 1–3 and 5 only); CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GEOFFREY A. KANDALL, Hamden, CT, USA; VEDULA N. MURTY, Dover, PA, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.