Pólya's Paragon

It Ain't So Complex (Part 3)

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Last month we noticed that when we multiply two complex numbers in polar form, the argument of the product is coterminal with the sum of the arguments; that is,

$$(\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2) \;=\; \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\,.$$

If we let $f(\theta) = \cos\theta + i\sin\theta$, then this equation can be rewritten as

$$f(\theta_1)\cdot f(\theta_2) \;=\; f(\theta_1+\theta_2)\,.$$

Thinking of all the functions you know, you might notice that our function f is behaving like an exponential function $f(x) = b^x$ (where $b > 0$ is the base). When $f(x) = b^x$, we have

$$f(\theta_1)f(\theta_2) \;=\; b^{\theta_1}\cdot b^{\theta_2} \;=\; b^{\theta_1+\theta_2} = f(\theta_1+\theta_2)\,.$$

But we defined $f(\theta)$ in terms of $\cos\theta$ and $\sin\theta$. How can f possibly be an exponential function? What is the base b?

Looking to calculus, we have the following power series expansions:

$$\begin{aligned} e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\,, \\ \sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\,, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\,, \end{aligned}$$

where the series are absolutely convergent for all real x. (For those of you who haven't taken calculus, you need to simply accept this.) Recall that $e \approx 2.71828\ldots$ is the base of the natural logarithm.

Let us use the series for $\sin x$ and $\cos x$ to calculate $\cos\theta + i\sin\theta$:

$$\begin{aligned} \cos\theta + i\sin\theta &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \end{aligned}$$

The resulting series above could be obtained by putting $x = i\theta$ in the series for e^x. So we *define* $e^{i\theta} = \cos\theta + i\sin\theta$. Thus, we can write any complex number $z = r(\cos\theta + i\sin\theta)$ in polar form as

$$z \;= re^{i\theta}\,.$$

This exponential notation leads to some interesting results, such as *de Moivre's Formula*, which states that, for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta .$$

Using exponential notation, the theorem simply says $\left(e^{i\theta}\right)^n = e^{in\theta}$, which seems quite obvious (using the basic properties of exponents). We can use de Moivre's Formula to work out other trigonometric identities; for example, when $n = 2$, we get

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta ,$$

Equating the real and imaginary parts, we get the double angle formulas for sine and cosine:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \text{and} \qquad \sin 2\theta = 2\sin\theta\cos\theta .$$

That will do for this issue. Next time, we will look at some applications to plane geometry. For homework, try the following:

1. Calculate $\cos 3\theta$ in terms of $\cos\theta$, and calculate $\sin 3\theta$ in terms of $\sin\theta$.
2. By using the exponential notation, obtain the formulas for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$ in terms of $\cos\theta_1$, $\cos\theta_2$, $\sin\theta_1$, and $\sin\theta_2$.
3. Use complex numbers to prove the *Triangle Inequality*. That is, for any triangle with sides a, b, c, prove that $c \le a + b$.
4. Use complex numbers to show that the medians of a triangle meet at a common point and determine how to find that point.

Finally, we consider the unanswered questions from last month's homework.

1. Converting the polar form to the exponential form, we have.

$$\begin{aligned}\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \\ &= \frac{r_1}{r_2}\big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\big) .\end{aligned}$$

3. Considering the equation in exponential form, we get $z^2 = i = e^{i\frac{\pi}{2}}$. If we let $z = re^{i\theta}$, then $z^2 = r^2 e^{2i\theta} = e^{i\frac{\pi}{2}}$. The usual convention is that $r > 0$ for all $z \neq 0$; thus, $r = 1$ in our case and one solution is $z_1 = e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. We approach the second solution by noting the redundancy in the polar form; that is,

$$e^{i\theta} = e^{i(\theta + 2\pi k)} ,$$

for any integer k. Thus, our other solution comes from considering $z^2 = e^{i\frac{\pi}{2}} = e^{i\frac{5\pi}{2}}$, which yields $z_2 = e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. For further exploration, plot these two solutions on the complex plane; how are they related geometrically? How are the the solutions to $z^3 = i$ related geometrically?

Until next month, happy problem solving.