Problem of the Month

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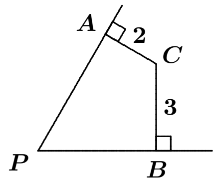
It has been a long time since we have done any geometry! Since contest season is on the horizon, it is probably time to brush up on this aspect of our mathematical repertoire.

Problem (1992 Canadian Invitational Mathematics Challenge, Grade 10)

A point C is situated inside an angle of 60° at a distance of 2 units and 3 units from its sides. Determine the distance from point P to point C.



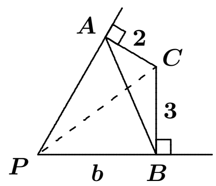
One of the wonderful things about geometry problems is the many different approaches that can be taken to solve the same problem. Here are three different approaches to this problem. We will keep the really nice approach for last to keep you reading until the end!

Solution 1: Since $PACB$ is a quadrilateral, the sum of its four interior angles is 360°. Thus, $\angle ACB = 120^\circ$. Join A to B. We can calculate the length of AB using the Cosine Law:

$$AB^2 = 2^2 + 3^2 - 2(2)(3)\cos 120^\circ = 4 + 9 - 2(2)(3)\left(-\tfrac{1}{2}\right) = 19\,.$$

Thus, $AB = \sqrt{19}$.

Now, let $PB = b$. By joining P to C, we form two right triangles sharing the hypotenuse PC. This means that $PA^2 + AC^2 = PC^2 = PB^2 + BC^2$; whence,



$$PA^2 = PB^2 + BC^2 - AC^2 = b^2 + 9 - 4 = b^2 + 5\,.$$

Therefore, $PA = \sqrt{b^2+5}$.

Next, we apply the Cosine Law again, this time in $\triangle PAB$, and solve for b. This requires a bit of patience.

$$\begin{aligned}
AB^2 &= PA^2 + PB^2 - 2(PA)(PB)\cos\angle APB\,,\\
19 &= (b^2+5) + b^2 - 2\left(\sqrt{b^2+5}\right)(b)\left(\tfrac{1}{2}\right),\\
14 - 2b^2 &= -b\sqrt{b^2+5}\,,\\
196 - 56b^2 + 4b^4 &= b^4 + 5b^2 \qquad \text{(squaring to get rid of square roots)},\\
3b^4 - 61b^2 + 196 &= 0\,.
\end{aligned}$$

We have obtained a quartic equation which is really a quadratic in disguise (didn't we see that last month?). We solve for b by factoring:

$$\begin{aligned}
(3b^2 - 49)(b^2 - 4) &= 0\,,\\
b^2 = \tfrac{49}{3} \qquad \text{or} \qquad b^2 &= 4\,.
\end{aligned}$$

Remembering that b must be positive, we get $b = 7/\sqrt{3}$ or $b = 2$. If $b = 2$, then $PA = 3$. Unfortunately, we have to reject this solution (why?). Hence, $b = 7/\sqrt{3}$. (It is interesting to wonder why we obtained the inadmissible solution $b = 2$.)

But we are seeking PC. Well, $PC^2 = PB^2 + BC^2 = \frac{49}{3} + 9 = \frac{76}{3}$, which means that $PC = \sqrt{\frac{76}{3}}$.

That required some persistence, but it worked out in the end. It was interesting that we ended up with some of the same algebraic issues that we discovered last month.

Solution 2: We could use coordinates! I'll get you started on this and leave you to work through the details. Again, it is not very pretty, but it works. And it is a good exercise in analytic geometry, especially since you already know the answer that you should get, which will help you track down any errors you make along the way.

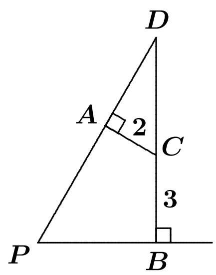
Set point P to be the origin $(0,0)$, with PB lying along the positive x–axis. Give B coordinates $(b,0)$. Then C has coordinates $(b,3)$. Since $\angle APB = 60^\circ$, the slope of the line through P and A is $\tan 60^\circ = \sqrt{3}$. Hence, this line has equation $y = \sqrt{3}x$.

From here, you need to find the equation of the line through A and C (*hint:* it is perpendicular to PA and passes through $(b,3)$), then find the coordinates of A (these will be in terms of b), and finally use the fact that the distance from A to C is 2. This will allow you to solve for b and then determine PC.

And now for the grand finale! After seeing the next solution, you will probably wish you had not seen either of the previous ones.

Solution 3: Extend the line through B and C up through C until it hits the line through P and A at D.

Since $\triangle DPB$ is right-angled at B, then $\angle PDB = 90^\circ - 60^\circ = 30^\circ$. Thus, $\triangle DPB$ is a 30°–60°–90° triangle. Therefore, $PB = \frac{1}{\sqrt{3}}DB$. But $\triangle DCA$ is also a 30°–60°–90° triangle; whence, $CD = 2CA = 4$, and $DB = 7$. Then $PB = \frac{7}{\sqrt{3}}$. Therefore,



$$PC^2 = PB^2 + BC^2 = \frac{49}{3} + 9 = \frac{76}{3},$$

and $PC = \sqrt{\frac{76}{3}}$.

I think you would agree that this last approach was very nice.

We have seen three very different solutions to the same geometry problem. The last solution involves a nifty construction. Many geometrical problems have really elegant solutions involving a construction, but these solutions are usually very hard to find. What is the best way to find them, you ask? Lots of practice!