

Mayhem Solutions

M146. Proposed by Mohammed Aassila, Strasbourg, France.

Let a, b, c be three positive numbers satisfying $a + b + c = 1$. Prove that

$$(ab)^{5/4} + (bc)^{5/4} + (ca)^{5/4} < \frac{1}{4}.$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Cauchy's Inequality gives us

$$\begin{aligned} & \left((ab)^{5/4} + (bc)^{5/4} + (ca)^{5/4} \right)^2 \\ & \leq \left(\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \right) (a^2b^2 + b^2c^2 + c^2a^2). \end{aligned}$$

Applying the AM–GM Inequality, we have

$$\sqrt{ab} + \sqrt{bc} + \sqrt{ca} \leq \frac{1}{2}(a+b) + \frac{1}{2}(b+c) + \frac{1}{2}(c+a) = a + b + c = 1.$$

Therefore, the inequality claimed will be established if we prove that

$$a^2b^2 + b^2c^2 + c^2a^2 < \frac{1}{16}. \quad (1)$$

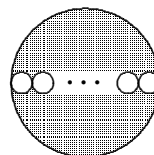
To prove (1), we may assume without loss of generality that $a \leq b \leq c$. Using the AM–GM Inequality, we get $\sqrt{(a+b)c} \leq \frac{1}{2}((a+b) + c) = \frac{1}{2}$. Then

$$\frac{1}{16} \geq c^2(a+b)^2 = a^2c^2 + b^2c^2 + 2abc^2 > a^2c^2 + b^2c^2 + abc^2.$$

Since $a \leq b \leq c$, we have $abc^2 > a^2b^2$, and then (1) follows.

M147. Proposed by the Mayhem staff.

The diameter of a large circle is broken into n equal parts to construct n smaller circles, as shown. Determine n so that the ratio of the shaded area to the unshaded area in the large circle is 3 : 1.



Solution by Gabriel Krimker, grade 10 student, Buenos Aires, Argentina.

Let r be the radius of the large circle. The radius of each smaller circle is $\frac{r}{n}$. The shaded area is $\pi r^2 - n\pi \left(\frac{r}{n}\right)^2 = \pi r^2 \left(1 - \frac{1}{n}\right)$, and the unshaded area is $n\pi \left(\frac{r}{n}\right)^2 = \pi r^2 \left(\frac{1}{n}\right)$. Then

$$3 = \frac{\pi r^2 \left(1 - \frac{1}{n}\right)}{\pi r^2 \left(\frac{1}{n}\right)} = n \left(1 - \frac{1}{n}\right) = n - 1.$$

Hence, $n = 4$.

Also solved by Roger He, grade 10 student, Prince of Wales Collegiate, St. John's, NL; Doug Newman, Lancaster, CA, USA; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M148. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit $x > 1$, $y > 1$, $z > 1$ et $x^2 = yz$. Trouver la valeur de

$$(\log_{zx} xy^4z) (\log_{xy} xyz^4) .$$

Solution par Houda Anoun, LaBri, Bordeaux, France.

Soient x , y et z des nombres réels tels que $x > 1$, $y > 1$, $z > 1$ et $x^2 = yz$. Posons $f = \log_{zx}(xy^4z)$ et $g = \log_{xy}(xyz^4)$. On a alors

$$(xz)^f = xy^4z = x^3y^3 = (xy)^3 . \quad (1)$$

D'autre part on a aussi

$$(xy)^g = xyz^4 = x^3z^3 = (xz)^3 . \quad (2)$$

D'après (1) et (2) on a alors

$$(xz)^{fg} = ((xy)^3)^g = ((xy)^g)^3 = (xz)^9 .$$

Or comme $xz > 1$ donc on en déduit que $fg = 9$.

En outre résolu par Marcie Fairchild, Daniel Mills, Laura Steil et Willie Ward, étudiants, Samford University, Birmingham, Alabama, USA; Shuang Han, étudiant, 12^{ième} catégorie, Holy Heart of Mary High School, St. John's, NL; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine; et Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M149. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right-angled Heron triangle ABC has the following property: the area is λ times the perimeter, where λ is a positive integer. Determine all solutions (a, b, λ) . (A Heron triangle is a triangle with integer sides and integer area.)

Solution by Marcie Fairchild, Daniel Mills, Laura Steil, and Willie Ward, students, Samford University, Birmingham, Alabama, USA.

The Heron triangles have to be right-angled with all sides of integer length. Thus, we know that the triangles we are looking for must have sides that make Pythagorean triples. We can list Pythagorean triples by using the following system:

$$\begin{aligned} a &= 2xyt, \\ b &= (x^2 - y^2)t, \\ c &= (x^2 + y^2)t, \end{aligned}$$

where x , y , and t are integers, x and y have opposite parity, $x > y$, and a and b are the legs of the triangle. Using this representation of the triangle sides, and letting P be the perimeter of the triangle and A the area, we have $P = a + b + c = 2xyt + (x^2 - y^2)t + (x^2 + y^2)t = 2xt(x + y)$ and $A = \frac{1}{2}ab = \frac{1}{2}(2xyt)(x^2 - y^2)t = xy t^2(x^2 - y^2)$. Now the problem stipulates that the area is λ times the perimeter, which implies that

$$xy t^2(x^2 - y^2) = \lambda(2xt(x + y)).$$

This equation can be solved for λ to yield $\lambda = \frac{1}{2}y(x - y)t$. Such λ will be an integer unless both t and y are odd. Therefore, all solutions are given by

$$(a, b, c) = \left(2xyt, (x^2 - y^2)t, \frac{y(x - y)t}{2} \right),$$

where x and y have opposite parity, $x > y$, and at least one of y and t is even.

One incomplete solution was received.

M150. Proposed by Arkady Alt, San Jose, CA, USA.

Let two complex numbers z_1 and z_2 satisfy the conditions

$$\begin{aligned} z_1 + z_2 &= -(i + 1), \\ z_1 \cdot z_2 &= -i. \end{aligned}$$

Without calculating z_1 and z_2 , find $z_1 \cdot \overline{z_2}$.

Solution by the proposer.

Note that $z_1 \cdot \overline{z_2} = \frac{z_1}{z_2} \cdot |z_2|^2$. From $(z_1 + z_2)^2 = 2i = -2z_1 \cdot z_2$, we immediately obtain $z_1^2 + 4z_1 z_2 + z_2^2 = 0$, or equivalently,

$$\left(\frac{z_1}{z_2} \right)^2 + 4 \left(\frac{z_1}{z_2} \right) + 1 = 0.$$

Thus, $\frac{z_1}{z_2}$ is real and negative. Therefore, $z_1 \cdot \overline{z_2}$ is also real and negative. Combining this with $|z_1 \cdot \overline{z_2}| = |z_1 \cdot z_2| = 1$, we see that $z_1 \cdot \overline{z_2} = -1$.