MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 April 2006**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M213. *Proposed by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

Set $S = (2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Evaluate $S^{\frac{1}{1024}}$ without using a calculator.

M214. *Proposed by Babis Stergiou, Chalkida, Greece.*

Two equilateral triangles ABC and CDE are on the same side of line BCD. If BE intersects AC at K and DA intersects CE at L, prove that KL is parallel to BD.

M215. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Find a rational number s such that s^2+5 and s^2-5 are both squares of rational numbers.

M216. *Proposed by K.R.S. Sastry, Bangalore, India.*

A Heron triangle has integer sides and area. Two sides of a Heron triangle are 442 and 649. If its area is 132396, find its perimeter.

M217. *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

Let a, b, c be integers such that 2005 divides into both $ab+9b+81$ and $bc+9c+81$. Prove that 2005 also divides into $ca+9a+81$.

M218. *Proposed by Neven Jurič, Zagreb, Croatia.*

Compute the sum

$$\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\,.$$

M213. *Proposé par Edward T.H. Wang, Université Wilfrid Laurier, Waterloo, ON.*

Soit $S=(2+1)(2^2+1)(2^4+1)(2^8+1)\cdots(2^{1024}+1)+1$. Calculer $S^{\frac{1}{1024}}$ sans l'aide d'une calculatrice.

M214. *Proposé par Babis Stergiou, Chalkida, Grèce.*

Deux triangles équilatéraux ABC et CDE sont du même côté de la droite BCD. Si BE coupe AC en K et DA coupe CE en L, montrer que KL est parallèle à BD.

M215. *Proposé par Bruce Shawyer, Université Memorial de Terre-Neuve, St. John's, NL.*

Trouver un nombre rationnel s tel que s^2+5 et s^2-5 sont tous deux des carrés de nombres rationnels.

M216. *Proposé par K.R.S. Sastry, Bangalore, Inde.*

Un triangle de Héron possède des côtés et une aire mesurés par des nombres entiers. Deux côtés d'un triangle de Héron mesurent 442 et 649. Si son aire est 132396, trouver son périmètre.

M217. *Proposé par Bill Sands, Université de Calgary, Calgary, AB.*

Soit a, b et c des entiers tels que 2005 soit divisible par $ab+9b+81$ et par $bc+9c+81$. Montrer que 2005 est aussi divisible par $ca+9a+81$.

M218. *Proposé par Neven Jurič, Zagreb, Croatie.*

Calculer la somme

$$\sum_{k=1}^{99} \frac{1}{k\sqrt{k+1}+(k+1)\sqrt{k}}\,.$$

Mayhem Solutions

M146. *Proposed by Mohammed Aassila, Strasbourg, France.*

Let a, b, c be three positive numbers satisfying $a+b+c=1$. Prove that

$$(ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4} \;<\; \frac{1}{4}\,.$$

Solution by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.

Cauchy's Inequality gives us

$$\left((ab)^{5/4}+(bc)^{5/4}+(ca)^{5/4}\right)^2 \;\leq\; \left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\left(a^2b^2+b^2c^2+c^2a^2\right).$$

Applying the AM–GM Inequality, we have

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \;\leq\; \tfrac{1}{2}(a+b)+\tfrac{1}{2}(b+c)+\tfrac{1}{2}(c+a) \;=\; a+b+c \;=\; 1\,.$$

Therefore, the inequality claimed will be established if we prove that

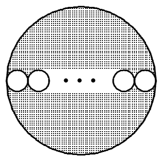
$$a^2b^2+b^2c^2+c^2a^2 \;<\; \frac{1}{16}\,. \qquad (1)$$

To prove (1), we may assume without loss of generality that $a\leq b\leq c$. Using the AM-GM Inequality, we get $\sqrt{(a+b)c} \leq \frac{1}{2}((a+b)+c) = \frac{1}{2}$. Then

$$\frac{1}{16} \;\geq\; c^2(a+b)^2 \;=\; a^2c^2+b^2c^2+2abc^2 \;>\; a^2c^2+b^2c^2+abc^2\,.$$

Since $a\leq b\leq c$, we have $abc^2>a^2b^2$, and then (1) follows.

M147. *Proposed by the Mayhem staff.*



The diameter of a large circle is broken into n equal parts to construct n smaller circles, as shown. Determine n so that the ratio of the shaded area to the unshaded area in the large circle is $3:1$.

Solution by Gabriel Krimker, grade 10 student, Buenos Aires, Argentina.

Let r be the radius of the large circle. The radius of each smaller circle is $\frac{r}{n}$. The shaded area is $\pi r^2 - n\pi\left(\frac{r}{n}\right)^2 = \pi r^2\left(1-\frac{1}{n}\right)$, and the unshaded area is $n\pi\left(\frac{r}{n}\right)^2 = \pi r^2\left(\frac{1}{n}\right)$. Then

$$3 \;=\; \frac{\pi r^2\left(1-\frac{1}{n}\right)}{\pi r^2\left(\frac{1}{n}\right)} \;=\; n\left(1-\frac{1}{n}\right) \;=\; n-1\,.$$

Hence, $n=4$.

Also solved by Roger He, grade 10 student, Prince of Wales Collegiate, St. John's, NL; Doug Newman, Lancaster, CA, USA; and Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M148. *Proposé par Vedula N. Murty, Dover, PA, USA.*

Soit $x > 1$, $y > 1$, $z > 1$ et $x^2 = yz$. Trouver la valeur de

$$\left(\log_{zx} xy^4z\right)\left(\log_{xy} xyz^4\right) .$$

Solution par Houda Anoun, LaBri, Bordeaux, France.

Soient x, y et z des nombres réels tels que $x > 1$, $y > 1$, $z > 1$ et $x^2 = yz$. Posons $f = \log_{zx}(xy^4z)$ et $g = \log_{xy}(xyz^4)$. On a alors

$$(xz)^f = xy^4z = x^3y^3 = (xy)^3 . \qquad (1)$$

D'autre part on a aussi

$$(xy)^g = xyz^4 = x^3z^3 = (xz)^3 . \qquad (2)$$

D'après (1) et (2) on a alors

$$(xz)^{fg} = \left((xy)^3\right)^g = \left((xy)^g\right)^3 = (xz)^9 .$$

Or comme $xz > 1$ donc on en déduit que $fg = 9$.

En outre résolu par Marcie Fairchild, Daniel Mills, Laura Steil et Willie Ward, étudiants, Samford University, Birmingham, Alabama, USA ; Shuang Han, étudiant, $12^{\text{ième}}$ catégorie, Holy Heart of Mary High School, St. John's, NL ; Gustavo Krimker, Universidad CAECE, Buenos Aires, Argentine ; et Luyan Zhong-Qiao, Columbia International College, Hamilton, ON.

M149. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A right-angled Heron triangle ABC has the following property: the area is λ times the perimeter, where λ is a positive integer. Determine all solutions (a, b, λ). (A Heron triangle is a triangle with integer sides and integer area.)

Solution by Marcie Fairchild, Daniel Mills, Laura Steil, and Willie Ward, students, Samford University, Birmingham, Alabama, USA.

The Heron triangles have to be right-angled with all sides of integer length. Thus, we know that the triangles we are looking for must have sides that make Pythagorean triples. We can list Pythagorean triples by using the following system:

$$\begin{aligned} a &= 2xyt , \\ b &= (x^2 - y^2)t , \\ c &= (x^2 + y^2)t , \end{aligned}$$

where x, y, and t are integers, x and y have opposite parity, $x > y$, and a and b are the legs of the triangle. Using this representation of the triangle sides, and letting P be the perimeter of the triangle and A the area, we have $P = a + b + c = 2xyt + (x^2 - y^2)t + (x^2 + y^2)t = 2xt(x + y)$ and $A = \frac{1}{2}ab = \frac{1}{2}(2xyt)(x^2 - y^2)t = xyt^2(x^2 - y^2)$. Now the problem stipulates that the area is λ times the perimeter, which implies that

$$xyt^2(x^2 - y^2) = \lambda\big(2xt(x + y)\big) .$$

This equation can be solved for λ to yield $\lambda = \frac{1}{2}y(x - y)t$. Such λ will be an integer unless both t and y are odd. Therefore, all solutions are given by

$$(a, b, c) = \left(2xyt,\ (x^2 - y^2)t,\ \frac{y(x - y)t}{2}\right) ,$$

where x and y have opposite parity, $x > y$, and at least one of y and t is even.

One incomplete solution was received.

M150. *Proposed by Arkady Alt, San Jose, CA, USA.*

Let two complex numbers z_1 and z_2 satisfy the conditions

$$\begin{aligned} z_1 + z_2 &= -(i + 1) , \\ z_1 \cdot z_2 &= -i . \end{aligned}$$

Without calculating z_1 and z_2, find $z_1 \cdot \overline{z_2}$.

Solution by the proposer.

Note that $z_1 \cdot \overline{z_2} = \frac{z_1}{z_2} \cdot |z_2|^2$. From $(z_1 + z_2)^2 = 2i = -2z_1 \cdot z_2$, we immediately obtain $z_1^2 + 4z_1z_2 + z_2^2 = 0$, or equivalently,

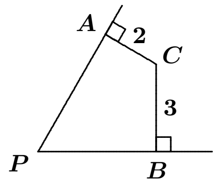
$$\left(\frac{z_1}{z_2}\right)^2 + 4\left(\frac{z_1}{z_2}\right) + 1 = 0 .$$

Thus, $\frac{z_1}{z_2}$ is real and negative. Therefore, $z_1 \cdot \overline{z_2}$ is also real and negative. Combining this with $|z_1 \cdot \overline{z_2}| = |z_1 \cdot z_2| = 1$, we see that $z_1 \cdot \overline{z_2} = -1$.

Problem of the Month

Ian VanderBurgh, University of Waterloo

It has been a long time since we have done any geometry! Since contest season is on the horizon, it is probably time to brush up on this aspect of our mathematical repertoire.



Problem (1992 Canadian Invitational Mathematics Challenge, Grade 10)

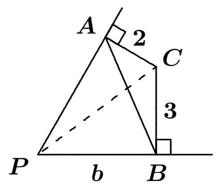
A point C is situated inside an angle of $60°$ at a distance of 2 units and 3 units from its sides. Determine the distance from point P to point C.

One of the wonderful things about geometry problems is the many different approaches that can be taken to solve the same problem. Here are three different approaches to this problem. We will keep the really nice approach for last to keep you reading until the end!

Solution 1: Since $PACB$ is a quadrilateral, the sum of its four interior angles is $360°$. Thus, $\angle ACB = 120°$. Join A to B. We can calculate the length of AB using the Cosine Law:

$$AB^2 = 2^2 + 3^2 - 2(2)(3)\cos 120° = 4 + 9 - 2(2)(3)\left(-\tfrac{1}{2}\right) = 19\,.$$

Thus, $AB = \sqrt{19}$.



Now, let $PB = b$. By joining P to C, we form two right triangles sharing the hypotenuse PC. This means that $PA^2 + AC^2 = PC^2 = PB^2 + BC^2$; whence,

$$PA^2 = PB^2 + BC^2 - AC^2 = b^2 + 9 - 4 = b^2 + 5\,.$$

Therefore, $PA = \sqrt{b^2+5}$.

Next, we apply the Cosine Law again, this time in $\triangle PAB$, and solve for b. This requires a bit of patience.

$$\begin{aligned} AB^2 &= PA^2 + PB^2 - 2(PA)(PB)\cos\angle APB\,, \\ 19 &= (b^2+5) + b^2 - 2\left(\sqrt{b^2+5}\right)(b)\left(\tfrac{1}{2}\right)\,, \\ 14 - 2b^2 &= -b\sqrt{b^2+5}\,, \\ 196 - 56b^2 + 4b^4 &= b^4 + 5b^2 \qquad \text{(squaring to get rid of square roots)}, \\ 3b^4 - 61b^2 + 196 &= 0\,. \end{aligned}$$

We have obtained a quartic equation which is really a quadratic in disguise (didn't we see that last month?). We solve for b by factoring:

$$\begin{aligned} (3b^2 - 49)(b^2 - 4) &= 0\,, \\ b^2 = \tfrac{49}{3} \quad \text{or} \quad b^2 &= 4\,. \end{aligned}$$

Remembering that b must be positive, we get $b = 7/\sqrt{3}$ or $b = 2$. If $b = 2$, then $PA = 3$. Unfortunately, we have to reject this solution (why?). Hence, $b = 7/\sqrt{3}$. (It is interesting to wonder why we obtained the inadmissible solution $b = 2$.)

But we are seeking PC. Well, $PC^2 = PB^2 + BC^2 = \frac{49}{3} + 9 = \frac{76}{3}$, which means that $PC = \sqrt{\frac{76}{3}}$.

That required some persistence, but it worked out in the end. It was interesting that we ended up with some of the same algebraic issues that we discovered last month.

Solution 2: We could use coordinates! I'll get you started on this and leave you to work through the details. Again, it is not very pretty, but it works. And it is a good exercise in analytic geometry, especially since you already know the answer that you should get, which will help you track down any errors you make along the way.

Set point P to be the origin $(0,0)$, with PB lying along the positive x–axis. Give B coordinates $(b,0)$. Then C has coordinates $(b,3)$. Since $\angle APB = 60°$, the slope of the line through P and A is $\tan 60° = \sqrt{3}$. Hence, this line has equation $y = \sqrt{3}x$.

From here, you need to find the equation of the line through A and C (*hint:* it is perpendicular to PA and passes through $(b,3)$), then find the coordinates of A (these will be in terms of b), and finally use the fact that the distance from A to C is 2. This will allow you to solve for b and then determine PC.

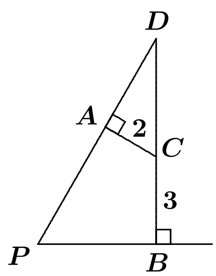
And now for the grand finale! After seeing the next solution, you will probably wish you had not seen either of the previous ones.

Solution 3: Extend the line through B and C up through C until it hits the line through P and A at D.

Since $\triangle DPB$ is right-angled at B, then $\angle PDB = 90° - 60° = 30°$. Thus, $\triangle DPB$ is a $30°$–$60°$–$90°$ triangle. Therefore, $PB = \frac{1}{\sqrt{3}}DB$. But $\triangle DCA$ is also a $30°$–$60°$–$90°$ triangle; whence, $CD = 2CA = 4$, and $DB = 7$. Then $PB = \frac{7}{\sqrt{3}}$. Therefore,

$$PC^2 = PB^2 + BC^2 = \frac{49}{3} + 9 = \frac{76}{3},$$

and $PC = \sqrt{\frac{76}{3}}$.



I think you would agree that this last approach was very nice.

We have seen three very different solutions to the same geometry problem. The last solution involves a nifty construction. Many geometrical problems have really elegant solutions involving a construction, but these solutions are usually very hard to find. What is the best way to find them, you ask? Lots of practice!

Pólya's Paragon

It Ain't So Complex (Part 3)

Shawn Godin

Last month we noticed that when we multiply two complex numbers in polar form, the argument of the product is coterminal with the sum of the arguments; that is,

$$(\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 + i\sin\theta_2) \;=\; \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\,.$$

If we let $f(\theta) = \cos\theta + i\sin\theta$, then this equation can be rewritten as

$$f(\theta_1)\cdot f(\theta_2) \;=\; f(\theta_1+\theta_2)\,.$$

Thinking of all the functions you know, you might notice that our function f is behaving like an exponential function $f(x) = b^x$ (where $b > 0$ is the base). When $f(x) = b^x$, we have

$$f(\theta_1)f(\theta_2) \;=\; b^{\theta_1}\cdot b^{\theta_2} \;=\; b^{\theta_1+\theta_2} = f(\theta_1+\theta_2)\,.$$

But we defined $f(\theta)$ in terms of $\cos\theta$ and $\sin\theta$. How can f possibly be an exponential function? What is the base b?

Looking to calculus, we have the following power series expansions:

$$\begin{aligned}
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\,,\\
\sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\,,\\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\,,
\end{aligned}$$

where the series are absolutely convergent for all real x. (For those of you who haven't taken calculus, you need to simply accept this.) Recall that $e \approx 2.71828\ldots$ is the base of the natural logarithm.

Let us use the series for $\sin x$ and $\cos x$ to calculate $\cos\theta + i\sin\theta$:

$$\begin{aligned}
\cos\theta + i\sin\theta &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\\
&= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots\\
&= 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots
\end{aligned}$$

The resulting series above could be obtained by putting $x = i\theta$ in the series for e^x. So we *define* $e^{i\theta} = \cos\theta + i\sin\theta$. Thus, we can write any complex number $z = r(\cos\theta + i\sin\theta)$ in polar form as

$$z \;= re^{i\theta}\,.$$

This exponential notation leads to some interesting results, such as *de Moivre's Formula*, which states that, for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta\,.$$

Using exponential notation, the theorem simply says $\left(e^{i\theta}\right)^n = e^{in\theta}$, which seems quite obvious (using the basic properties of exponents). We can use de Moivre's Formula to work out other trigonometric identities; for example, when $n = 2$, we get

$$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta\,,$$

Equating the real and imaginary parts, we get the double angle formulas for sine and cosine:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \text{and} \qquad \sin 2\theta = 2\sin\theta\cos\theta\,.$$

That will do for this issue. Next time, we will look at some applications to plane geometry. For homework, try the following:

1. Calculate $\cos 3\theta$ in terms of $\cos\theta$, and calculate $\sin 3\theta$ in terms of $\sin\theta$.
2. By using the exponential notation, obtain the formulas for $\cos(\theta_1 + \theta_2)$ and $\sin(\theta_1 + \theta_2)$ in terms of $\cos\theta_1$, $\cos\theta_2$, $\sin\theta_1$, and $\sin\theta_2$.
3. Use complex numbers to prove the *Triangle Inequality*. That is, for any triangle with sides a, b, c, prove that $c \le a + b$.
4. Use complex numbers to show that the medians of a triangle meet at a common point and determine how to find that point.

Finally, we consider the unanswered questions from last month's homework.

1. Converting the polar form to the exponential form, we have.

$$\begin{aligned}\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} &= \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}\\ &= \frac{r_1}{r_2}\big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\big)\,.\end{aligned}$$

3. Considering the equation in exponential form, we get $z^2 = i = e^{i\frac{\pi}{2}}$. If we let $z = re^{i\theta}$, then $z^2 = r^2e^{2i\theta} = e^{i\frac{\pi}{2}}$. The usual convention is that $r > 0$ for all $z \ne 0$; thus, $r = 1$ in our case and one solution is $z_1 = e^{i\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$. We approach the second solution by noting the redundancy in the polar form; that is,

$$e^{i\theta} = e^{i(\theta+2\pi k)}\,,$$

for any integer k. Thus, our other solution comes from considering $z^2 = e^{i\frac{\pi}{2}} = e^{i\frac{5\pi}{2}}$, which yields $z_2 = e^{i\frac{5\pi}{4}} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$. For further exploration, plot these two solutions on the complex plane; how are they related geometrically? How are the the solutions to $z^3 = i$ related geometrically?

Until next month, happy problem solving.