SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2963. [2004 : 367, 370; 2005 : 350–352] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any acute-angled triangle. Let r and R be the inradius and circumradius, respectively, and let s be the semiperimeter; that is, $s = \frac{1}{2}(a+b+c)$. Let m_a be the length of the median from A to BC, and let w_a be the length of the internal bisector of $\angle A$ from A to the side BC. We define m_b, m_c, w_b and w_c similarly. Prove that

(a) $\dfrac{3s^2-r^2-4Rr}{8sRr} \le \sum\limits_{\text{cyclic}} \dfrac{m_a}{aw_a} \le \dfrac{s^2-r^2-4Rr}{7sRr}$;

(b) $\dfrac{3}{4} \le \sum\limits_{\text{cyclic}} \dfrac{m_a^2}{b^2+c^2} \le \dfrac{4R+r}{4R}$.

The editor apologizes for misplacing the solution of Li Zhou, Polk Community College, Winter Haven, FL, USA. In his solution Zhou actually proves that the lower bound of 3/4 in part (b) can be increased to 1, which was conjectured by Walther Janous, Ursulinengymnasium, Innsbruck, Austria, as mentioned in the comments following the featured solutions to this problem [2005 : 350, 352]. *We present Zhou's proof below.*

We prove that the lower bound of $\frac{3}{4}$ in inequality (b) can be increased to 1. Since ABC is an acute-angled triangle, we have $b^2+c^2 > a^2$, so that $b^2+c^2 > \frac{1}{2}(a^2+b^2+c^2)$. Hence,

$$\begin{aligned}\sum_{\text{cyclic}} \frac{m_a^2}{b^2+c^2} &= \sum_{\text{cyclic}} \frac{2(b^2+c^2)-a^2}{4(b^2+c^2)} \\ &= \frac{3}{2} - \frac{1}{4}\sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} > \frac{3}{2} - \frac{1}{4}\sum_{\text{cyclic}} \frac{2a^2}{a^2+b^2+c^2} = 1,\end{aligned}$$

which completes the proof.

2972. [2004 : 369, 372] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

(a) Prove that if $0 \le \lambda \le 4$, then, for all positive real numbers x, y, z, t,

$$\begin{aligned}&(t^2+1)(x^3+y^3+z^3)+3(1-t^2)xyz \\ &\quad\ge (1+\lambda t)(x^2y+y^2z+z^2x)+(1-\lambda t)(xy^2+yz^2+zx^2).\end{aligned}$$

(b) For $t = \frac{1}{4}$ and $\lambda = 4$, the above inequality becomes

$$17(x^3 + y^3 + z^3) + 45xyz \ \geq \ 32(x^2y + y^2z + z^2x)\,.$$

Find all positive values of δ such that the inequality

$$x^3 + y^3 + z^3 + 3\delta xyz \ \geq \ (1+\delta)(x^2y + y^2z + z^2x)$$

holds for all x, y, z which are: (i) positive real numbers; (ii) side lengths of a triangle.

Composite of solutions by Li Zhou, Polk Community College, Winter Haven, FL, USA and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

(a) Without loss of generality, assume $x \leq y$ and $x \leq z$. Let $p = y - x$ and $q = z - x$. Then

$$\begin{aligned} D &= (1+t^2)(x^3+y^3+z^3) + 3(1-t^2)xyz \\ &\quad - (1+\lambda t)(x^2y + y^2z + z^2x) - (1-\lambda t)(xy^2 + yz^2 + zx^2) \\ &= (1+3t^2)(p^2 - pq + q^2)x + (p^3+q^3)t^2 \\ &\quad - \lambda pq(p-q)t + (p+q)(p-q)^2\,. \end{aligned}$$

The first term of this expression is clearly non-negative, and the last three terms form a quadratic function of t with discriminant

$$\begin{aligned} &\lambda^2p^2q^2(p-q)^2 - 4(p^3+q^3)(p+q)(p-q)^2 \\ &\qquad \leq \ 4(p-q)^2\big(4p^2q^2 - (p^3+q^3)(p+q)\big) \\ &\qquad = \ -4(p-q)^4(p^2+3pq+q^2) \ \leq \ 0\,. \end{aligned}$$

Hence, $D \geq 0$.

(b)(i) Similarly, assume that $p = y - x \geq 0$ and $q = z - x \geq 0$. Then

$$\begin{aligned} D &= x^3 + y^3 + z^3 + 3\delta xyz - (1+\delta)(x^2y + y^2z + z^2x) \\ &= (2-\delta)(p^2 - pq + q^2)x + p^3 - (1+\delta)p^2q + q^3\,. \end{aligned}$$

Since x can be arbitrarily close to 0, a necessary condition for $D \geq 0$ is

$$p^3 - (1+\delta)p^2q + q^3 \ \geq \ 0\,.$$

Define $f(r) = r^3 - (1+\delta)r^2 + 1$ for $r \geq 0$. Then $f'(r) = 3r^2 - 2(1+\delta)r = 0$ at $r = \frac{2}{3}(1+\delta)$. Setting $f\left(\frac{2}{3}(1+\delta)\right) \geq 0$, we get $-\frac{4}{27}(1+\delta)^3 + 1 \geq 0$; that is, $\delta \leq \frac{3\sqrt[3]{2}}{2} - 1 \approx 0.88988$.

Conversely, suppose $0 \leq \delta \leq \frac{3\sqrt[3]{2}}{2} - 1$. Since $(2-\delta)(p^2 - pq + q^2)x \geq 0$, the condition $p^3 - (1+\delta)p^2q + q^3 \geq 0$ is also sufficient for $D \geq 0$. Observe that $p^3 - (1+\delta)p^2q + q^3 \geq p^3 - \frac{3\sqrt[3]{2}}{2}p^2q + q^3$, and, by the AM–GM Inequality,

$$p^3 + q^3 \ = \ \tfrac{1}{2}p^3 + \tfrac{1}{2}p^3 + q^3 \ \geq \ \tfrac{3\sqrt[3]{2}}{2}p^2q\,.$$

(b)(ii) Let $u = \frac{1}{2}(z + x - y)$, $v = \frac{1}{2}(x + y - z)$, and $w = \frac{1}{2}(y + z - x)$. Then $x = u + v$, $y = v + w$, $z = w + u$, and u, v, $w \geq 0$. Hence,

$$\begin{aligned} D &= x^3+y^3+z^3+3\delta xyz-(1+\delta)(x^2y+y^2z+z^2x) \\ &= (1-\delta)(u^3+v^3+w^3)-6uvw+(1+\delta)(u^2v+v^2w+w^2u)\,. \end{aligned}$$

Assume similarly that $p = v - u \geq 0$ and $q = w - u \geq 0$. Then

$$D \;=\; 2(2-\delta)(p^2-pq+q^2)u+(1-\delta)p^3+(1+\delta)p^2q+(1-\delta)q^3\,.$$

Again, a necessary condition for $D \geq 0$ is

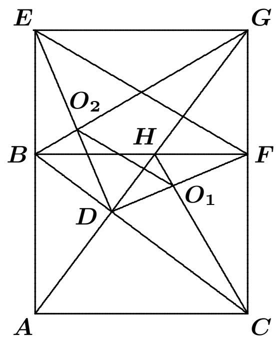
$$(1-\delta)p^3+(1+\delta)p^2q+(1-\delta)q^3 \;\geq\; 0\,,$$

which forces $\delta \leq 1$. Conversely, if $0 \leq \delta \leq 1$, then $D \geq 0$.

Also solved by the proposer.

2973. [2004 : 369, 372] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain (dedicated to Toshio Seimiya).*

Let ABC be a non-isosceles right triangle with right angle at A and $AC > AB$. Let D be the foot of the altitude from A to the side BC. Let G be the point of intersection of the line AD (extended) with the line through C which is parallel to AB. Let E be the point such that $ACGE$ is a rectangle, and let F be the point such that $BFGE$ is a rectangle. Let H be the intersection of AG and BF. Let O_1 be the intersection of the diagonals of the quadrilateral $CDHF$, and let O_2 be the intersection of the diagonals of the quadrilateral $BDGE$.

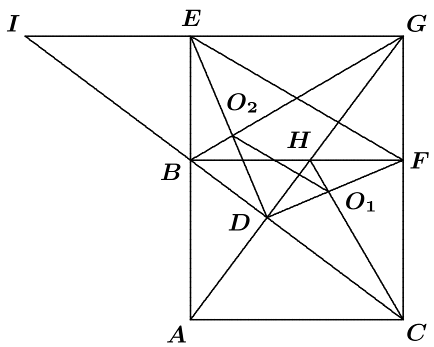


Prove that the triangles ABC, DFE, and DO_1O_2 are similar.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since $\angle BEG = \angle BDG = \angle BFG = 90^\circ$, the points B, E, G, F, and D are concyclic, implying that $\angle EDF = \angle EBF = \angle BAC$ and that $\angle DEF = \angle DBF = \angle BCA$. Thus, $\triangle ABC \sim \triangle DEF$, proving the first part.

Extend CB to meet line EG at the point I. Since BG is a transversal of $\triangle DEI$, we have, by Menelaus' Theorem,



$$\frac{DO_2}{O_2E}\cdot\frac{EG}{GI}\cdot\frac{IB}{BD} \;=\; -1\,.$$

Similarly, since HC is a transversal of $\triangle DFG$, we have

$$\frac{DO_1}{O_1F}\cdot\frac{FC}{CG}\cdot\frac{GH}{HD} \;=\; -1\,.$$

Thus,

$$\frac{DO_2}{O_2E}\cdot\frac{EG}{GI}\cdot\frac{IB}{BD} = \frac{DO_1}{O_1F}\cdot\frac{FC}{CG}\cdot\frac{GH}{HD}. \qquad (1)$$

Since $\triangle CFB \sim \triangle CGI$, we have

$$\frac{EG}{GI} = \frac{BF}{GI} = \frac{FC}{CG}.$$

Since $BH \parallel IG$, we have $\frac{IB}{BD} = \frac{GH}{HD}$. Therefore, equation (1) reduces to

$$\frac{DO_2}{O_2E} = \frac{DO_1}{O_1F},$$

implying that $O_1O_2 \parallel EF$. Thus, $\triangle DO_1O_2 \sim \triangle DEF$, and we are done.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (two solutions); MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Other solvers used a variety of methods including coordinates and vectors. Woo emphasized that his proof needed no trigonometry. In fact, only Zvonaru used trigonometry, and only one application of the Cosine Rule at that. Janous commented: "a lovely problem".

2974. [2004 : 369, 372] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let P be any point on the line BC in $\triangle ABC$. Let A_1 be the intersection of AP (possibly extended) with the line through B which is parallel to AC, and let A_2 be the intersection of AP (possibly extended) with the line through C which is parallel to AB.

Prove that the area of $\triangle ABC$ is the geometric mean of the areas of $\triangle A_1BC$ and $\triangle A_2BC$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.

Let $[ABC]$ denote the area of triangle ABC. Let points H, H_1, and H_2 be the feet of the perpendiculars to the line BC from points A, A_1, and A_2, respectively. From the pairs of similar triangles A_1BP, ACP, and ABP, A_2CP, we have

$$\frac{AH}{A_1H_1} = \frac{CP}{BP} \quad\text{and}\quad \frac{AH}{A_2H_2} = \frac{BP}{CP}.$$

Multiplying these two equalities, we find

$$(AH)^2 = A_1H_1\cdot A_2H_2. \qquad (1)$$

Since AH, A_1H_1, and A_2H_2 are just the altitudes to the side BC of triangles ABC, A_1BC, and A_2BC, respectively, and the areas of triangles with equal bases are proportional to the altitudes of the triangles, the equality (1) yields

$$[ABC]^2 = [A_1BC] \cdot [A_2BC],$$

which is the desired result.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); MICHEL BATAILLE, Rouen, France; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; ANDY PHAM, California State University, Fullerton, CA, USA; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2975. [2004 : 370, 372] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Given an inscribed convex quadrilateral with sides of length m, n, p, q, taken in order around the quadrilateral, and diagonals of length d and d', prove that $\sqrt{mp+nq} \le \frac{1}{2}(d+d')$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; Michel Bataille, Rouen, France; Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain; Kin Fung Chung, student, University of Toronto, Toronto, ON; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Richard B. Eden, Ateneo de Manila University, The Philippines; John G. Heuver, Grande Prairie, AB; Walther Janous, Ursulinengymnasium, Innsbruck, Austria; Michael Parmenter, Memorial University of Newfoundland, St. John's, NL; Victor Pambuccian, Arizona State University West, Phoenix, AZ, USA; Bogdan Suceavă, California State University, Fullerton, CA, USA; Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany; Peter Y. Woo, Biola University, La Mirada, CA, USA; Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

By Ptolemy's Theorem and the AM–GM Inequality, we have

$$\sqrt{mp+nq} = \sqrt{dd'} \le \tfrac{1}{2}(d+d').$$

Equality occurs if and only if $d = d'$; that is, if and only if the inscribed quadrilateral is an isosceles trapezium.

Also solved by RICHARD I. HESS, Rancho Palos Verdes, CA, USA; GEORGE TSAPAKIDIS, Agrinio, Greece; and the proposer.

2976. [2004 : 429, 432] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Let a, b, $c \in \mathbb{R}$. Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \;\geq\; (ab+bc+ca)^3\,.$$

Solution by Kee-Wai Lau, Hong Kong, China.

It can be readily checked that

$$\begin{aligned}(a^2+ab+b^2)&(b^2+bc+c^2)(c^2+ca+a^2)-(ab+bc+ca)^3\\ &= \frac{1}{6}\Big[2(ab+bc+ca)^2\sum_{\text{cyclic}}(a-b)^2+(a+b+c)^2\sum_{\text{cyclic}}a^2(b-c)^2\Big]\,.\end{aligned}$$

Clearly, the last expression is non-negative, and the result is immediate.

Also solved by ARKADY ALT, San Jose, CA, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania (two solutions); VASILE CÎRTOAJE, University of Ploiesti, Romania; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (two solutions); ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; TITU ZVONARU, Comăneşti, Romania; and the proposer.

Bencze and Cîrtoaje proved the two stronger inequalities

$$\begin{aligned}(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) &\geq \frac{27}{64}(a+b)^2(b+c)^2(a+b)^2\\ &\geq (ab+bc+ca)^3\,,\end{aligned}$$

from which the given inequality follows. Bencze also gave a generalization and stated some related problems, while Cîrtoaje mentioned that he had proposed this problem earlier and gave a reference to the book of L. Panaitopol, V. Băndilă and M. Lascu, Inequalities*, GIL, Zalău (Romania) 1995, p. 147.*

2977. [2004 : 429, 432] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let a_1, a_2, $\ldots$, a_n be positive real numbers, let $r=\sqrt[n]{a_1a_2\cdots a_n}$, and let

$$E_n \;=\; \frac{1}{a_1(1+a_2)}+\frac{1}{a_2(1+a_3)}+\cdots+\frac{1}{a_n(1+a_1)}-\frac{n}{r(1+r)}\,.$$

(a) Prove that $E_n \geq 0$ for

- (a_1) $n=3$;
- (a_2) $n=4$ and $r\leq 1$;
- (a_3) $n=5$ and $\frac{1}{2}\leq r\leq 2$;
- (a_4) $n=6$ and $r=1$.

(b)★ Prove or disprove that $E_n \geq 0$ for

- (b_1) $n=5$ and $r>0$;
- (b_2) $n=6$ and $r\leq 1$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

(a) For convenience, we take all indices modulo n. Let

$$F_n = \sum_{i=1}^{n} \frac{1}{a_i(1+a_{i+1})}.$$

Put $k = \sum\limits_{i=1}^{n} \sqrt[n]{a_i a_{i+1}^2 \cdots a_{i+n-2}^{n-1}}$ and $x_i = \frac{1}{k}\sqrt[n]{a_i a_{i+1}^2 \cdots a_{i+n-2}^{n-1}}$ for $1 \le i \le n$. Then

$$\sum_{i=1}^{n} x_i = 1 \qquad \text{and} \qquad F_n = \sum_{i=1}^{n} \frac{x_i}{r(x_{i+1}+rx_{i+2})}.$$

Applying Jensen's Inequality to the convex function $1/t$ on $(0,\infty)$, we get

$$F_n \ge \frac{1}{r(S+rT)},$$

where $S = \sum\limits_{i=1}^{n} x_i x_{i+1}$ and $T = \sum\limits_{i=1}^{n} x_i x_{i+2}$.

(a_1) When $n=3$, it is easy to see that

$$S = T \le \tfrac{1}{3}\left(\sum_{i=1}^{3} x_i\right)^2 = \tfrac{1}{3}.$$

Hence, $F_3 \ge \dfrac{3}{r(1+r)}$.

(a_2) For $n=4$, let $M = (x_1+x_2)(x_3+x_4)+(x_2+x_3)(x_4+x_1)$ and $N = (x_1+x_3)(x_2+x_4)$. Then $S = N$ and $T = M - N$. By the AM–GM Inequality,

$$M \le 2\left(\tfrac{1}{2}(x_1+x_2+x_3+x_4)\right)^2 = \tfrac{1}{2}$$
$$\text{and} \qquad N \le \left(\tfrac{1}{2}(x_1+x_2+x_3+x_4)\right)^2 = \tfrac{1}{4}.$$

Hence, for $r \le 1$, we have

$$F_4 \ge \frac{1}{r((1-r)N+rM)} \ge \frac{4}{r((1-r)+2r)} = \frac{4}{r(1+r)}.$$

(a_3) We improve this part to $E_5 \ge 0$ for $\frac{3}{7} \le r \le \frac{7}{3}$. Let

$$Q = \sum_{i=1}^{5} x_i^2, \qquad M = \sum_{i=1}^{5}\left(x_i + x_{i+1} + \tfrac{1}{2}x_{i+2}\right)\left(\tfrac{1}{2}x_{i+2} + x_{i+3} + x_{i+4}\right),$$
$$\text{and} \qquad N = \sum_{i=1}^{5}\left(x_i + x_{i+2} + \tfrac{1}{2}x_{i+4}\right)\left(\tfrac{1}{2}x_{i+4} + x_{i+6} + x_{i+8}\right).$$

Then it is purely computational to verify that

$$1 = \left(\sum_{i=1}^{5} x_i\right)^2 = Q + 2S + 2T,$$
$$M = \tfrac{1}{4}Q + 2S + 4T, \qquad \text{and} \qquad N = \tfrac{1}{4}Q + 4S + 2T.$$

Solving the system, we get

$$S \;=\; \tfrac{1}{20}(7N - 3M - 1) \qquad \text{and} \qquad T \;=\; \tfrac{1}{20}(7M - 3N - 1)\,.$$

Also, by the AM–GM Inequality,

$$M \;\le\; \sum_{i=1}^{5} \left(\tfrac{1}{2}(x_i + x_{i+1} + x_{i+2} + x_{i+3} + x_{i+4})\right)^2 \;=\; \tfrac{5}{4}\,.$$

Likewise, $N \le \frac{5}{4}$. Hence,

$$\begin{aligned} S + rT &= \tfrac{1}{20}\big((7r-3)M + (7-3r)N - (r+1)\big) \\ &\le \tfrac{1}{80}\big(5(7r-3) + 5(7-3r) - 4(r+1)\big) \;=\; \tfrac{1}{5}(1+r)\,. \end{aligned}$$

Thus, $F_5 \ge \dfrac{5}{r(1+r)}$ for $\frac{3}{7} \le r \le \frac{7}{3}$.

(a$_4$) By the AM–GM Inequality,

$$\begin{aligned} 1 \;=\; \left(\sum_{i=1}^{6} x_i\right)^2 &= (x_1+x_4)^2 + (x_2+x_5)^2 + (x_3+x_6)^2 + 2(S+T) \\ &\ge (x_1+x_4)(x_2+x_5) + (x_2+x_5)(x_3+x_6) \\ &\qquad + (x_3+x_6)(x_1+x_4) + 2(S+T) \\ &= 3(S+T)\,. \end{aligned}$$

Hence, $F_6 \ge \dfrac{1}{S+T} \ge 3$ for $r = 1$.

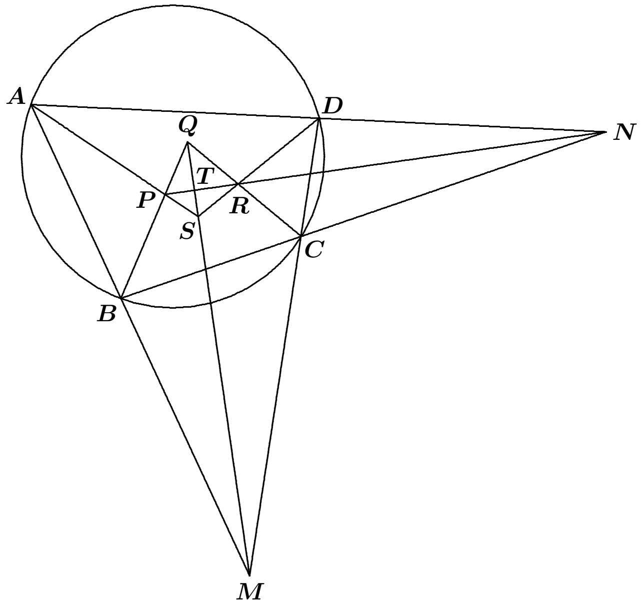
Also solved by the proposer. Part (a$_1$) *alone was solved by MIHÁLY BENCZE, Brasov, Romania and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany. The starred part (part* (b)) *remain open.*

2978★. [2004 : 429, 432] *Proposed by Christopher J. Bradley, Bristol, UK.*

Let $ABCD$ be a cyclic quadrilateral. The internal bisectors of angles A and B meet at P. Points Q, R, S are similarly defined by a cyclic change of letters. It is easy to show that $PQRS$ is a cyclic quadrilateral. Suppose that the circles $ABCD$ and $PQRS$ have centres O and X, respectively. Let AC meet BD at E. Prove that O, E, and X are collinear. Prove also that $PR \perp QS$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, with some additional detail taken from the solutions marked with an asterisk ($*$) *in the list of solvers below.*

We assume that the quadrilateral $ABCD$ is convex. If $AB \parallel CD$ or $BC \parallel AD$, then $ABCD$ is an isosceles trapezoid, and O, E, and X lie on its axis of symmetry. Thus, we may suppose that the lines AB and CD meet at a point M and the lines BC and AD meet at a point N (see the diagram on the next page). We denote the internal angles at the vertices of the quadrilaterals $ABCD$ and $PQRS$ by the same symbols as the vertices themselves. We also denote the angles AMD and BNA simply by M and N, respectively.



To show that the quadrilateral $PQRS$ is cyclic, we check that the opposite angles Q and S are supplementary:

$$Q + S \;=\; \left(180^\circ - \tfrac{1}{2}C - \tfrac{1}{2}B\right) + \left(180^\circ - \tfrac{1}{2}A - \tfrac{1}{2}D\right) \;=\; 180^\circ\,,$$

since $A + B + C + D = 360^\circ$. This shows that $PQRS$ is cyclic whether or not $ABCD$ is cyclic.

Next, we note that Q and S both lie on the internal bisector of angle AMD, because Q is an excentre of $\triangle BMC$ while S is the incentre of $\triangle AMD$. (The internal and external bisectors of the three angles of a triangle meet by threes in four points, which are the incentre and the three excentres of the triangle.) Similarly, P and R both lie on the bisector of $\triangle BNA$.

Now,

$$\begin{aligned}\angle SDC \;=\; \tfrac{1}{2}D \;&=\; \tfrac{1}{2}\,(180^\circ - A - M) \;=\; \tfrac{1}{2}(C - M) \\ &=\; \angle QCD - \angle QMD \;=\; \angle MQC\,.\end{aligned}$$

It follows that opposite angles in the quadrilateral $QCDS$ are supplementary; hence, this quadrilateral is cyclic. Then $MC \cdot MD = MQ \cdot MS$. Similarly, $NC \cdot NB = NP \cdot NR$. Therefore, MN is the radical axis of the circles $ABCD$ and $PQRS$, which implies that $OX \perp MN$. Moreover, MN is the polar of E with respect to the circle $ABCD$, and consequently, $OE \perp MN$. Thus, O, X, and E are collinear.

Let T be the point at which PR intersects QS. Then

$$\begin{aligned}\angle MTN &= \angle AMT + A + \angle ANT \;=\; \tfrac{1}{2}(M+A) + \tfrac{1}{2}(A+N) \\ &= \tfrac{1}{2}(180^\circ - D) + \tfrac{1}{2}(180^\circ - B) \;=\; 90^\circ\,,\end{aligned}$$

since $B + D = 180^\circ$. Thus, $PR \perp QS$.

Also solved by ∗MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; ∗VÁCLAV KONEČNÝ, Big Rapids, MI, USA; and DAVID MONK, Edinburgh, Scotland, UK.

2979. [2004 : 430, 432] *Proposed by Ovidiu Furdui, student, Western Michigan University, Kalamazoo, MI, USA.*

If $e_n = \left(1+\frac{1}{n}\right)^n$, find $\displaystyle\lim_{n\to\infty}\left(\frac{2n(e-e_n)}{e}\right)^n$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since $n\ln\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + \frac{1}{3n^2} + O\left(\frac{1}{n^3}\right)$, we obtain

$$e_n \;=\; \left(1+\frac{1}{n}\right)^n \;=\; e^{n\ln\left(1+\frac{1}{n}\right)} \;=\; e^{1-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)}\,.$$

Then

$$\begin{aligned}\frac{e-e_n}{e} &= 1 - \frac{e_n}{e} \;=\; 1 - e^{-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)} \\ &= -\left(-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)\right) \\ &\qquad -\frac{1}{2}\left(-\frac{1}{2n}+\frac{1}{3n^2}+O\left(\frac{1}{n^3}\right)\right)^2 + O\left(\frac{1}{n^3}\right) \\ &= \frac{1}{2n} - \frac{11}{24n^2} + O\left(\frac{1}{n^3}\right)\,.\end{aligned}$$

Hence,

$$\begin{aligned}n\ln\frac{2n(e-e_n)}{e} &= n\ln\left(1-\frac{11}{12n}+O\left(\frac{1}{n^2}\right)\right) \;=\; n\left(-\frac{11}{12n}+O\left(\frac{1}{n^2}\right)\right) \\ &= -\frac{11}{12} + O\left(\frac{1}{n}\right)\,,\end{aligned}$$

and therefore,

$$\lim_{n\to\infty}\left(\frac{2n(e-e_n)}{e}\right)^n \;=\; \lim_{n\to\infty} e^{n\ln\frac{2n(e-e_n)}{e}} \;=\; e^{-\frac{11}{12}}\,.$$

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State University, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY RICARDO, Medgar Evers College (CUNY), Brooklyn, NY, USA; JOEL SCHLOSBERG, Bayside, NY, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and the proposer.

2980. [2004 : 430, 432] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Let Γ be a semicircle with centre O on diameter AB. Let C be the mid-point of the semicircular arc $\widehat{AB}$. Let P be an arbitrary point on the semicircle different from both A and B.

Determine all points Q on the semicircle such that if the lines BP and AQ intersect at a point S, then C is the orthocentre of $\triangle SPQ$.

Solution by Michel Bataille, Rouen, France.

Since Q is defined such that QC is an altitude in $\triangle SPQ$, Q must be the point where the perpendicular to BP through C intersects Γ. Conversely, suppose first that P is on the arc $\widehat{AC}$ of Γ; then the perpendicular to BP from C meets the line segment AB and does not meet Γ, in which case there exists no suitable point Q. Suppose now that P is on the arc $\widehat{BC}$ of Γ (B and C excluded) and let Q and M be the points of intersection of the perpendicular to BP through C with Γ and BP, respectively. Being perpendicular to BP, AP and CM are parallel, implying that $\angle PCM = \angle APC = 45^\circ$. If N is the intersection point of PC and AQ, then $BQ \perp QN$ and $\angle BQC = 45^\circ$, implying that $\angle CQN = 45^\circ$. Since we also have $\angle QCN = \angle PCM = 45^\circ$, it follows that $\angle QNC = 90^\circ$; that is, $PC \perp AQ$. The lines PC and QC are two altitudes of the triangle PSQ, which makes C its orthocentre. Thus, for a point P situated on the arc $\widehat{BC}$, there is one suitable point Q, namely the intersection of Γ with the perpendicular to BP through C.

Also solved by JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOEL SCHLOSBERG, Bayside, NY, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incomplete solution.

Solvers provided two other characterizations for the point Q: For each point P on arc $\widehat{BC}$, Q is the unique point of arc $\widehat{AC}$ for which $\angle POQ = 90^\circ$; also, Q is the unique point of arc $\widehat{AC}$ for which QC is parallel to AP. Both these characterizations follow easily from the featured solution. Also, from the known angles at N, C, and M, we find that $\angle ASB$ $(= \angle NSM) = 45^\circ$. It follows that as P moves along arc $\widehat{BC}$ of Γ, the point S traces out an arc of the circle, say Ω, whose chord AB subtends an angle of 45°. This observation provides yet another characterization of the point Q: for each point P, define S to be the point where the line BP intersects Ω, in which case Q is the point of intersection of SA with Γ.

As a final observation, note that there is no reason to restrict Γ to a semicircle: Were P allowed to move about the entire circle ABC, all our characterizations of Q remain valid, and C continues to be the orthocentre of triangle PQS except when the triangle degenerates (when P coincides with A or when Q coincides with B). The only surprise is that when P and Q both lie on the half of circle ABC opposite point C, $\angle BSA = 135^\circ$; thus, in the more general problem, the locus of S consists of two arcs of a single circle on chord AB.

2981★. [2004 : 430, 433] *Proposed by Wu Wei Chao, Guang Zhou University (New), Guang Zhou City, Guang Dong Province, China.*

Find all pairs of positive integers a and b such that a divides b^2+b+1, and b divides a^2+a+1.

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

Define a sequence $\{a_n\}$ by $a_0 = a_1 = 1$ and $a_{n+2} = 5a_{n+1} - a_n - 1$ for $n \geq 0$. It is easily shown by induction that $a_{n+1} > a_n \geq 1$ for all $n \geq 1$. We claim that $\{a_n\}$ satisfies the following recurrence relation:

$$a_{n-1}a_{n+1} \;=\; a_n^2 + a_n + 1 \tag{1}$$

for all $n \geq 1$. Since $a_0 = a_1 = 1$ and $a_2 = 3$, equation (1) is true for $n = 1$. Suppose that (1) holds for some $n \geq 1$. Then

$$\begin{aligned} a_n a_{n+2} &= a_n(5a_{n+1} - a_n - 1) \;=\; 5a_n a_{n+1} - a_n^2 - a_n \\ &= 5a_n a_{n+1} - a_{n-1}a_{n+1} + 1 \;=\; (5a_n - a_{n-1})a_{n+1} + 1 \\ &= (a_{n+1} + 1)a_{n+1} + 1 \;=\; a_{n+1}^2 + a_{n+1} + 1\,, \end{aligned}$$

completing the induction.

As a result, a_n divides $a_{n+1}^2 + a_{n+1} + 1$ and a_{n+1} divides $a_n^2 + a_n + 1$ for all $n \geq 1$. Since a_0 clearly divides $a_1^2 + a_1 + 1$ and a_1 divides $a_0^2 + a_0 + 1$, we conclude that the set

$$\mathbb{S} \;=\; \{(a_n, a_{n+1}) \mid n \geq 0\} \cup \{(a_{n+1}, a_n) \mid n \geq 0\}$$

provides infinitely many solutions.

We now prove that $\mathbb{S}$ actually contains all the solutions.

Suppose that a and b are positive integers satisfying the given conditions. If $a = b$, then the given conditions imply that $a = b = 1$. Hence, $(a,b) = (a_0, a_1) \in \mathbb{S}$.

If $a \neq b$, we may assume without loss of generality that $a < b$. Note that a and b must be relatively prime, since if $k = \gcd(a,b)$, then k divides both a and $a^2 + a + 1$, which clearly implies that $k = 1$. From the given conditions, there exist positive integers x_1 and y_1 such that $a^2 + a + 1 = bx_1$ and $b^2 + b + 1 = ay_1$.

If $x_1 > a$, then $x_1 \geq a + 1$, which, together with $b \geq a + 1$, imply that $a^2 + a + 1 = bx_1 \geq (a+1)^2$, a contradiction (since $a > 0$).

If $x_1 = a$, then $a^2 + a + 1 = ba$, which implies that $a = 1$. Since b divides $a^2 + a + 1$, it follows that $b = 3$ and we have $(a,b) = (a_1, a_2) \in \mathbb{S}$.

Thus, we are only left with the case $x_1 < a$.

Note that

$$\begin{aligned} &b^2(x_1^2 + x_1 + 1) \\ &\quad= (a^2 + a + 1)^2 + b(a^2 + a + 1) + b^2 \\ &\quad= a^2(a+1)^2 + 2a(a+1) + ba(a+1) + b^2 + b + 1\,. \end{aligned} \tag{2}$$

Since a divides $b^2 + b + 1$ and a and b are relatively prime, we deduce from (2) that a divides $x_1^2 + x_1 + 1$. It follows that x_1 and a also satisfy the given conditions.

If $x_1 > 1$, then we may repeat this procedure to find a positive integer x_2 with $x_2 < x_1$ such that x_1 divides $x_2^2 + x_2 + 1$ and $x_1^2 + x_1 + 1 = ax_2$.

Continuing, we obtain positive integers x_1, x_2, x_3, ... with $x_{i+1} < x_i$, such that x_i divides $x_{i+1}^2 + x_{i+1} + 1$, and $x_i^2 + x_i + 1 = x_{i-1}x_{i+1}$ for all $i = 1, 2, 3, \ldots$ where $x_0 = a$. Since $a > x_1 > x_2 > \cdots \geq 1$, the process must terminate at $x_m = 1 = a_1$, for some m.

Since $x_{m-1} > x_m$ and x_{m-1} divides $x_m^2 + x_m + 1$, we see that $x_{m-1} = 3 = a_2$. Also,

$$a_1 x_{m-2} \;=\; x_m x_{m-2} \;=\; x_{m-1}^2 + x_{m-1} + 1 \;=\; a_2^2 + a_2 + 1 \;=\; a_1 a_2\,,$$

which implies that $x_{m-2} = a_3$. Continuing this argument, we then obtain $x_2 = a_{m-1}$, $x_1 = a_m$, and $a = a_{m+1}$.

Finally, from $a_m b = x_1 b = a^2 + a + 1 = a_{m+1}^2 + a_{m+1} + 1 = a_m a_{m+2}$, we conclude that $b = a_{m+2}$.

Therefore, $(a, b) = (a_{m+1}, a_{m+2}) \in \mathbb{S}$, and the proof is complete.

Also solved by MANUEL BENITO, ÓSCAR CIAURRI, and EMILIO FERNÁNDEZ, Logroño, Spain; IRVINE ROBINSON, Math Challenge at Western, London, Ontario; MARIAN TETIVA, Bîrlad, Romania; and YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON. There was also one incorrect and two incomplete solutions.

Tetiva informed us that this problem has appeared before in the September, 1988 issue of Komal Magazine *(vol. 48, no. 6), and it was proposed by Ervin Fried of Budapest.*

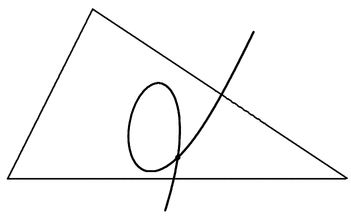
2982★. [2004 : 430, 433] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

In a given triangle ABC, points D, E, F are taken on the sides BC, CA, AB, respectively, such that

$$BD : DC \;=\; CE : EA \;=\; AF : FB \;=\; \frac{1-\lambda}{\lambda}\,,$$

where λ is a constant. (If $0 < \lambda < 1$, then the points are interior to the sides; if $\lambda < 0$ or $\lambda > 1$, then the points are exterior to the sides; if $\lambda = 0$ or $\lambda = 1$, then the points are coincident with the vertices A, B, C.)

It is easy to see that the centroid of $\triangle DEF$ is a fixed point as λ varies. The curve in the figure is the locus of the circumcentre of $\triangle DEF$ as λ varies. Determine this curve.



Solution by Michel Bataille, Rouen, France.

Let $BC = a$, $CA = b$, $AB = c$, $EF = d$, $FD = e$, $DE = f$. From the hypotheses, we have $\overrightarrow{AF} = (1-\lambda)\overrightarrow{AB}$ and $\overrightarrow{AE} = \lambda\overrightarrow{AC}$, so that

$$\begin{aligned} d^2 &= \left(\overrightarrow{AF} - \overrightarrow{AE}\right)^2 = AF^2 + AE^2 - 2\overrightarrow{AF}\cdot\overrightarrow{AE} \\ &= (1-\lambda)^2c^2 + \lambda^2b^2 - 2\lambda(1-\lambda)\overrightarrow{AB}\cdot\overrightarrow{AC}\,. \end{aligned}$$

Since $2\overrightarrow{AB}\cdot\overrightarrow{AC} = b^2 + c^2 - a^2$, we easily obtain that

$$d^2 = \lambda^2\left(2b^2 + 2c^2 - a^2\right) + \lambda\left(a^2 - b^2 - 3c^2\right) + c^2\,.$$

Similar relations (with cyclic permutation of the letters a, b, c) hold for e^2 and f^2.

Now, the circumcentre, Ω, of $\triangle DEF$ has areal coordinates (α, β, γ) with respect to (D, E, F) with

$$\begin{aligned} \alpha &= \alpha(\lambda) = d^2\left(e^2 + f^2 - d^2\right)\,, \\ \beta &= \beta(\lambda) = e^2\left(f^2 + d^2 - e^2\right)\,, \\ \gamma &= \gamma(\lambda) = f^2\left(d^2 + e^2 - f^2\right)\,. \end{aligned}$$

Note that $\alpha+\beta+\gamma = 16[DEF]^2$, where $[X]$ denotes the area of polygon X.

From

$$D = \lambda B + (1-\lambda)C\,,\quad E = \lambda C + (1-\lambda)A\,,\quad F = \lambda A + (1-\lambda)B\,,$$

we deduce two results. First, the ratio $\dfrac{[DEF]}{[ABC]}$, being the modulus of the determinant

$$\begin{vmatrix} 0 & 1-\lambda & \lambda \\ \lambda & 0 & 1-\lambda \\ 1-\lambda & \lambda & 0 \end{vmatrix}\,,$$

is $3\lambda^2 - 3\lambda + 1$. Secondly, the areal coordinates of Ω with respect to (A, B, C) are

$$\left((1-\lambda)\beta + \lambda\gamma,\ \lambda\alpha + (1-\lambda)\gamma,\ (1-\lambda)\alpha + \lambda\beta\right)\,.$$

As a result, we have

$$16K^2\left(3\lambda^2 - 3\lambda + 1\right)^2\overrightarrow{a\Omega} = \left(\lambda\alpha + (1-\lambda)\gamma\right)\overrightarrow{AB} + \left((1-\lambda)\alpha + \lambda\beta\right)\overrightarrow{AC}\,,$$

where $K = [ABC]$.

In the system of oblique axes with origin A, x–axis AB (with unit direction vector $\overrightarrow{AB}$) and y–axis AC (with unit direction vector $\overrightarrow{AC}$), the coordinates of Ω are given by

$$x(\lambda) = \frac{\lambda\alpha(\lambda) + (1-\lambda)\gamma(\lambda)}{16K^2\left(3\lambda^2 - 3\lambda + 1\right)^2}\,,\quad y(\lambda) = \frac{(1-\lambda)\alpha(\lambda) + \lambda\beta(\lambda)}{16K^2\left(3\lambda^2 - 3\lambda + 1\right)^2}\,. \qquad (1)$$

A lengthy (but easy) calculation yields

$$\begin{aligned}\alpha(\lambda) &= d^2\left(e^2+f^2-d^2\right)\\ &= \Big(\lambda^2\left(\lambda 2b^2+2c^2-a^2\right)+\lambda\left(a^2-b^2-3c^2\right)+c^2\Big) \qquad (2)\\ &\quad\cdot\Big(\lambda^2\left(5a^2-b^2-c^2\right)+\lambda\left(3c^2-b^2-5a^2\right)+a^2+b^2-c^2\Big),\\ \beta(\lambda) &= d^2\left(f^2+d^2-e^2\right)\\ &= \Big(\lambda^2\left(\lambda 2c^2+2a^2-b^2\right)+\lambda\left(b^2-c^2-3a^2\right)+a^2\Big) \qquad (3)\\ &\quad\cdot\Big(\lambda^2\left(5b^2-c^2-a^2\right)+\lambda\left(3a^2-c^2-5b^2\right)+b^2+c^2-a^2\Big),\\ \alpha(\lambda) &= d^2\left(e^2+f^2-d^2\right)\\ &= \Big(\lambda^2\left(\lambda 2a^2+2b^2-c^2\right)+\lambda\left(c^2-a^2-3b^2\right)+b^2\Big) \qquad (4)\\ &\quad\cdot\Big(\lambda^2\left(5c^2-a^2-b^2\right)+\lambda\left(3b^2-a^2-5c^2\right)+c^2+a^2-b^2\Big),\end{aligned}$$

and the desired locus of Ω is the parametrized curve (with parameter λ) defined by $x = x(\lambda)$, $y = y(\lambda)$ (given by (1) with $\alpha(\lambda)$, $\beta(\lambda)$, $\gamma(\lambda)$ given by (2), (3), (4), respectively).

In particular, if $\triangle ABC$ is isosceles, right angled at A with $b = c = 1$ (so that $a^2 = 2$ and $K = \frac{1}{2}$), we obtain that

$$\begin{aligned}x(\lambda) &= \frac{3\lambda^5+3\lambda^4-14\lambda^3-6\lambda+1}{2\left(3\lambda^2-3\lambda+1\right)^2},\\ y(\lambda) &= \frac{-3\lambda^5+18\lambda^4-28\lambda^3+20\lambda^2-7\lambda+1}{2\left(3\lambda^2-3\lambda+1\right)^2}.\end{aligned}$$

Plotting this using MAPLE® gives a curve that looks like the proposer's.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (who, however, did not check that his curve looked like the proposer's).

2983. [2004 : 430, 433] *Proposed by Vasile Cîrtoaje, University of Ploiesti, Romania.*

Let $a_1, a_2, \ldots, a_n < 1$ be non-negative real numbers satisfying

$$a = \sqrt{\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}} \geq \frac{\sqrt{3}}{3}.$$

Prove that

$$\frac{a_1}{1-a_1^2}+\frac{a_2}{1-a_2^2}+\cdots+\frac{a_n}{1-a_n^2} \geq \frac{na}{1-a^2}.$$

Solution by Michel Bataille, Rouen, France.

The inequality to be proved can be rewritten as

$$\frac{a_1^2}{na^2}f(a_1) + \frac{a_2^2}{na^2}f(a_2) + \cdots + \frac{a_n^2}{na^2}f(a_n) \;\geq\; f(a)\,, \tag{1}$$

where f denotes the function defined on $[0,1)$ by $f(x) = \frac{1}{x(1-x^2)}$. The first two derivatives of f are given by

$$f'(x) \;=\; \frac{3x^2-1}{\big(x(1-x^2)\big)^2} \qquad \text{and} \qquad f''(x) \;=\; \frac{2(6x^4-3x^2+1)}{\big(x(1-x^2)\big)^3}\,,$$

showing that $f'(x) \geq 0$ for $x \in \big[\sqrt{3}/3, 1\big)$ and $f''(x) > 0$ for $x \in [0,1)$. From the latter, f is convex. Noticing that $\sum\limits_{k=1}^{n} \frac{a_k^2}{na^2} = 1$, we apply Jensen's Inequality to get

$$\frac{a_1^2}{na^2}f(a_1) \;+\; \frac{a_2^2}{na^2}f(a_2) \;+\; \cdots \;+\; \frac{a_n^2}{na^2}f(a_n) \;\geq\; f\left(\frac{a_1^3+a_2^3+\cdots+a_n^3}{na^2}\right).$$

From the Power Mean Inequality, we have

$$\left(\frac{a_1^3+a_2^3+\cdots+a_n^3}{n}\right)^{1/3} \;\geq\; \left(\frac{a_1^2+a_2^2+\cdots+a_n^2}{n}\right)^{1/2},$$

and from the hypothesis on a, we obtain

$$\frac{a_1^3+a_2^3+\cdots+a_n^3}{na^2} \;\geq\; a \;\geq\; \frac{\sqrt{3}}{3}\,.$$

Since f is increasing on $\big[\sqrt{3}/3, 1\big)$, we thus have

$$f\left(\frac{a_1^3+a_2^3+\cdots+a_n^3}{na^2}\right) \;\geq\; f(a)\,.$$

Using this result above, we obtain (1).

Also solved by ARKADY ALT, San Jose, CA, USA; MIHÁLY BENCZE, Brasov, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; B.J. VENKATACHALA, Indian Institute of Science, Bangalore, India; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2984. [2004 : 431, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{ij(i+j)} \;=\; 2\sum_{n=1}^{\infty}\frac{1}{n^3}\,.$$

I. *Solution by Roger Zarnowski, Angelo State University, San Angelo, TX, USA.*

Denoting the expression on the left by S, we have

$$S \;=\; \sum_{i=1}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{\infty}\left(\frac{1}{j}-\frac{1}{i+j}\right)\right) \;=\; \sum_{i=1}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}\right).$$

Hence,

$$\begin{aligned} 2S &= 2\left(1+\sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i}\frac{1}{j}\right)\right) \;=\; 2\left(1+\sum_{i=2}^{\infty}\left(\frac{1}{i^2}\left(\frac{1}{i}+\sum_{j=1}^{i-1}\frac{1}{j}\right)\right)\right) \\ &= 2\left(1+\sum_{i=2}^{\infty}\frac{1}{i^3}+\sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i-1}\frac{1}{j}\right)\right) \\ &= 2\sum_{i=1}^{\infty}\frac{1}{i^3}+2\sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i-1}\frac{1}{j}\right). \qquad (1) \end{aligned}$$

Next, we set $k=i+j$ in the original double summation. Since k ranges from 2 to infinity and, for each fixed k, j ranges from 1 to $k-1$ in order for $i=k-j$ to remain positive, we have

$$\begin{aligned} S &= \sum_{k=2}^{\infty}\sum_{j=1}^{k-1}\frac{1}{(k-j)jk} \;=\; \sum_{k=2}^{\infty}\left(\frac{1}{k^2}\sum_{j=1}^{k-1}\left(\frac{1}{k-j}+\frac{1}{j}\right)\right) \\ &= \sum_{k=2}^{\infty}\left(\frac{2}{k^2}\sum_{j=1}^{k-1}\frac{1}{j}\right) \;=\; 2\sum_{i=2}^{\infty}\left(\frac{1}{i^2}\sum_{j=1}^{i-1}\frac{1}{j}\right). \qquad (2) \end{aligned}$$

Substituting (2) into (1), the result follows immediately.

II. *Composite of essentially the same solutions by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let S denote the given double summation. Then

$$\begin{aligned} S &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\int_0^1 x^{i+j-1}\,dx \;=\; \int_0^1\left(\frac{1}{x}\left(\sum_{i=1}^{\infty}\frac{x^i}{i}\right)\left(\sum_{j=1}^{\infty}\frac{x^j}{j}\right)\right)dx \\ &= \int_0^1\frac{\ln^2(1-x)}{x}\,dx. \qquad (1) \end{aligned}$$

Changing variable via $t = -\ln(1-x)$, we have $x = 1-e^{-t}$ and $dx = e^{-t}dt$; whence,

$$\int_0^1 \frac{\ln^2(1-x)}{x}\,dx = \int_0^\infty \frac{t^2e^{-t}}{1-e^{-t}}\,dt = \int_0^\infty t^2\left(\sum_{n=1}^\infty e^{-nt}\,dt\right) = \sum_{n=1}^\infty \int_0^\infty t^2e^{-nt}\,dt\,. \qquad (2)$$

Applying the usual integration by parts twice, we find, after some routine computations involving improper integrals, that

$$\int_0^\infty t^2e^{-nt}\,dt = \frac{2}{n^3}\,. \qquad (3)$$

The desired result now follows from (1), (2), and (3).

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; and the proposer.

Both Curtis and Janous pointed out that this problem is not new. Curtis cited the book The Red Book of Mathematical Problems *by K.S. Williams and K. Hardy, Dover, 1996; and Janous gave the reference* Mathematical Constants *by Steven R. Finch, Cambridge University Press, 2003.*

Alt obtained the following identity as a by-product:

$$\sum_{n=1}^\infty \frac{H_n}{n^2} = 2\sum_{n=1}^\infty \frac{1}{n^3}\,,$$

where $H_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$.

The proposer gave the following comments: if we let

$$P(k) = \sum_{i_1=1}^\infty \cdots \sum_{i_k=1}^\infty \frac{1}{i_1i_2\cdots i_k(i_1+i_2+\cdots+i_k)}\,,$$

for $k = 1, 2, 3, \ldots$, *then clearly,* $P(1) = \zeta(2)$, *and the current problem shows that* $P(2) = 2\zeta(3)$, *where* $\zeta(s) = \sum\limits_{n=1}^\infty \frac{1}{n^s}$ *denotes the Riemann Zeta function. He offered the conjecture that* $\zeta(k) = k\zeta(k+1)$ *for all* $k \in \mathbb{N}$. [Ed: *Here,* $P(1)$ *is interpreted to be* $\sum\limits_{i_1=1}^\infty \frac{1}{i_1^2}$.]