THE OLYMPIAD CORNER

No. 249

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We begin this number of the *Corner* with the two days of the Bosnia and Herzegovina 7th National Olympiad Selection Test, written May 2002. Thanks go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for providing us with a copy of the contest.

BOSNIA AND HERZEGOVINA
7th NATIONAL OLYMPIAD — SELECTION TEST
First Day - May 2002

1. Let x, y, and z be real numbers such that

$$x + y + z = 3 \qquad \text{and} \qquad xy + yz + xz = a$$

(a is a real parameter). Determine the value of the parameter a for which the difference between the maximum and minimum possible values of x equals 8.

2. Triangle ABC is given in a plane. Draw the bisectors of all three of its angles. Then draw the line that connects the points where the bisectors of angles ABC and ACB meet the sides AC and AB, respectively. Through the point of intersection of the bisector of angle BAC and the previously drawn line, draw another line, parallel to the side BC. Let this line intersect the sides AB and AC in points M and N. Prove that $2MN = BM + CN$.

3. If n is a natural number, prove that the number $(n+1)(n+2)\cdots(n+10)$ is not a perfect square.

Second Day - May 2002

4. Let a, b, and c be real numbers such that $a^2 + b^2 + c^2 = 1$. Prove the inequality

$$\frac{a^2}{1+2bc} + \frac{b^2}{1+2ca} + \frac{c^2}{1+2ab} \geq \frac{3}{5}\,.$$

5. Let p and q be different prime numbers. Solve the following system of equations in the set of integers:

$$\frac{z+p}{x} + \frac{z-p}{y} = q\,,$$
$$\frac{z+p}{y} - \frac{z-p}{x} = q\,.$$

6. Let the vertices of the convex quadrilateral $ABCD$ and the intersecting point S of its diagonals be integer points in the plane. Let P be the area of the quadrilateral $ABCD$ and P_1 the area of triangle ABS. Prove the following inequality:

$$\sqrt{P} \geq \sqrt{P_1} + \frac{\sqrt{2}}{2}.$$

Next we give the four problems of the Fourth Hong Kong (China) Mathematical Olympiad, written December 2001. Thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for collecting them for our use.

4th HONG KONG (CHINA) MATHEMATICAL OLYMPIAD

December 22, 2001

Time: 3 hours

1. A triangle ABC is given. A circle Γ passes through vertex A and is tangent to side BC at point P. The circle Γ intersects sides AB and AC at points M and N, respectively. Prove that (minor) arcs MP and NP are equal if and only if Γ is tangent to the circumcircle of $\triangle ABC$ at A.

2. Find all positive integers n such that the equation $x^3+y^3+z^3 = nx^2y^2z^2$ has positive integer solutions. Be sure to give a proof.

3. For each integer $k \geq 4$, prove that if $F(x)$ is a polynomial with integer coefficients which satisfies the condition $0 \leq F(c) \leq k$ for every $c = 0, 1, \ldots, k+1$, then $F(0) = F(1) = \cdots = F(k+1)$.

4. There are 212 points inside or on a circle with radius 1. Prove that there are at least 2001 pairs of these points having distances at most 1.

As a final set of questions we give the Fifteenth Irish Mathematical Olympiad, First and Second Paper, written May 2002. My thanks again go to Bill Sands, Chair of the International Olympiad Committee of the Canadian Mathematical Society, for obtaining them for our use.

15th IRISH MATHEMATICAL OLYMPIAD

First Paper

11 May 2002 – morning

1. In a triangle ABC, $AB = 20$, $AC = 21$, and $BC = 29$. The points D and E lie on the line segment BC, with $BD = 8$ and $EC = 9$. Calculate the angle $\angle DAE$.

2. (a) A group of people attends a party. Each person has at most three acquaintances in the group, and if two people do not know each other, then they have a mutual acquaintance in the group. What is the maximum number of people present?

(b) If, in addition, the group contains three mutual acquaintances (that is, three people each of whom knows the other two), what is the maximum number of people?

3. Find all triples of positive integers (p, q, n), with p and q prime, such that

$$p(p+3)+q(q+3) \;=\; n(n+3)\,.$$

4. Let the sequence a_1, a_2, a_3, a_4, $\ldots$ be defined by

$$a_1 \;=\; 1\,, \quad a_2 \;=\; 1\,, \quad a_3 \;=\; 1\,, \quad \text{and} \quad a_{n+1}a_{n-2} - a_n a_{n-1} \;=\; 2\,,$$

for all $n \geq 3$. Prove that a_n is a positive *integer* for all $n \geq 1$.

5. Let $0 < a$, b, $c < 1$. Prove that

$$\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} \;\geq\; \frac{3\sqrt[3]{abc}}{1-\sqrt[3]{abc}}\,.$$

Determine the case of equality.

Second Paper

11 May 2002 – afternoon

6. A $3 \times n$ grid is filled as follows. The first row consists of the numbers from 1 to n arranged from left to right in ascending order. The second row is a cyclic shift of the top row. Thus, the order goes

$$i,\, i+1,\, \ldots,\, n-1,\, n,\, 1,\, 2,\, \ldots,\, i-1$$

for some i. The third row has the numbers 1 to n in some order, subject to the rule that in each of the n columns, the sum of the three numbers is the same.

For which values of n is it possible to fill the grid according to the above rules? For an n for which this is possible, determine the number of different ways of filling the grid.

7. Suppose n is a product of four distinct primes a, b, c, d such that

(a) $a+c=d$;

(b) $a(a+b+c+d)=c(d-b)$;

(c) $1+bc+d=bd$.

Determine n.

8. Denote by $\mathbb{Q}$ the set of rational numbers. Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + f(y)) = y + f(x), \quad \text{for all } x, y \in \mathbb{Q}.$$

9. For each real number x, define $\lfloor x \rfloor$ to be the greatest integer less than or equal to x. Let $\alpha = 2 + \sqrt{3}$. Prove that

$$\alpha^n - \lfloor \alpha^n \rfloor = 1 - \alpha^{-n}, \quad \text{for } n = 0, 1, 2, \ldots.$$

10. Let ABC be a triangle whose side lengths are all integers, and let D and E be the points at which the incircle of ABC touches BC and AC, respectively. If $|AD^2 - BE^2| \leq 2$, show that $AC = BC$.

Before turning to readers' solutions, I want to point out that we are gradually catching up on our backlog of solutions, and I hope that we will soon be in a position that will see readers' solutions appear within a year of publication of the problem sets. This means a bit of a challenge to solvers to send me the solutions within about 8 months to allow the time to review solutions, select and edit them.

The first set of reader's solutions we present are to the selected problems of the Ukrainian Mathematical Olympiad written March 2001, for which problems were given in [2003 : 497–498].

1. (Grade 9) All 5-digit positive integers with digits in increasing order (from left to right) are given. Is it possible to take away one digit from each number so that we obtain all 4-digit positive integers with digits in increasing order?

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Each positive integer whose digits are strictly increasing is uniquely determined by the set of its digits, which cannot include 0. Thus, each 5–digit positive integer with digits in increasing order corresponds to (and will be identified with) a 5–element subset of $\{1, 2, \ldots, 9\}$, and similarly for 4–digit integers. Let A be the set of all 5-element subsets of $\{1, 2, \ldots, 9\}$ and B the set of all 4-element subsets of $\{1, 2, \ldots, 9\}$. Note that $|A| = |B| = \binom{9}{4}$. The problem asks whether there exists a bijective mapping $f : A \to B$ such that $f(X) \subset X$ for all $X \in A$. We will construct such a mapping.

Let $X \in A$. Write the numbers $1, 2, \ldots, 9$ around a circle, and colour all the numbers red that are not in X. (There will be four red numbers.) Then, for each red number, move clockwise to the next uncoloured number, and colour it blue. The four blue numbers are the set $f(X)$.

The inverse mapping is defined as follows. Let $Y \in B$. Again write the numbers 1, 2, …, 9 around a circle. Colour all the numbers in Y blue. For each blue number, move counter-clockwise to the next uncoloured number, and colour it red. Then $f^{-1}(Y)$ is the set of all numbers not coloured red.

It is easy to see that f satisfies all conditions, and we are done.

3. (Grade 10) Let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \ \geq\ n^2 \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \ \leq\ n^3 + 1\,.$$

Prove that $n - 1 \leq a_k \leq n + 1$ for all k.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bataille's solution.

First, note that the inequality $a_1 + a_2 + \cdots + a_n \geq n^2$ can be rewritten as

$$(a_1 - n) + (a_2 - n) + \cdots + (a_n - n) \ \geq\ 0\,. \tag{1}$$

Now,

$$\begin{aligned} & (a_1 - n)^2 + (a_2 - n)^2 + \cdots + (a_n - n)^2 \\ =\ & a_1^2 + a_2^2 + \cdots + a_n^2 - n^3 - 2n((a_1 - n) + (a_2 - n) + \cdots + (a_n - n)) \\ \leq\ & 1\,, \end{aligned}$$

using (1) and the given inequality $a_1^2 + a_2^2 + \cdots + a_n^2 \leq n^3 + 1$. Thus, we certainly have $(a_k - n)^2 \leq 1$; that is, $n - 1 \leq a_k \leq n + 1$ for all k.

4. (Grade 10) There are n mathematicians in each of three countries. Each mathematician corresponds with at least $n{+}1$ foreign mathematicians. Prove that there exist three mathematicians who correspond with each other.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's solution.

Let the three countries be A, B, and C. Let X be the mathematician who corresponds with the greatest number of mathematicians from one country, and let that number be j. Note that $j \leq n$, since there are only n mathematicians in each country. Without loss of generality, let X be in A and let X correspond with j mathematicians from B. Then X corresponds with at least $n + 1 - j > 0$ mathematicians in C. Let Y be a mathematician in C such that X corresponds with Y.

Suppose there do not exist three mathematicians who correspond with each other. Then Y does not correspond with any mathematicians in B who correspond with X; hence, Y corresponds with at most $n - j$ mathematicians in B, which implies that Y corresponds with at least $(n+1) - (n-j) = j+1$ mathematicians in A. This contradicts the choice of X since $j + 1 > j$. Therefore, there exist three mathematicians who correspond with each other.

5. (Grade 11) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$ the following equality holds?

$$f(xy) = \max\{f(x), y\} + \min\{f(y), x\} .$$

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Suppose that there exists such a function. Then, setting $x = y = 1$ in the given equation, we get

$$f(1) = \max\{f(1), 1\} + \min\{f(1), 1\} = f(1) + 1 ,$$

a contradiction. Thus, no such function exists.

6. (Grade 11) Positive integers a and n are such that n divides $a^2 + 1$. Prove that there exists a positive integer b such that $n(n^2 + 1)$ divides $b^2 + 1$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Choose b to be $(n^2 + 1)a + n$. Then

$$b^2 + 1 = \left((n^2 + 1)a + n\right)^2 + 1 \equiv a^2 + 1 \equiv 0 \pmod{n} ,$$

and $$b^2 + 1 = \left((n^2 + 1)a + n\right)^2 + 1 \equiv n^2 + 1 \equiv 0 \left(\bmod\ n^2 + 1\right) .$$

Since n and $n^2 + 1$ are relatively prime, we see that $n(n^2 + 1)$ divides $b^2 + 1$, and the result follows.

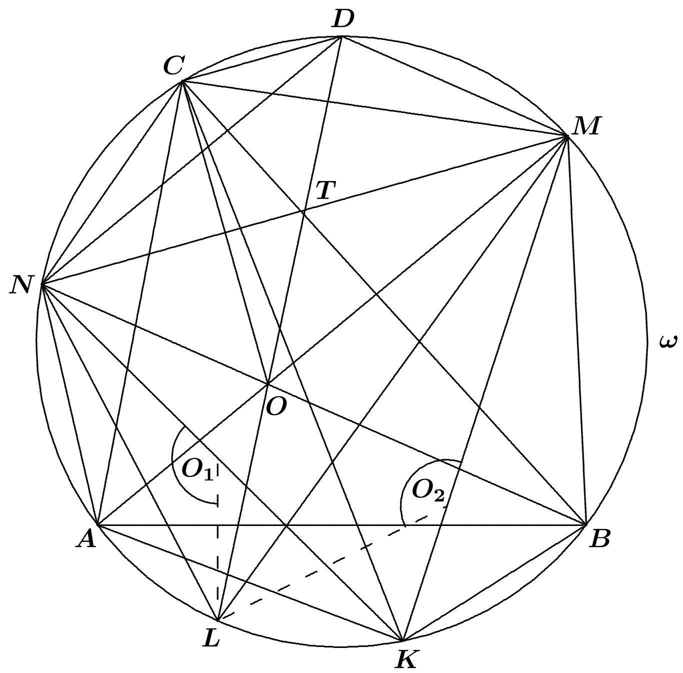
7. (Grade 11) An acute triangle ABC, with $AC \neq BC$, is inscribed in a circle ω. The points A, B, C divide the circle into disjoint arcs $\widehat{AB}$, $\widehat{BC}$, and $\widehat{CA}$. Let M and N be the mid-points of $\widehat{BC}$ and $\widehat{AC}$, respectively, and let K be an arbitrary point of $\widehat{AB}$. Let D be the point of $\widehat{MN}$ such that $CD \parallel NM$. Let O, O_1, O_2 be the incentres of triangles ABC, CAK, CBK, respectively. Let L be the intersection point of the line DO and the circle ω, where $L \neq D$. Prove that the points K, O_1, O_2, L are concyclic.

Solution by Toshio Seimiya, Kawasaki, Japan.

Since M and N are the mid-points of $\widehat{BC}$ and $\widehat{AC}$, respectively, we see that AM and BN are the bisectors of $\angle CAB$ and $\angle CBA$, respectively. Thus, the intersection of AM and BN is the incentre O of $\triangle ABC$. Since O is the incentre of $\triangle ABC$, we have $\angle ACO = \angle BCO$. Hence,

$$\begin{aligned} \angle MOC = \angle ACO + \angle CAM &= \angle BCO + \angle BAM \\ &= \angle BCO + \angle BCM = \angle MCO . \end{aligned}$$

Thus, $MO = MC = MB$. Similarly, we get $NO = NC = NA$.



Since O_1 and O_2 are the incentres of $\triangle CAK$ and $\triangle CBK$, respectively, we similarly obtain

$$NO_1 = NC = NA, \qquad \text{and} \qquad MO_2 = MC = MB.$$

Since $CD \parallel MN$, we have $MD = NC = NO$, and $DN = CM = MO$. Hence, quadrilateral $MDNO$ is a parallelogram. Let T be the intersection of DO and MN. Then $NT = TM$.

Since $\triangle NLT \sim \triangle DMT$, and $\triangle MLT \sim \triangle DNT$, we obtain

$$NL : DM = LT : MT = LT : NT = ML : DN.$$

Hence, $NL : ML = DM : DN$. Since $DM = CN = NO_1$, and $DN = CM = MO_2$, we have

$$NL : ML = NO_1 : MO_2. \tag{1}$$

Since $\angle LNO_1 = \angle LNK = \angle LMK = \angle LMO_2$, we get from (1)

$$\triangle NLO_1 \sim \triangle MLO_2.$$

Thus, $\angle NO_1L = \angle MO_2L$. Hence,

$$\angle LO_1K = 180^\circ - \angle NO_1L = 180^\circ - \angle MO_2L = \angle LO_2K.$$

Therefore, K, O_1, O_2, L are concyclic.

8. (Grade 11) Let a, b, c and α, β, γ be positive real numbers such that $\alpha+\beta+\gamma=1$. Prove the inequality

$$\alpha a+\beta b+\gamma c+2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)} \;\le\; a+b+c\,.$$

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Zhao's write-up.

Since the inequality is homogeneous in a, b, and c, we may assume that $a+b+c=1$ (without loss of generality). Then, using the AM–GM Inequality,

$$\begin{aligned}
&2\sqrt{(\alpha\beta+\beta\gamma+\gamma\alpha)(ab+bc+ca)}\\
&\qquad\le\; \alpha\beta+\beta\gamma+\gamma\alpha+ab+bc+ca\\
&\qquad=\; \frac{(\alpha+\beta+\gamma)^2-\alpha^2-\beta^2-\gamma^2}{2}+\frac{(a+b+c)^2-a^2-b^2-c^2}{2}\\
&\qquad=\; 1-\frac{a^2+\alpha^2}{2}-\frac{b^2+\beta^2}{2}-\frac{c^2+\gamma^2}{2}\\
&\qquad\le\; 1-\sqrt{a^2\alpha^2}-\sqrt{b^2\beta^2}-\sqrt{c^2\gamma^2}\\
&\qquad=\; a+b+c-a\alpha-b\beta-c\gamma\,.
\end{aligned}$$

The result follows immediately.

Next we turn to our file of readers' solutions to problems of the 2004 numbers of ***Crux Mathematicorum***, starting with the XVII National Mathematical Contest of Italy [2004 : 18–19].

2. In a basketball tournament, each team played twice against each other team. Two points were awarded for a win and no points for a loss. (No game could finish in a draw.) A single team won the tournament with 26 points, and exactly two teams finished last with 20 points. How many teams participated in the tournament?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Edward T.H. Wang and Kaiming Zhao, Wilfrid Laurier University, Waterloo, ON. We give the solution by Wang and Zhao.

The answer is 12. Let n be the number of the teams. Then $n\ge 3$. Let s_k be the points scored by team T_k, for $k=1, 2, \ldots, n$. Without loss of generality, we assume that

$$20 \;=\; s_1 \;=\; s_2 \;<\; s_3 \;\le\; s_4 \;\le\cdots\le\; s_{n-1} \;<\; s_n \;=\; 26\,.$$

By assumption, T_n played a total of $2(n-1)$ games and won 13 games; thus, it lost $2n-15$ games. Thus, $2n-15<13$ from which we get $n\le 13$.

Similarly, T_1 won 10 games and lost $2n - 12$ games; thus, $2n - 12 > 10$ from which we get $n \geq 12$. Hence, $n = 12$ or 13.

Since each s_k is even, it follows that for all $k \in I = \{3, 4, \ldots, n-1\}$, we must have $s_k = 22$ or $s_k = 24$. Suppose $s_k = 22$ for the first ℓ values of $k \in I$, and $s_k = 24$ for the other $n - \ell - 3$ values of $k \in I$, where $0 \leq \ell \leq n - 3$.

Since the total number of games played was $2\binom{n}{2} = n(n-1)$, the total number of points scored was $2n(n-1)$. Hence,

$$20 + 20 + 22\ell + 24(n - \ell - 3) + 26 \;=\; 2n(n-1)\,,$$

which simplifies to $\ell + 3 = n(13 - n)$. Clearly, $n = 13$ is impossible. Therefore, $n = 12$ and $\ell = 9$. That is, the points scored by the 12 teams were 20, 20, 22, $\ldots$, 22, 26. We note that this situation can actually be realized if every team "draws" every other team (that is, wins one game and loses one in the two games played between them), except team T_{12}, which beats T_1 and T_2 in both games.

3. Given the equation $x^{2001} = y^x$,

(a) find all solution pairs (x, y) consisting of positive integers with x prime;

(b) find all solution pairs (x, y) consisting of positive integers.

(Recall that $2001 = 3 \cdot 23 \cdot 29$.)

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Let us solve (b) directly.

If $x = 1$, we find the solution $(x, y) = (1, 1)$. Now, we assume that $x \geq 2$; then $y \geq 2$. Clearly, x and y have the same prime divisors.

Let $x = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ and $y = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}$ be the prime decompositions of x and y. Thus, for each i, we have

$$2001\alpha_i \;=\; x\beta_i\,. \tag{1}$$

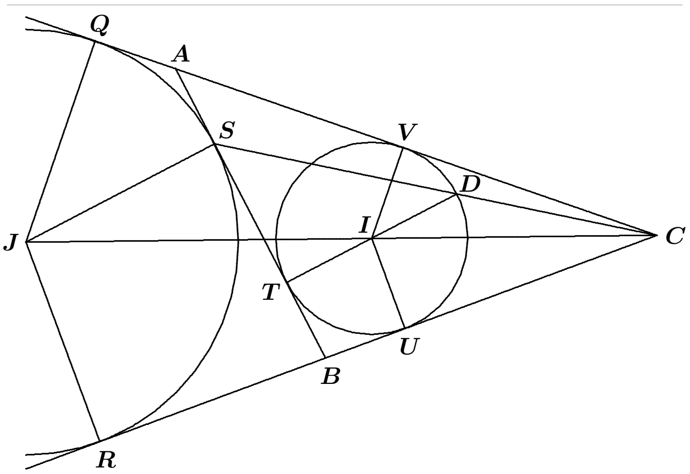
If $\gcd(p_i, 2001) = 1$, then, from (1), we deduce that p_i divides α_i; indeed, we even have $p_i^{\alpha_i}$ divides α_i. But it is easy to prove by induction that $2^k > k$ for each positive integer k. Hence, $p_i^{\alpha_i} \geq 2^{\alpha_i} > \alpha_i$. Thus, for each i, we have $\gcd(p_i, 2001) = p_i$.

Since $2001 = 3 \times 23 \times 29$, it follows that $x = 3^\alpha \times 23^\beta \times 29^\gamma$ for some non-negative integers α, β, γ. From (1), if $\alpha > 0$, we deduce (as above) that $3^{\alpha-1}$ divides α, which leads to $\alpha = 1$. Similarly, we have β, $\gamma \in \{0, 1\}$. Thus, x divides 2001.

Conversely, if $2001 = kx$ for some positive integer k, then the given equation $x^{2001} = y^x$ is equivalent to $y = x^k$. Thus, the set of solutions (x, y) is the set of all ordered pairs $(x, x^{2001/x})$, where x is any positive divisor of 2001; that is, $x \in \{1, 3, 23, 29, 69, 87, 667, 2001\}$.

5. The incircle γ of triangle ABC touches the side AB at T. Let D be the point on γ diametrically opposite to T, and let S be the intersection of the line through C and D with the side AB. Show that $AT = SB$.

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Michel Bataille, Rouen, France; and Toshio Seimiya, Kawasaki, Japan. We give the write-up of Bataille.



Let γ touch BC at U and CA at V. Let I be the incentre and J be the point of intersection of the line CI and the perpendicular to AB at S. Let Q, R be the projections of J onto the lines CA, CB, respectively (see figure). Note that $ID \parallel JS$, $UI \parallel JR$, $IV \parallel JQ$. It follows that $\frac{CI}{CJ} = \frac{IU}{JR} = \frac{ID}{JS}$. Since $IU = ID$ (the inradius), we obtain $JR = JS$. Similarly, $JQ = JS$. Hence, $JQ = JR = JS$ (and clearly $J \neq I$). Thus, J is the excentre in $\angle ACB$, and the excircle with centre J touches CA, AB, CB at Q, S, R, respectively.

Now, it is well known that $AT = BS = s - a$ (with the standard notations) [briefly, $b - a = BR - QA = BS - SA = 2BS - c$ and $b - a = AV - BU = AT - BT = 2AT - c$, from which we obtain $BS = AT = \frac{1}{2}(b + c - a) = s - a$].

Next we turn to solutions of problems of the 52nd Polish Mathematical Olympiad given [2004 : 19].

1. Show that the inequality

$$\sum_{i=1}^{n} ix_i \;\leq\; \binom{n}{2} + \sum_{i=1}^{n} x_i^i$$

holds for every integer $n \geq 2$ and all real numbers $x_1, x_2, \ldots, x_n \geq 0$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Maisons-Laffitte, France; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give the solution by Bornsztein.

From the AM–GM Inequality, for each non-negative real number x and for each positive integer k, we have

$$x^k + k - 1 \;=\; x^k + 1 + \cdots + 1 \;\geq\; k\sqrt[k]{x^k \times 1 \times \cdots \times 1} \;=\; kx\,.$$

Equality occurs, for $k \geq 2$, if and only if $x = 1$.

Using this for each x_i and summing leads to

$$\sum_{i=1}^{n} ix_i \;\leq\; \sum_{i=1}^{n}\left((i-1) + x_i^i\right) \;=\; \binom{n}{2} + \sum_{i=1}^{n} x_i^i\,,$$

as desired.

Equality occurs if and only if $x_2 = \cdots = x_n = 1$ (and $x_1 \geq 0$ is arbitrary).

2. Consider an arbitrary point P inside the regular tetrahedron with an edge of length 1. Show that the sum of the distances from P to the vertices of the tetrahedron does not exceed 3.

Solved by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

We will first consider a two-dimensional version of the problem.

Lemma. Supose that ABC is an equilateral triangle with side length 1 and that P is a point inside it. Then $PA + PB + PC \leq 2$.

Proof: Consider an ellipse with foci A and B that passes through P. Let the ellipse intersect AC at Q. Then the convexity of the ellipse implies that $PC \leq QC$. Since P and Q both lie on the ellipse, $PA + PB = QA + QB$. Since the longest chord in a triangle is the longest side, we have $QB \leq 1$. Hence,

$$PA + PB + PC \;\leq\; QA + QB + QC \;\leq\; QA + QC + 1 \;=\; 2\,.$$

Returning to the original problem, let the tetrahedron be $ABCD$. Let $\mathcal{N}$ be the ellipsoid with foci A and B which passes through P, and let $\mathcal{M}$ be the ellipsoid with foci C and D which passes through P. Obviously, if a point X lies within the ellipsoid $\mathcal{N}$, then $XA + XB \leq PA + PB$. The inequality is reversed if X is not inside $\mathcal{N}$.

Consider the intersection of the surface of $\mathcal{N}$ with the surface of the tetrahedron. Suppose that the intersection is completely inside the ellipsoid $\mathcal{M}$. Then, due to the convexity of ellipsoids, we see that the surfaces of $\mathcal{N}$ and $\mathcal{M}$ do not meet inside the tetrahedron, implying that P could not exist, a contradiction. Thus, there is a point Q (say on face ABC) such that Q is on $\mathcal{N}$ but not inside $\mathcal{M}$.

Then $PA + PB = QA + QB$ and $PC + PD \leq QC + QD$; hence,

$$PA + PB + PC + PD \;\leq\; QA + QB + QC + QD\,.$$

By our lemma, we have $QA + QB + QC \leq 2$, and clearly $QD \leq 1$. Adding the two inequalities yields the result.

3. The sequence x_1, x_2, x_3, ... is defined recursively by

$$x_1 \;=\; a\,,\quad x_2 \;=\; b\,,\quad \text{and}\quad x_{n+2} \;=\; x_{n+1} + x_n \quad \text{for } n = 1,\,2,\,3,\,\ldots,$$

where a and b are real numbers. A number c will be called a *repeated value* if $x_k = x_l = c$ for at least two distinct indices k and l. Prove that the initial terms a and b can be chosen so that there are more than 2000 repeated values, but it is impossible to choose a and b so that there are infinitely many repeated values.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then α and β are the zeros of $x^2 - x - 1$ and $\alpha\beta = -1$. Suppose that p is an odd positive integer. Then

$$\begin{aligned}\frac{\alpha^p - \alpha^{8000-p}}{\beta^p - \beta^{8000-p}} &= \frac{\alpha^{8000}(\alpha^p - \alpha^{80000-p})}{\alpha^{8000}\beta^p - \alpha^{8000}\beta^{8000-p}} \\ &= \frac{\alpha^{8000}(\alpha^p - \alpha^{8000-p})}{-\alpha^{8000-p} + \alpha^p} \;=\; \alpha^{8000}\,.\end{aligned}$$

Rearranging the above relationship, we get

$$\alpha^p - \alpha^{8000}\beta^p \;=\; \alpha^{8000-p} - \alpha^{8000}\beta^{8000-p}\,.$$

Define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_n = \alpha^n - \alpha^{8000}\beta^n$. Then $x_1 = x_{7999}$, $x_3 = x_{7997}$, ..., $x_{3999} = x_{4001}$. Furthermore, x_1, x_2, ..., x_{3999} are distinct, since, if p and q are odd numbers such that $x_p = x_q$, then

$$0 \;=\; x_p - x_q \;=\; \alpha^p + \alpha^{8000-p} - \alpha^q - \alpha^{8000-q} \;=\; (\alpha^p - \alpha^q)(1 - \alpha^{8000-p-q})\,,$$

and hence $p = q$ or $p + q = 8000$; thus, p and q cannot be distinct elements of $\{1, 3, \ldots, 3999\}$. Moreover, by the theory of linear recursive sequences, the sequence $\{x_n\}$ satisfies $x_{n+2} = x_{n+1} + x_n$. Therefore, the chosen sequence has at least 2000 repeated values (namely, x_1, x_3, ..., x_{3999}).

On the other hand, if $\{x_n\}_{n=1}^{\infty}$ is a sequence with the properties given in the problem, then there exist constants a and b such that $x_n = a\alpha^n + b\beta^n$ for all n. We may assume that neither a nor b is zero, for otherwise there would obviously be only a finite number of repeated values. Since $|\alpha| > 1$ and $|\beta| < 1$, the sequence would eventually be strictly monotonic (increasing if $a > 0$, and decreasing if $a < 0$). Hence, there is a value m such that none of the numbers x_m, x_{m+1}, ... appear again in the sequence $\{x_1, x_2, \ldots\}$. Therefore, the number of repeated values must be finite.

4. The integers a and b have the property that, for every non-negative integer n, the number $2^n a + b$ is the square of an integer. Show that $a = 0$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Bornsztein's write-up.

Suppose, for a contradiction, that $a \neq 0$. For n sufficiently large, $2^n a + b$ has the same sign as a; thus $a > 0$. For each positive integer n, let x_n be the positive integer such that $x_n^2 = 2^n a + b$.

Suppose first that $b > 0$. Clearly, $\lim\limits_{n \to +\infty} x_n = +\infty$. Thus, there exists n_0 such that, for all $n \geq n_0$, we have $3b - 4x_n + 1 < 0$. Then, for $n \geq n_0$,

$$\begin{aligned}(2x_n - 1)^2 &= 4x_n^2 - 4x_n + 1 \\ &= 2^{n+2}a + 4b - 4x_n + 1 \;<\; 2^{n+2}a + b \;=\; x_{n+2}^2 \,.\end{aligned}$$

Therefore, $2x_n - 1 < x_{n+2}$. On the other hand,

$$(2x_n)^2 \;=\; 4x_n^2 \;=\; 2^{n+2}a + 4b \;>\; 2^{n+2}a + b \;=\; x_{n+2}^2 \,,$$

and therefore $2x_n > x_{n+2}$. Thus, we have $2x_n > x_{n+2} > 2x_n - 1$. This is a contradiction, since x_n and x_{n+2} are integers.

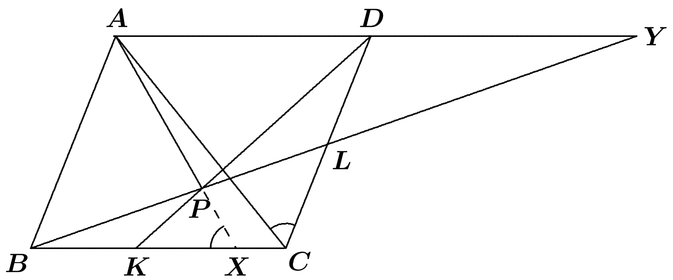
Suppose next that $b < 0$. Similar reasoning leads to a similar contradiction that $2x_n < x_{n+2} < 2x_n + 1$.

Thus, $b = 0$. Let $a = 2^\alpha \beta$, where $\alpha \geq 0$ and $\beta \geq 1$ is odd. Let n have opposite parity to α. Then $n + \alpha$ is odd, and $2^n a = 2^{n+\alpha}\beta$ cannot be a square, a contradiction.

Since we get a contradiction in each case, it follows that $a = 0$.

5. Let $ABCD$ be a parallelogram, and let K and L be points lying on the sides BC and CD, respectively, such that $BK \cdot AD = DL \cdot AB$. The segments DK and BL intersect at P. Show that $\angle DAP = \angle BAC$.

Solved by Toshio Seimiya, Kawasaki, Japan; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON. We give Seimiya's solution.



Let X and Y be the intersections of AP and BL with BC and AD, respectively. Since $BX \parallel AY$, we have

$$BK : BX \;=\; YD : YA \,.$$

Since $DL \parallel AB$, we see that $YD : YA = DL : AB$, and then $BK : BX = DL : AB$; that is,

$$BK : DL \;=\; BX : AB \,. \qquad (1)$$

Since $BK \cdot AD = DL \cdot AB$, we have

$$BK : DL \;=\; AB : AD\,. \tag{2}$$

It follows from (1) and (2) that

$$BX : AB \;=\; AB : AD \;=\; DC : AD\,. \tag{3}$$

Since $\angle ABX = \angle ADC$, we have from (3)

$$\triangle ABX \sim \triangle ADC\,.$$

Thus,

$$\angle AXB \;=\; \angle ACD\,. \tag{4}$$

Since $AD \parallel BX$, and $AB \parallel DC$, we have $\angle AXB = \angle DAX = \angle DAP$ and $\angle ACD = \angle BAC$. Therefore, using (4), we get $\angle DAP = \angle BAC$.

6. Let $n_1 < n_2 < \cdots < n_{2000} < 10^{100}$ be positive integers. Prove that the set $\{n_1, n_2, \ldots, n_{2000}\}$ has two non-empty disjoint subsets A and B with equally many elements, equal sums of their elements, and equal sums of the squares of their elements.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Let $\mathcal{S}$ be the set of all 1000-element subsets of $\{n_1, n_2, \ldots, n_{2000}\}$. For any set $S \in \mathcal{S}$, we see that the sum of the elements of S must be less than $10^{100} \cdot 1000 = 10^{103}$, and the sum of the squares of the elements of S must be less than $10^{200} \cdot 1000 = 10^{203}$. Let us create $10^{103} \cdot 10^{203} = 10^{306}$ pigeonholes, each representing a distinct combination of a sum and a sum of squares. Now

$$\begin{aligned} |\mathcal{S}| \;&=\; \binom{2000}{1000} \;=\; \prod_{i=1}^{1000} \frac{1000+i}{i} \;=\; \prod_{i=1}^{500} \frac{1000+i}{i} \prod_{i=501}^{1000} \frac{1000+i}{i} \\ &> \prod_{i=1}^{500} 3 \prod_{i=501}^{1000} 2 = 6^{500} \;=\; (\sqrt{216})^{1000/3} > 10^{306}\,. \end{aligned}$$

Hence, if we place all the elements of $\mathcal{S}$ into the 10^{306} pigeonholes, there must exist two distinct sets X, Y, belonging to the same pigeonhole. That is, the sum of their elements is equal, and the sum of the squares of their elements is equal. Choose $A = X \setminus Y$ and $B = Y \setminus X$. Then A and B obviously satisfy all the requirements in the problem.

That completes this issue. Please send in solutions to problems from 2004–2005 quickly to help keep our schedule up. Also send your Olympiad materials.