SKOLIAD No. 89

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Please send your solutions to the problems in this issue by ***March 1, 2006***. A copy of **MATHEMATICAL MAYHEM Vol. 2** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

Our featured contest this issue is the 1999 New Zealand Junior Mathematics Competition, for which I thank Derek Holton and Warren Palmer, both from the University of Otago in New Zealand.

1999 New Zealand Junior Mathematics Competition
Sponsored by the University of Otago

1. Morris Muddledit multiplies two-digit numbers by multiplying together the ones and tens digit separately and adding the results. Let this erroneous multiplication be noted by $(\times)$. For example:

$$\begin{aligned} 36(\times)47 &= 42+12 = 54\,, \\ 23(\times)40 &= 0+8 = 8\,, \\ \text{and}\quad 65(\times)31 &= 5+18 = 23\,. \end{aligned}$$

Let's call this operation the "Morris product".

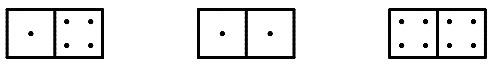
(a) What are the Morris products $11(\times)18$, $91(\times)19$, and $35(\times)62$?

(b) What is the largest possible Morris product of two two-digit numbers?

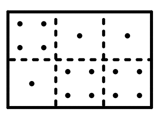
(c) Find all two-digit numbers ab such that $32(\times)ab = 32$.

(d) What is the largest actual product of two two-digit numbers whose Morris product is less than 10?

2. Bored at the beach, Barbara is idly arranging dominoes on the table. Over the years a few dominoes have been lost from the set. In fact, only three are left. These happen to be:



(a) How are the dominoes arranged to give this rectangle?



(b) Give an example of a similar rectangle which can be formed in exactly 2 different ways from these three dominoes.

(c) Give an example of a similar rectangle which can be formed in exactly 3 different ways from these three dominoes.

(d) Are there any rectangles which can be formed in four ways?

3. The mythical country of EnZed is divided into a north and south island. Inhabitants of the north island never tell the truth, while those from the south always do. Furthermore, on the south island they produce and drink a magical brown nectar called Spites. A thirsty traveler once entered a bar seeking a drink of this wondrous brew, only to find three full glasses on the counter, and five people lounging around. Somehow he knew that exactly one of the glasses contained Spites, while the other two contained pale and unappetising imitations. Not unexpectedly, each of the people around the bar made a single statement:

Andy: The left-most glass contains Spites.

Brenda: The right-most glass contains Spites.

Carol: Andy and Brenda are not both from the north island.

Deirdre: Either Andy is from the north island or Brenda is from the south island.

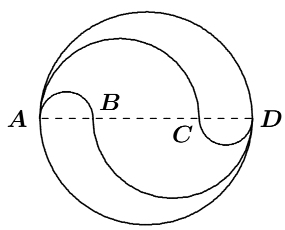
Ed: Either I am from the north island, or Carol and Deirdre are both from the same island.

(a) Remembering that to EnZedians (and to mathematicians everywhere) a statement of the type "Either X or Y" is true if either X or Y or both are true, what can be concluded from Ed's statement?

(b) Which glass (left-most, middle, or right-most) should the traveller take?

4. A circular plate is divided into 20 equal sectors. Ten sectors are painted blue, and ten are painted yellow. Show that somewhere on the plate there must be ten consecutive sectors, five of which are blue and five yellow (no matter how the blue and yellow sectors have been chosen).

5. King Lear, having come to his senses, intends to divide his fortune equally among his three daughters. Among his possessions is a large circular golden disc 1 m in diameter. For aesthetic reasons, he plans to have his goldsmith cut it up into three pieces of equal area using semicircular arcs along a diameter AD as shown below (but not to scale). If AB and CD are to have the same length, what should that length be (exactly)?



(The disc is to be divided along the semicircular solid arcs. The dotted diameter AD is for reference only.)

Compétition Junior de Mathématiques de Nouvelle-Zélande 1999
Organisé par l'Université d'Otago

1. Morris DuConfu multiplie les nombres à 2 chiffres en multipliant ensemble les chiffres des unités et des dizaines séparément puis en additionnant les résultats. On notera cette multiplication erronée par $(\times)$. Par exemple :

$$\begin{aligned} 36(\times)47 &= 42+12 = 54\,, \\ 23(\times)40 &= 0+8 = 8\,, \\ \text{et}\quad 65(\times)31 &= 5+18 = 23\,. \end{aligned}$$

Appelons cette opération le "Morris-produit".

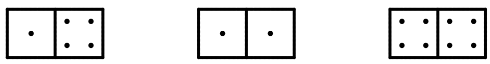
(a) Que valent les Morris-produits $11(\times)18$, $91(\times)19$ et $35(\times)62$?

(b) Quel est le plus grand Morris-produit de 2 nombres à 2 chiffres ?

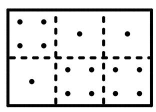
(c) Trouver tous les nombres à 2 chiffres ab tels que $32(\times)ab = 32$.

(d) Quel est le plus grand produit réel de 2 nombres à 2 chiffres dont le Morris-produit est inférieur à 10 ?

2. S'ennuyant à la plage, Barbara arrange des dominos sur la table. Avec les années, quelques dominos ont été perdus à tel point qu'il n'en reste que trois. Notamment :



(a) Comment arranger les dominos pour obtenir le rectangle ?



(b) Donner un exemple d'un rectangle similaire que l'on peut former exactement de 2 manières différentes avec ces trois dominos.

(c) Donner un exemple d'un rectangle similaire que l'on peut former exactement de 3 manières différentes avec ces trois dominos.

(d) Y a-t-il des rectangles que l'on peut former de 4 manières ?

3. Le pays magique de EnZed est formé des îles Nord et Sud. Les habitants de l'île nord ne disent jamais la vérité, alors que ceux du sud le font tout le temps. Sur l'île du sud, ils produisent une potion magique brune appelée Spites. Un voyageur assoiffé entra un jour dans une taverne à la recherche de cette potion. Sur le comptoir il trouva 3 verres pleins et 5 personnes assis autour du bar. Son intuition lui disait que seulement un des verres contenait de la potion, alors que les autres contenaient une imitation peu savoureuse. Sans surprise, chacune des personnes autour du bar ne fit qu'un commentaire :

Andy : Le verre de gauche contient du Spites.

Brenda : Le verre de droite contient du Spites.

Carol : Andy et Brenda ne sont pas de la même île.

Deirdre : Soit Andy est de l'île nord ou Brenda est de l'île sud.

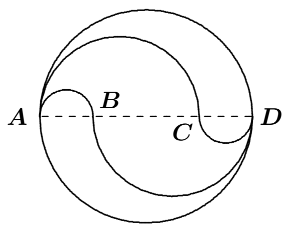
Ed : Soit je suis de l'île nord, ou Carol et Deirdre sont de la même île.

(a) En se rappelant que pour les EnZediens (et pour les mathématiciens partout) une expression du type "Soit X ou Y" est vrai si soit X ou Y ou les deux sont vrais, que peut-on conclure du commentaire à Ed ?

(b) Quel verre (gauche, centre or droite) le voyageur devrait-il prendre ?

4. Une assiette circulaire est divisée en 20 secteurs égaux. Dix secteurs sont peints en bleu, et dix en jaune. Montrez que quelque part sur l'assiette il doit y avoir dix secteurs consécutifs, cinq étant bleus et cinq jaunes (quelque soit la manière que l'on a fait pour choisir les secteurs).

5. Le roi Lear a l'intention de séparer sa fortune également parmi ses trois filles. Parmi ses possessions, on retrouve un large disque doré de 1m de diamètre. Pour des raisons esthétiques, il planifie que son forgeron le coupe en trois morceaux de même aire en utilisant des arcs semi-circulaires le long du diamètre AD comme sur le dessin (pas à l'échelle). Si AB et CD ont la même longueur, quelle devrait être cette longueur (exactement) ?



(Le disque va être partagé selon les lignes pleines. La ligne en pointillés est seulement là pour indiquer le diamètre AD.)

Next we give the solutions to the 4th Annual CNU Regional High School Mathematics Contest [2005 : 129–132].

4th Annual CNU Regional High School Mathematics Contest
Saturday, December 6, 2003

1. (∗) If 64 is divided into three parts proportional to 2, 4, and 6, the smallest part is:

(A) $5\frac{1}{3}$ (B) 11 (C) $10\frac{2}{3}$ (D) none of these

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON, modified by the editor.

The answer is C. Since the sum of the integers 2, 4, and 6 in the proportion is 12, we let $x = 64/12 = 16/3$. Then the three parts into which 64 is divided are $2x$, $4x$, and $6x$. The smallest part is then $2x = 32/3 = 10\frac{2}{3}$.

2. ($*$) If, in applying the quadratic formula to a quadratic equation $f(x) = ax^2 + bx + c = 0$, it happens that $c = \dfrac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly:

(A) have a maximum (B) have a minimum

(C) be tangent to the x–axis (D) be tangent to the y–axis

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is C. If $c = \dfrac{b^2}{4a}$, then $f(x) = ax^2+bx+\dfrac{b^2}{4a} = a\left(x + \dfrac{b}{2a}\right)^2$. From here, one can see that the vertex of the parabola is $\left(\dfrac{-b}{2a}, 0\right)$; hence, it lies on the x–axis. Because this is a vertex, and we have a parabola, $\left(\dfrac{-b}{2a}, 0\right)$ will be the only point at which the parabola touches the x–axis. Thus, the parabola is tangent to the x–axis.

3. ($*$) Let $\{a_n\}$ be a geometric sequence. If $a_1 = 8$ and $a_7 = 5832$, then a_5 is:

(A) 648 (B) 832 (C) 1168 (D) 1944

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON.

The answer is A. Let r be the multiplier in the sequence. Writing the first few terms of the sequence starting with $a_1 = 8$, we get

$$a_2 = a_1 r = 8r\,, \quad a_3 = a_1 r^2 = 8r^2\,, \ \ldots\,, \quad a_7 = a_1 r^6 = 8r^6 = 5832\,.$$

Hence, $r^6 = 5832/8 = 729$. Therefore, $r^3 = 27$, implying that $r = 3$. Then $a_5 = a_1 r^4 = 8 \times 81 = 648$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

4. ($*$) The area enclosed by $|x| + |y| = 1$ is:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is C. For $|x| + |y| = 1$, we must consider four cases (each in its own quadrant).

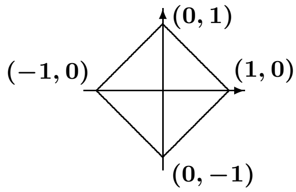
I $x \geq 0$ and $y \geq 0$. This gives us $x + y = 1$ or $y = 1 - x$.

II $x \geq 0$ and $y < 0$. This gives us $x - y = 1$ or $y = x - 1$.

III $x < 0$ and $y \geq 0$. This gives us $-x + y = 1$ or $y = x + 1$.

IV $x < 0$ and $y < 0$. This gives us $-x - y = 1$ or $y = -x - 1$.

Now, we can draw the graph. It is a square consisting of 4 congruent isosceles right triangles of side 1. The area of each of these triangles is $\frac{1\times 1}{2} = \frac{1}{2}$. Hence, the square has area 2.



5. ($*$) If the graph of $f(x) = \big||x-2|-a\big| - 3$ has exactly three x–intercepts, then a equals:

(A) 3 (B) 4 (C) 0 (D) -3

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON, modified by the editor.

The answer is A. An x–intercept occurs when $f(x) = 0$. Thus, we have $\big||x-2|-a\big| = 3$. We know that if the absolute value of a number is k, then the number is either k or $-k$. Therefore, either

$$\begin{aligned} |x-2|-a &= 3 \quad \text{or} \quad |x-2|-a = -3\,; \\ |x-2| &= a+3 \quad \text{or} \quad |x-2| = a-3\,; \end{aligned} \tag{1}$$

Both of the above equations have the form $|x-2| = k$ (in the first equation, $k = a+3$; in the second equation, $k = a-3$). If $k < 0$, the equation $|x-2| = k$ has no solutions; if $k = 0$, it has exactly one solution, $x = 2$; and if $k > 0$, then the equation is equivalent to $x - 2 = \pm k$, which has two solutions, $x = 2 \pm k$.

In order to get exactly 3 solutions for x in (1), we need one of the equations in (1) to have two solutions and the other to have exactly one solution. Thus, we need either $a+3 > 0$ and $a-3 = 0$, or $a+3 = 0$ and $a-3 > 0$. The first of these two alternatives occurs when $a = 3$, while the second is not possible (since $a+3 = 0$ implies that $a = -3$ which does not satisfy $a-3 > 0$). We conclude that $a = 3$.

(The three solutions for x are then $x = 2 \pm (3+3)$ and $x = 2$; that is, $x = 8$, $x = -4$ and $x = 2$.)

6. ($*$) If $\dfrac{m}{n} = \dfrac{4}{3}$ and $\dfrac{r}{t} = \dfrac{9}{14}$, then the value of $\dfrac{3mr - nt}{4nt - 7mr}$ is:

(A) $-5\frac{1}{2}$ (B) $-\frac{11}{14}$ (C) $-\frac{2}{3}$ (D) $-1\frac{1}{4}$

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is B. We have $m = 4n/3$ and $r = 9t/14$. Substituting this into $\dfrac{3mr - nt}{4nt - 7mr}$, we get

$$\frac{3mr-nt}{4nt-7mr} = \frac{3\left(\frac{4n}{3}\right)\left(\frac{9t}{14}\right)-nt}{4nt-7\left(\frac{4n}{3}\right)\left(\frac{9t}{14}\right)} = \frac{\frac{18tn}{7}-nt}{4nt-6nt}$$

$$= \frac{nt(\frac{18}{7}-1)}{nt(4-6)} = \frac{\frac{11}{7}}{-2} = \frac{-11}{14}\,.$$

One incomplete solution was also submitted.

7. $(*)$ Which functions satisfy $f\left(\frac{x+y}{2}\right) = \frac{1}{2}(f(x)+f(y))$?

(A) $\ln x$ (B) $\frac{1}{x}$ (C) $2x$ (D) 2^x

Solution by the editor.

The answer is C. The given equation is supposed to hold for all x and y in the domain of the function f. To show that it does not hold, for some function f, we just have to find one pair of numbers x and y in the domain of f for which it does not hold.

(A) Let $f(x) = \ln x$. Taking $x = \frac{1}{2}$ and $y = \frac{3}{2}$, we get

$$f\left(\frac{x+y}{2}\right) = \ln\left(\frac{\frac{1}{2}+\frac{3}{2}}{2}\right) = \ln 1 = 0$$

and

$$\tfrac{1}{2}(f(x)+f(y)) = \tfrac{1}{2}\left(\ln\tfrac{1}{2}+\ln\tfrac{3}{2}\right) = \tfrac{1}{2}(\ln 3-\ln 4) \neq 0\,.$$

Therefore, this function f fails to satisfy the required equation.

(B) Let $f(x) = 1/x$. Taking $x = \frac{1}{2}$ and $y = \frac{3}{2}$, we get

$$f\left(\frac{x+y}{2}\right) = \frac{2}{\frac{1}{2}+\frac{3}{2}} = 1$$

and

$$\tfrac{1}{2}(f(x)+f(y)) = \tfrac{1}{2}\left(2+\tfrac{2}{3}\right) = \tfrac{4}{3} \neq 1\,.$$

Therefore, this function f also fails to satisfy the equation.

(C) Let $f(x) = 2x$. Then, for all real numbers x and y,

$$f\left(\frac{x+y}{2}\right) = 2\left(\frac{x+y}{2}\right) = x+y = \tfrac{1}{2}(2x+2y) = \tfrac{1}{2}(f(x)+f(y))\,.$$

Therefore, this function f satisfies the equation.

(D) Let $f(x) = 2^x$. Taking $x = 1$ and $y = 3$, we get

$$f\left(\frac{x+y}{2}\right) = 2^{\frac{1+3}{2}} = 2^2 = 4$$

and

$$\tfrac{1}{2}(f(x)+f(y)) = \tfrac{1}{2}(2^1+2^3) = \tfrac{1}{2}(10) = 5 \neq 4\,.$$

Thus, this function f fails to satisfy the required equation.

Note: The reader who is familiar with the graphs of the given functions can see the answer to this problem immediately. Any function whose graph is a straight line (any function of the form $f(x) = mx + b$) satisfies the given equation. If a function f has a graph that is concave up (curving in an upward direction, such as 2^x for all x and $\frac{1}{x}$ for $x > 0$), then

$$f\left(\frac{x+y}{2}\right) < \tfrac{1}{2}\left(f(x) + f(y)\right).$$

If a function f has a graph that is concave down (curving in a downward direction, such as $\ln x$ for all $x > 0$ and $\frac{1}{x}$ for $x < 0$), then

$$f\left(\frac{x+y}{2}\right) > \tfrac{1}{2}\left(f(x) + f(y)\right).$$

These assertions follow from the fact that if $P\big(x, f(x)\big)$ and $Q\big(y, f(y)\big)$ are any two points on the graph of f, then the mid-point of the line segment PQ is $\left(\frac{x+y}{2}, \frac{f(x)+f(y)}{2}\right)$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

8. $(*)$ Let x, $y > 0$, $x > y$, and $z \neq 0$. The inequality which is not always correct is:

(A) $x + z > y + z$ (B) $x - z > y - z$

(C) $xz > yz$ (D) $xz^2 > yz^2$

Solution by the editor.

The answer is C. The inequalities A and B are always true when $x > y$, and D is true when $x > y$ and $z \neq 0$, since $z^2 > 0$ for $z \neq 0$. But C is not always true, because if we take $z < 0$ with $x > y$, we get $xz < yz$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

9. $(*)$ Si $a^x = c^q = b$ et $c^y = a^z = d$, alors :

(A) $xy = qz$ (B) $x + y = q + z$ (C) $x - y = q - z$ (D) $x^y = q^z$

Solution par le rédacteur.

La réponse est A. Élevons la première équation à la puissance y. On obtient $a^{xy} = c^{qy}$. Élevons la seconde équation à la puissance q. On obtient $c^{yq} = a^{zq}$. Par transitivité de l'égalité, on obtient $a^{xy} = a^{zq}$. Puisque la base est la même des 2 côtés du signe égal, on obtient $xy = qz$.

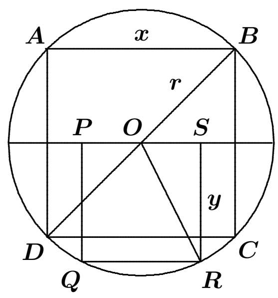
Une solution incorrecte a été soumise.

10. $(*)$ The area of a square inscribed in a semicircle is to the area of the square inscribed in the entire circle as:

(A) $1:2$ (B) $2:3$ (C) $2:5$ (D) $3:4$

Solution by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON, modified by the editor.

The answer is C. In right triangle BCD in the figure to the right, we see that the hypotenuse is $2r$. If we let the two sides have length x and use the Pythagorean Theorem, we get the equation $2x^2 = 4r^2$. Hence, $x^2 = 2r^2$. In right triangle ORS in the figure, the hypotenuse is r, one side is y, and the other side is $y/2$. Using the Pythagorean Theorem, we get $5y^2/4 = r^2$. Thus,



$$y^2 = \frac{4r^2}{5} = \frac{2x^2}{5}.$$

We know that the ratio of the areas is the same as the ratio of the square of the sides. Thus, the required ratio is $y^2 : x^2 = 2 : 5$.

Also solved by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

11. (∗) If $0 < \alpha, \beta < \frac{\pi}{2}$ and $\alpha > \beta$, then:

(A) $\sin(\alpha - \beta) > \sin\alpha - \sin\beta$ (B) $\sin(\alpha - \beta) < \sin\alpha - \sin\beta$

(C) $\sin(\alpha - \beta) = \sin\alpha - \sin\beta$ (D) none of these

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON, modified by the editor.

The answer is A. We must compare $\sin(\alpha - \beta)$ and $\sin\alpha - \sin\beta$. But

$$\sin(\alpha - \beta) = 2\cos\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\text{and} \quad \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right).$$

Since $0 < \beta < \alpha < \frac{\pi}{2}$, we have $0 < \alpha - \beta < \alpha + \beta < \pi$, and therefore $0 < \frac{\alpha-\beta}{2} < \frac{\alpha+\beta}{2} < \frac{\pi}{2}$. Note that $\sin\left(\frac{\alpha-\beta}{2}\right)$ appears on the right in both identies above, and $\sin\left(\frac{\alpha-\beta}{2}\right) > 0$ since $0 < \frac{\alpha-\beta}{2} < \frac{\pi}{2}$. Hence, we need only compare $\cos\left(\frac{\alpha+\beta}{2}\right)$ and $\cos\left(\frac{\alpha-\beta}{2}\right)$ to get the answer.

Since $0 < \frac{\alpha-\beta}{2} < \frac{\alpha+\beta}{2} < \frac{\pi}{2}$, and since $\cos x$ is strictly decreasing on $\left[0, \frac{\pi}{2}\right]$, we see at once that

$$\cos\left(\frac{\alpha+\beta}{2}\right) < \cos\left(\frac{\alpha-\beta}{2}\right).$$

It follows that $\sin\alpha - \sin\beta < \sin(\alpha - \beta)$.

12. (∗) Let $f(x) = 3^x + 5$. Then the domain of f^{-1} is:

(A) $(0, +\infty)$ (B) $(5, +\infty)$ (C) $(8, +\infty)$ (D) $(-\infty, +\infty)$

Solution by Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

The answer is B. From $f(x) = 3^x + 5$, we get $f^{-1}(x) = \log_3(x - 5)$. Hence, we have $x - 5 > 0$ or $x > 5$. The domain is $(5, +\infty)$.

13. (∗) A man has a pocket full of change, but can not make change for a dollar. What is the greatest value of coins he could have?

(A) \$0.99 (B) \$1.09 (C) \$1.19 (D) \$1.29

Solution by the editor.

The answer is C. The total of \$1.19 can be obtained with 1 quarter, 9 dimes, and 4 pennies; with 3 quarters, 4 dimes, and 4 pennies; or with 1 fifty-cent piece, 1 quarter, 4 dimes, and 4 pennies. [*Editor:* Obviously, this problem is ignoring the existence of coins of \$1 value or more. Since there is nothing to be gained by allowing a fifty-cent piece in place of two quarters, we assume that the man does not have a fifty-cent piece.]

To show that \$1.19 is the maximum amount the man can have, we first note that the number of quarters he has can be no greater than 3, since 4 quarters makes \$1. Assuming he has 3 quarters, he cannot have more than 4 dimes, because 2 quarters plus 5 dimes makes \$1. If we allow him to have a nickel or 5 pennies, then he can have only 1 dime, because 3 quarters plus 2 dimes plus the 5 cents makes \$1. But, he can have 4 pennies. Therefore, as long as he has 3 quarters, the greatest total he can have is \$1.19.

If he has only 2 quarters, then he still cannot have more than 4 dimes, and the largest total he can have is only \$0.99.

And if he has no quarters at all, then it is easy to see that he can have no more than \$0.99.

Now suppose that he has exactly 1 quarter. He can have 9 dimes, but not 10 dimes, since 10 dimes makes \$1. If we allow him to have a nickel or 5 pennies, then he can have only 6 dimes, because one quarter plus 7 dimes plus 5 cents makes \$1. But he can have 4 pennies. Thus, once again, we obtain \$1.19 for the maximum.

Also solved by Karthik Natarajan, student, Edgewater Park Public School, Thunder Bay, ON; and Alexander Remorov, student, Waterloo Collegiate Institute, Waterloo, ON.

That brings us to the end of another issue.