Problem of the Month

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Problem (1997-1998 Scottish Mathematical Challenges) Tim organized a bus trip to the seaside. Initially, more than twenty of his friends said they would go on the outing. Tim calculated the individual cost by dividing the total cost by the number of participants, and was pleased to find that it was a whole number of dollars each. He announced the cost and four people dropped out. He recalculated the individual cost from the same total cost and started to collect the money. All went well until the last two people, who now said they couldn't come. On the day of the trip, Tim had to collect another $3 from each of the remaining participants. They all had a splendid day out, including the bus driver. How much did it cost each of the participants?

Have you ever tried to organize a trip before? If so, you and Tim have likely had similar experiences.

I really enjoy the entertaining problem style from these Scottish Mathematical Challenges. This particular problem has been "translated" a bit from its original format, with pounds replaced by dollars and "bus" replacing "coach".

This problem is similar to problems that we all saw when we first started learning algebra, but it turns out to be a fair bit more complicated than it first appears.

Solution. Let T be the total cost of the trip, and let N be number of people (including Tim) who still agreed to go after the initial price was announced. This means that $N+4$ people initially said they would go on the trip, where $N+4$ is at least 22 (that is, N is at least 18), and that $N-2$ people finally went on the trip.

When there were N people going on the trip, the individual price was T/N. After the last two people dropped out, the individual price became $T/(N-2)$.

From the given information, we have $\frac{T}{N}+3=\frac{T}{N-2}$; that is,

$$\begin{aligned} T(N-2)+3N(N-2) &= TN\,, \\ 3N^2-6N &= 2T\,. \end{aligned}$$

Since T must be a whole number (the initial cost per person was a whole number of dollars), the right side is even. Hence, the left side is even. Since $6N$ is even, we see that $3N^2$ must also be even, implying that N is even.

We set $N=2n$ where n is an integer (and n is at least 9, since N is at least 18), and we see that $2T=3(2n)^2-6(2n)=12n^2-12n$, or $T=6n^2-6n$.

It is now a bit tricky to figure out where to go. What is the crucial piece of information that we have yet to use to its fullest extent? The missing link is not the colour of the bus, but rather that the initial cost per person was a whole number of dollars; that is,

$$\frac{T}{N+4} = \frac{6n^2-6n}{2n+4} = \frac{3n^2-3n}{n+2}$$

is a whole number. Maybe this will help!

When we have a rational expression like this one, it is often helpful to "long-divide" the denominator into the numerator—you will see why in a minute! If you know how to long-divide polynomials, great; if not, after a bit of fiddling around, you can figure out that $3n^2-3n = (3n-9)(n+2)+18$, which implies that $\frac{3n^2-3n}{n+2} = 3n-9+\frac{18}{n+2}$.

Where do we go from here? Since $\frac{3n^2-3n}{n+2}$ is a whole number, we see that $3n-9+\frac{18}{n+2}$ is a whole number. We already know that $3n-9$ is a whole number; thus, $\frac{18}{n+2}$ is a whole number. Then $n+2$ must be a divisor of 18. But n is at least 9. Hence, $n+2$ is at least 11. Thus, $n+2$ must be 18, since it must be a divisor of 18. Therefore, $n=16$. Hence, $N=32$ and $T = 6n^2-6n = 1440$.

At this point, we have to remember what it was that we were originally asked! We need to know how much it cost each of the participants. We find this cost by calculating $\frac{T}{N-2} = \frac{1440}{30} = 48$. This means that the final cost for each of the participants was $\$48$.

We should go back at this stage to check our information: we have a total cost of $\$1440$, along with 32 participants who agreed to go after finding out the initial cost. Thus, there were 36 who initially agreed to go, giving an individual cost of $\$1440 \div 36 = \$40$, and 32 who agreed to go after learning the price, giving an individual cost of $\$1440 \div 32 = \$45$. Finally, there were 30 who went, giving the final individual cost of $\$1440 \div 30 = \$48$. All of this information agrees with what we were given.

What is the moral of this story? If you're going to organize a trip to the seaside, make sure you hone up on your algebraic skills first! You never know when they will come in handy. And make sure that the bus driver has a good time too!