Mayhem Solutions

M130. *Proposed by the Mayhem Staff.*

Tickets are numbered 1, 2, 3, 4, $\ldots$, N. Exactly half of the tickets have the digit 1 on them.

(a) If N is a three-digit number, determine all possible values of N.

(b) Determine some possible values for N if N is a four-digit number, or a five-digit number, etc.

Solution by Doug Newman, Lancaster, CA, USA, modified by the editor.

(a) We begin by counting the numbers less than 100 that contain the digit 1. Among the nine single-digit numbers, only one contains 1; among the two-digit numbers, those from 10 to 19 each contain 1 in the tens position, while there are only 8 further two-digit numbers which contain 1, namely 21, 31, $\ldots$, 91. Thus, there are 19 numbers less than 100 that contain 1.

From 100 to 199, every number contains 1 in the hundreds position. Thus, from 100 to 199, there are 100 numbers containing 1. From 200 to 299, we have a repetition of the scenario from 1 to 99; that is, exactly 19 of the numbers contain 1. Similarly, in each of the ranges 300–399, 400–499, $\ldots$, 900–999, exactly 19 numbers contain 1.

Let N be the number of tickets, and let T be the number of tickets which contain the digit 1. We want to find N and T such that $T = N/2$, or, equivalently, $T/N = \frac{1}{2}$. From above, the ratio T/N is approximately 0.192 when $N = 99$; it is approximately 0.598 when $N = 199$; and it is approximately 0.462 when $N = 299$. Since the growth rate of T is only greater than $\frac{1}{2}$ in the first 20 numbers of each set of 100, we can rule out any higher values of N by simply considering $N = 319$. For $N = 319$, the number of tickets containing the digit 1 is $19 + 100 + 19 + 11 = 149$. Then $T/N = 149/319 < \frac{1}{2}$. Thus, if $T/N = \frac{1}{2}$, then $100 \leq N \leq 299$.

When $100 \leq N \leq 199$, we note that T is increased by 1 each time N is increased by 1 (since all the numbers in this range have a 1 in the hundreds position). Therefore, we are seeking an integer n such that

$$\frac{T}{N} = \frac{20+n}{100+n} = \frac{1}{2}\,.$$

Solving, we find that $n = 60$, which yields $N = 100 + 60 = 160$ and $T = 20 + 60 = 80$. Hence, 160 is one of our answers.

If $200 \le N \le 300$, then increases in T are not proportional to increases in N. Let us first consider the range $200 \le N \le 209$. Then $T = 119$ for $N = 200$, and $T = 120$ for $201 \le N \le 209$. For all such values of N, we have $T/N > \frac{1}{2}$. Next consider the range $210 \le N \le 219$. In this range, the ratio T/N increases, since T increases by the same amount as N, which means that we still have $T/N > \frac{1}{2}$.

For $N \ge 219$ we have $T \ge 130$, which means that we need $N \ge 260$ in order to get $T/N = \frac{1}{2}$. However, when $N = 260$, we find that $T = 134$. Thus, we need $N \ge 268$, at which point $T = 135$, further refining N to be at least 270. Indeed, $N = 270$ is a solution. Since N must be even, the next possibility is $N = 272$, in which case $T = 136$, which means that $N = 272$ is also a solution. Now, the next number to contain the digit 1 is 281, at which point $T = 137$, which has $T/N < 0.488$. Since the value of T/N will continue to decline from this point on, we must have found all the solutions.

In summary, the solution set for N is $\{160, 270, 272\}$.

(b) By first finding the values of T for different ranges of N (see the table below) and then using the above methodology, one can determine that $N \in \{1458, 3398, 13120, 44686\}$.

From	To	# with "1s"	Cumulative Total
1	9	1	1
10	19	10	11
20	99	8	19
100	199	100	119
200	999	152 [= 8(19)]	271
1 000	1 999	1 000	1 271
2 000	9 999	2 168 [= 8(271)]	3 439
10 000	19 999	10 000	13 439
20 000	99 999	27 512 [= 8(3439)]	40 593

A solution where T only counted numbers with exactly one digit 1 was submitted by Robert Bilinski, Outremont, QC.

M131. *Proposed by the Mayhem Staff.*

The triangular array of numbers shown has the following properties:

1. The bottom row contains each of the numbers 1, 2, . . . , 8 exactly once.
2. Each number in a row above the bottom row is the sum of the two neighbouring numbers in the row immediately below, if this sum is less than 10; otherwise, 9 is subtracted from this sum.

$$\begin{array}{c} 6 \\ 7\quad 8 \\ 3\quad 4\quad 4 \\ 8\quad 4\quad 9\quad 4 \\ 7\quad 1\quad 3\quad 6\quad 7 \\ 4\quad 3\quad 7\quad 5\quad 1\quad 6 \\ 5\quad 8\quad 4\quad 3\quad 2\quad 8\quad 7 \\ 2\quad 3\quad 5\quad 8\quad 4\quad 7\quad 1\quad 6 \end{array}$$

Is it possible to create a triangular array with the above properties using each number from 1 to 9 exactly four times?

Solution by Doug Newman, Lancaster, CA, USA.

The simple answer is yes. Through trial and error one can find the following two triangular arrays that satisfy the required conditions.

```
       9                      9
      4 5                    8 1
     4 9 5                  8 9 1
    3 1 8 6                6 2 7 3
   2 1 9 8 7              4 2 9 7 5
  8 3 7 2 6 1            7 6 5 4 3 2
 3 5 7 9 2 4 6          6 1 5 9 4 8 3
1 2 3 4 5 6 7 8        2 4 6 8 1 3 5 7
```

By recognizing that the arithmetic is performed modulo 9 and by noting certain relations that hold on the base row of numbers, such as

$$a_1 + a_8 \;=\; a_2 + a_7 \;=\; a_3 + a_6 \;=\; a_4 + a_5 \;=\; 9\,,$$
$$a_3,\, a_5 \in \{3,\,6\}\,,$$
$$a_2 + a_3 + a_4 \;=\; a_5 + a_6 + a_7 \;=\; 9\,,$$
$$a_1 + a_2 \;=\; a_3 \quad \text{and} \quad a_7 + a_8 \;=\; a_6\,,$$

one can create other such triangular arrays:

```
       9                      9
      4 5                    8 1
     4 9 5                  8 9 1
    6 7 2 3                3 5 4 6
   8 7 9 2 1              7 5 9 4 2
  2 6 1 8 3 7            4 3 2 7 6 5
 6 5 1 9 8 4 3          3 1 2 9 7 8 6
7 8 6 4 5 3 1 2        5 7 3 8 1 6 2 4

       9                      9
      2 7                    2 7
     2 9 7                  2 9 7
    6 5 4 3                3 8 1 6
   1 5 9 4 8              4 8 9 1 5
  4 6 8 1 2 3            1 3 5 4 6 8
 6 7 8 9 1 2 3          3 7 5 9 4 2 6
5 1 6 2 7 3 8 4        8 4 3 2 7 6 5 1
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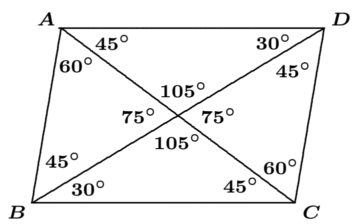
Also solved by Robert Bilinski, Outremont, QC.

M132. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) What is wrong with the diagram of the parallelogram? The angle measures are in degrees.

(b) Suggest a modification that would make the diagram plausible. You are not allowed to modify the three lines through B.



Solution by Zhao Xin Hao, student, and Luyun Zhong-Qiao, Columbia International College, Hamilton, ON.

(a) Let E be the intersection of the of the diagonals AC and BD. Let $a = AE = EC$ and let $b = BE = ED$. The Law of Sines applied to $\triangle ABE$ gives us

$$\frac{b}{a} = \frac{\sin 60^\circ}{\sin 45^\circ} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

Applying the Law of Sines to $\triangle BCE$ yields

$$\frac{b}{a} = \frac{\sin 45^\circ}{\sin 30^\circ} = \sqrt{2}.$$

Since $\sqrt{2} \neq \frac{1}{2}\sqrt{6}$, the diagram is flawed.

(b) If we are to keep the three lines through B fixed, then the angles between must also be fixed. As above, we let E be the intersection of the diagonals AC and BD, and let $a = BE = EC$ and $b = BE = ED$. If we set $\alpha = \angle BAC$ and $\beta = \angle BCA$, then, by applying the Law of Sines to $\triangle ABE$ and $\triangle CBE$, we have

$$\frac{\sin 45^\circ}{\sin\alpha} = \frac{a}{b} = \frac{\sin 30^\circ}{\sin\beta}.$$

Thus,

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin 45^\circ}{\sin 30^\circ} = \frac{\sqrt{2}/2}{1/2} = \sqrt{2}.$$

In $\triangle ABC$, we have $\alpha + \beta + 75^\circ = 180^\circ$. Hence, $\alpha = 105^\circ - \beta$, and $\sin\alpha = \sin(105^\circ - \beta) = \sin 105^\circ \cos\beta - \cos 106^\circ \sin\beta$. Then

$$\begin{aligned}\sqrt{2} &= \frac{\sin 105^\circ \cos\beta - \cos 106^\circ \sin\beta}{\sin\beta} \\ &= \sin 105^\circ \cot\beta - \cos 105^\circ \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\cot\beta - \frac{\sqrt{2}-\sqrt{6}}{4}.\end{aligned}$$

(We have calculated $\sin 105^\circ$ and $\cos 105^\circ$ by noting that $105^\circ = 60^\circ + 45^\circ$, and applying the addition formulas.) Solving the above for $\cot\beta$ yields

$$\begin{aligned}\cot\beta &= \frac{\frac{4\sqrt{2}+\sqrt{2}-\sqrt{6}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{5\sqrt{2}-\sqrt{6}}{\sqrt{6}+\sqrt{2}} \\ &= \frac{(5\sqrt{2}-\sqrt{6})(\sqrt{6}-\sqrt{2})}{4} = \frac{12\sqrt{3}-16}{4} = 3\sqrt{3}-4.\end{aligned}$$

Therefore, $\beta = \cot^{-1}(3\sqrt{3}-4)$, which is approximately 39.896° (instead of 45°, as shown in the diagram). All the remaining angles can be determined from β.