MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

The Mayhem Editor is Shawn Godin (Ottawa Carleton District School Board). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Amy Cameron (Carleton University), Dan MacKinnon (Ottawa Carleton District School Board), and Ian VanderBurgh (University of Waterloo).

Mayhem Problems

Please send your solutions to the problems in this edition by **1 September 2005**. *Solutions received after this date will only be considered if there is time before publication of the solutions.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Martin Goldstein of the University of Montreal for translations of the problems.

M188. *Proposed by Charalampos Stergiou, Chalkida, Greece.*

Consider triangle ABC in which $\angle B = \angle C = 35^\circ$. In the interior of the triangle we take a point M such that $\angle MBC = 25^\circ$ and $\angle MCB = 30^\circ$. Prove, without trigonometry, that $\angle AMC = 150^\circ$.

M189. *Proposed by Mihály Bencze, Brasov, Romania.*

Find all real solutions of the following system of equations:

$$\begin{aligned} x + \sqrt{x^2+1} &= 10^{y-x}\,, \\ y + \sqrt{y^2+1} &= 10^{z-y}\,, \\ z + \sqrt{z^2+1} &= 10^{x-z}\,. \end{aligned}$$

M190. *Proposed by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Given any three points in a unit square, show that a pair of them must be no further apart than $\sqrt{6} - \sqrt{2}$.

M191. *Proposed by the Mayhem Staff.*

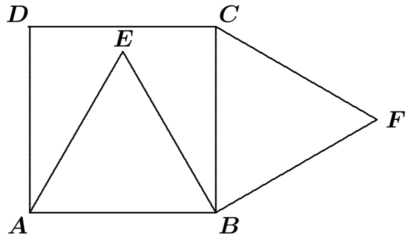
The surface areas of the six faces of a rectangular prism (box) are 1254, 1254, 770, 770, 1995, and 1995 cm^2. Determine the volume of the prism.

M192. *Proposed by Victor Oxman, Western Galilee College, Israel.*

In triangles $A_1B_1C_1$ and $A_2B_2C_2$, we are given that $A_1C_1 = A_2C_2$, that the medians B_1M_1 and B_2M_2 are equal, and that the bisectors A_1D_1 and A_2D_2 are equal. Prove that the triangles are congruent.

M193. *Proposed by Robert Bilinski, Outremont, QC.*

On square $ABCD$, an equilateral triangle ABE is constructed internally and an equilateral triangle BCF is constructed externally. Prove that the points D, E, and F are collinear.



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M188. *Proposé par Charalampos Stergiou, Chalkida, Grèce.*

Dans un triangle ABC les angles B et C mesurent $35°$. On choisit un point M à l'intérieur du triangle de sorte que les angles MBC et MCB valent respectivement $25°$ et $30°$. Sans trigonométrie, montrer que l'angle AMC vaut $150°$.

M189. *Proposé par Mihály Bencze, Brasov, Roumanie.*

Trouver toutes les solutions réelles du système d'équations suivant :

$$\begin{aligned} x + \sqrt{x^2+1} &= 10^{y-x}\,, \\ y + \sqrt{y^2+1} &= 10^{z-y}\,, \\ z + \sqrt{z^2+1} &= 10^{x-z}\,. \end{aligned}$$

M190. *Proposé par Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Etant donné trois points dans un carré unité, montrer qu'une paire d'entre eux ne peuvent être distants de plus de $\sqrt{6}-\sqrt{2}$.

M191. *Proposé par Équipe de Mayhem.*

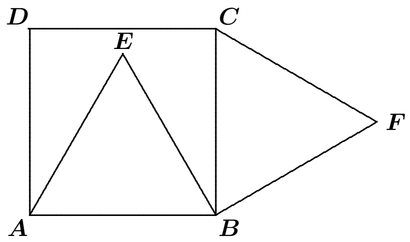
Les aires des six faces d'un prisme rectangulaire (une boîte) valent 1254, 1254, 770, 770, 1995 et 1995 cm^2. Calculer le volume du prisme.

M192. *Proposé par Victor Oxman, Western Galilee College, Israël.*

Montrer que les triangles $A_1B_1C_1$ et $A_2B_2C_2$ sont congruents, sachant que $A_1C_1 = A_2C_2$, que les médianes B_1M_1 et B_2M_2 ainsi que les bissectrices A_1D_1 et A_2D_2 sont égales.

M193. *Proposé par Robert Bilinski, Outremont, QC.*

On trace à l'intérieur du carré $ABCD$ un triangle équilatéral ABE et à l'extérieur un triangle équilatéral BCF. Montrer que les points D, E et F sont alignés.



Mayhem Solutions

M130. *Proposed by the Mayhem Staff.*

Tickets are numbered 1, 2, 3, 4, ..., N. Exactly half of the tickets have the digit 1 on them.

(a) If N is a three-digit number, determine all possible values of N.

(b) Determine some possible values for N if N is a four-digit number, or a five-digit number, etc.

Solution by Doug Newman, Lancaster, CA, USA, modified by the editor.

(a) We begin by counting the numbers less than 100 that contain the digit 1. Among the nine single-digit numbers, only one contains 1; among the two-digit numbers, those from 10 to 19 each contain 1 in the tens position, while there are only 8 further two-digit numbers which contain 1, namely 21, 31, ..., 91. Thus, there are 19 numbers less than 100 that contain 1.

From 100 to 199, every number contains 1 in the hundreds position. Thus, from 100 to 199, there are 100 numbers containing 1. From 200 to 299, we have a repetition of the scenario from 1 to 99; that is, exactly 19 of the numbers contain 1. Similarly, in each of the ranges 300–399, 400–499, ..., 900–999, exactly 19 numbers contain 1.

Let N be the number of tickets, and let T be the number of tickets which contain the digit 1. We want to find N and T such that $T = N/2$, or, equivalently, $T/N = \frac{1}{2}$. From above, the ratio T/N is approximately 0.192 when $N = 99$; it is approximately 0.598 when $N = 199$; and it is approximately 0.462 when $N = 299$. Since the growth rate of T is only greater than $\frac{1}{2}$ in the first 20 numbers of each set of 100, we can rule out any higher values of N by simply considering $N = 319$. For $N = 319$, the number of tickets containing the digit 1 is $19 + 100 + 19 + 11 = 149$. Then $T/N = 149/319 < \frac{1}{2}$. Thus, if $T/N = \frac{1}{2}$, then $100 \leq N \leq 299$.

When $100 \leq N \leq 199$, we note that T is increased by 1 each time N is increased by 1 (since all the numbers in this range have a 1 in the hundreds position). Therefore, we are seeking an integer n such that

$$\frac{T}{N} = \frac{20+n}{100+n} = \frac{1}{2}\,.$$

Solving, we find that $n = 60$, which yields $N = 100 + 60 = 160$ and $T = 20 + 60 = 80$. Hence, 160 is one of our answers.

If $200 \leq N \leq 300$, then increases in T are not proportional to increases in N. Let us first consider the range $200 \leq N \leq 209$. Then $T = 119$ for $N = 200$, and $T = 120$ for $201 \leq N \leq 209$. For all such values of N, we have $T/N > \frac{1}{2}$. Next consider the range $210 \leq N \leq 219$. In this range, the ratio T/N increases, since T increases by the same amount as N, which means that we still have $T/N > \frac{1}{2}$.

For $N \geq 219$ we have $T \geq 130$, which means that we need $N \geq 260$ in order to get $T/N = \frac{1}{2}$. However, when $N = 260$, we find that $T = 134$. Thus, we need $N \geq 268$, at which point $T = 135$, further refining N to be at least 270. Indeed, $N = 270$ is a solution. Since N must be even, the next possibility is $N = 272$, in which case $T = 136$, which means that $N = 272$ is also a solution. Now, the next number to contain the digit 1 is 281, at which point $T = 137$, which has $T/N < 0.488$. Since the value of T/N will continue to decline from this point on, we must have found all the solutions.

In summary, the solution set for N is $\{160, 270, 272\}$.

(b) By first finding the values of T for different ranges of N (see the table below) and then using the above methodology, one can determine that $N \in \{1458, 3398, 13120, 44686\}$.

From	To	# with “1s”	Cumulative Total
1	9	1	1
10	19	10	11
20	99	8	19
100	199	100	119
200	999	152 [= 8(19)]	271
1 000	1 999	1 000	1 271
2 000	9 999	2 168 [= 8(271)]	3 439
10 000	19 999	10 000	13 439
20 000	99 999	27 512 [= 8(3439)]	40 593

A solution where T only counted numbers with exactly one digit 1 was submitted by Robert Bilinski, Outremont, QC.

M131. *Proposed by the Mayhem Staff.*

The triangular array of numbers shown has the following properties:

1. The bottom row contains each of the numbers 1, 2, ..., 8 exactly once.
2. Each number in a row above the bottom row is the sum of the two neighbouring numbers in the row immediately below, if this sum is less than 10; otherwise, 9 is subtracted from this sum.

```
              6
            7   8
          3   4   4
        8   4   9   4
      7   1   3   6   7
    4   3   7   5   1   6
  5   8   4   3   2   8   7
2   3   5   8   4   7   1   6
```

Is it possible to create a triangular array with the above properties using each number from 1 to 9 exactly four times?

Solution by Doug Newman, Lancaster, CA, USA.

The simple answer is yes. Through trial and error one can find the following two triangular arrays that satisfy the required conditions.

9
4 5
4 9 5
3 1 8 6
2 1 9 8 7
8 3 7 2 6 1
3 5 7 9 2 4 6
1 2 3 4 5 6 7 8

9
8 1
8 9 1
6 2 7 3
4 2 9 7 5
7 6 5 4 3 2
6 1 5 9 4 8 3
2 4 6 8 1 3 5 7

By recognizing that the arithmetic is performed modulo 9 and by noting certain relations that hold on the base row of numbers, such as

$$a_1 + a_8 = a_2 + a_7 = a_3 + a_6 = a_4 + a_5 = 9\,,$$
$$a_3,\, a_5 \in \{3,\, 6\}\,,$$
$$a_2 + a_3 + a_4 = a_5 + a_6 + a_7 = 9\,,$$
$$a_1 + a_2 = a_3 \quad \text{and} \quad a_7 + a_8 = a_6\,,$$

one can create other such triangular arrays:

9
4 5
4 9 5
6 7 2 3
8 7 9 2 1
2 6 1 8 3 7
6 5 1 9 8 4 3
7 8 6 4 5 3 1 2

9
8 1
8 9 1
3 5 4 6
7 5 9 4 2
4 3 2 7 6 5
3 1 2 9 7 8 6
5 7 3 8 1 6 2 4

9
2 7
2 9 7
6 5 4 3
1 5 9 4 8
4 6 8 1 2 3
6 7 8 9 1 2 3
5 1 6 2 7 3 8 4

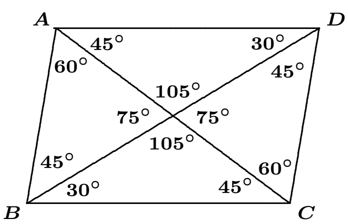
9
2 7
2 9 7
3 8 1 6
4 8 9 1 5
1 3 5 4 6 8
3 7 5 9 4 2 6
8 4 3 2 7 6 5 1

Also solved by Robert Bilinski, Outremont, QC.

M132. *Proposed by Peter Y. Woo, Biola University, La Mirada, CA, USA.*

(a) What is wrong with the diagram of the parallelogram? The angle measures are in degrees.

(b) Suggest a modification that would make the diagram plausible. You are not allowed to modify the three lines through B.



Solution by Zhao Xin Hao, student, and Luyun Zhong-Qiao, Columbia International College, Hamilton, ON.

(a) Let E be the intersection of the of the diagonals AC and BD. Let $a = AE = EC$ and let $b = BE = ED$. The Law of Sines applied to $\triangle ABE$ gives us

$$\frac{b}{a} = \frac{\sin 60^\circ}{\sin 45^\circ} = \frac{\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{6}}{2}.$$

Applying the Law of Sines to $\triangle BCE$ yields

$$\frac{b}{a} = \frac{\sin 45^\circ}{\sin 30^\circ} = \sqrt{2}.$$

Since $\sqrt{2} \neq \frac{1}{2}\sqrt{6}$, the diagram is flawed.

(b) If we are to keep the three lines through B fixed, then the angles between must also be fixed. As above, we let E be the intersection of the diagonals AC and BD, and let $a = BE = EC$ and $b = BE = ED$. If we set $\alpha = \angle BAC$ and $\beta = \angle BCA$, then, by applying the Law of Sines to $\triangle ABE$ and $\triangle CBE$, we have

$$\frac{\sin 45^\circ}{\sin\alpha} = \frac{a}{b} = \frac{\sin 30^\circ}{\sin\beta}.$$

Thus,

$$\frac{\sin\alpha}{\sin\beta} = \frac{\sin 45^\circ}{\sin 30^\circ} = \frac{\sqrt{2}/2}{1/2} = \sqrt{2}.$$

In $\triangle ABC$, we have $\alpha + \beta + 75^\circ = 180^\circ$. Hence, $\alpha = 105^\circ - \beta$, and $\sin\alpha = \sin(105^\circ - \beta) = \sin 105^\circ \cos\beta - \cos 106^\circ \sin\beta$. Then

$$\begin{aligned} \sqrt{2} &= \frac{\sin 105^\circ \cos\beta - \cos 106^\circ \sin\beta}{\sin\beta} \\ &= \sin 105^\circ \cot\beta - \cos 105^\circ \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}\cot\beta - \frac{\sqrt{2}-\sqrt{6}}{4}. \end{aligned}$$

(We have calculated $\sin 105^\circ$ and $\cos 105^\circ$ by noting that $105^\circ = 60^\circ + 45^\circ$, and applying the addition formulas.) Solving the above for $\cot\beta$ yields

$$\begin{aligned} \cot\beta &= \frac{\frac{4\sqrt{2}+\sqrt{2}-\sqrt{6}}{4}}{\frac{\sqrt{6}+\sqrt{2}}{4}} = \frac{5\sqrt{2}-\sqrt{6}}{\sqrt{6}+\sqrt{2}} \\ &= \frac{(5\sqrt{2}-\sqrt{6})(\sqrt{6}-\sqrt{2})}{4} = \frac{12\sqrt{3}-16}{4} = 3\sqrt{3}-4. \end{aligned}$$

Therefore, $\beta = \cot^{-1}(3\sqrt{3}-4)$, which is approximately 39.896° (instead of 45°, as shown in the diagram). All the remaining angles can be determined from β.

Problem of the Month

Ian VanderBurgh, University of Waterloo

Problem (1997-1998 Scottish Mathematical Challenges) Tim organized a bus trip to the seaside. Initially, more than twenty of his friends said they would go on the outing. Tim calculated the individual cost by dividing the total cost by the number of participants, and was pleased to find that it was a whole number of dollars each. He announced the cost and four people dropped out. He recalculated the individual cost from the same total cost and started to collect the money. All went well until the last two people, who now said they couldn't come. On the day of the trip, Tim had to collect another $3 from each of the remaining participants. They all had a splendid day out, including the bus driver. How much did it cost each of the participants?

Have you ever tried to organize a trip before? If so, you and Tim have likely had similar experiences.

I really enjoy the entertaining problem style from these Scottish Mathematical Challenges. This particular problem has been "translated" a bit from its original format, with pounds replaced by dollars and "bus" replacing "coach".

This problem is similar to problems that we all saw when we first started learning algebra, but it turns out to be a fair bit more complicated than it first appears.

Solution. Let T be the total cost of the trip, and let N be number of people (including Tim) who still agreed to go after the initial price was announced. This means that $N+4$ people initially said they would go on the trip, where $N+4$ is at least 22 (that is, N is at least 18), and that $N-2$ people finally went on the trip.

When there were N people going on the trip, the individual price was T/N. After the last two people dropped out, the individual price became $T/(N-2)$.

From the given information, we have $\frac{T}{N}+3=\frac{T}{N-2}$; that is,

$$\begin{aligned} T(N-2)+3N(N-2) &= TN\,, \\ 3N^2-6N &= 2T\,. \end{aligned}$$

Since T must be a whole number (the initial cost per person was a whole number of dollars), the right side is even. Hence, the left side is even. Since $6N$ is even, we see that $3N^2$ must also be even, implying that N is even.

We set $N=2n$ where n is an integer (and n is at least 9, since N is at least 18), and we see that $2T=3(2n)^2-6(2n)=12n^2-12n$, or $T=6n^2-6n$.

It is now a bit tricky to figure out where to go. What is the crucial piece of information that we have yet to use to its fullest extent? The missing link is not the colour of the bus, but rather that the initial cost per person was a whole number of dollars; that is,

$$\frac{T}{N+4} = \frac{6n^2 - 6n}{2n+4} = \frac{3n^2 - 3n}{n+2}$$

is a whole number. Maybe this will help!

When we have a rational expression like this one, it is often helpful to "long-divide" the denominator into the numerator—you will see why in a minute! If you know how to long-divide polynomials, great; if not, after a bit of fiddling around, you can figure out that $3n^2 - 3n = (3n-9)(n+2) + 18$, which implies that $\frac{3n^2 - 3n}{n+2} = 3n - 9 + \frac{18}{n+2}$.

Where do we go from here? Since $\frac{3n^2 - 3n}{n+2}$ is a whole number, we see that $3n - 9 + \frac{18}{n+2}$ is a whole number. We already know that $3n - 9$ is a whole number; thus, $\frac{18}{n+2}$ is a whole number. Then $n+2$ must be a divisor of 18. But n is at least 9. Hence, $n+2$ is at least 11. Thus, $n+2$ must be 18, since it must be a divisor of 18. Therefore, $n = 16$. Hence, $N = 32$ and $T = 6n^2 - 6n = 1440$.

At this point, we have to remember what it was that we were originally asked! We need to know how much it cost each of the participants. We find this cost by calculating $\frac{T}{N-2} = \frac{1440}{30} = 48$. This means that the final cost for each of the participants was \$48.

We should go back at this stage to check our information: we have a total cost of \$1440, along with 32 participants who agreed to go after finding out the initial cost. Thus, there were 36 who initially agreed to go, giving an individual cost of $\$1440 \div 36 = \$40$, and 32 who agreed to go after learning the price, giving an individual cost of $\$1440 \div 32 = \$45$. Finally, there were 30 who went, giving the final individual cost of $\$1440 \div 30 = \$48$. All of this information agrees with what we were given.

What is the moral of this story? If you're going to organize a trip to the seaside, make sure you hone up on your algebraic skills first! You never know when they will come in handy. And make sure that the bus driver has a good time too!

Pólya's Paragon

Fun With Numbers (Part 3)

Shawn Godin

Last time I left you with the task of looking for patterns in the following table:

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	46	47	48	49
50	51	52	53	54	55	56
57	58	59	60	61	62	63
64	65	66	67	68	69	70
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Notice the interesting pattern that shows up, each column just forms an arithmetic progression with common difference 7. Thus, if two numbers are in the same column, they must differ by a multiple of 7. They must also have the same remainder when you divide them by 7. Mathematicians say that each column in the table forms an *equivalence class*. The numbers in any one column are *equivalent* in that they yield the same remainder when you divide them by 7.

Calculate each of the following and try to see how they are related.

$$3+5\,,\qquad 10+19\,,\qquad 31+12\,,\qquad 45+20\,.$$

I hope you have come up with some ideas. In each case, we were adding a number from column 3 to a number from column 5. We ended up with a number in column 1. We can easily justify this by noticing that each pair of numbers can be written as $7a+3$ and $7b+5$ for some integers a and b. Their sum is

$$(7a+3)+(7b+5) \;=\; 7(a+b+1)+1\,.$$

Similar things happen when you look at subtraction and multiplication. (You should check this out yourself.)

Mathematicians, being extremely lazy beasts, are always looking for a short way of writing things. To show that the numbers 33 and 5 are in the same equivalence class, they write

$$33 \;\equiv\; 5 \pmod 7\,.$$

We read this as "33 is congruent to 5 modulo 7". What this means is that if we divide 33 and 5 by 7, we get the same remainder, or (equivalently) 33 and 5 differ by a multiple of 7.

We have been investigating some of the basic properties of congruences. We can write them down as follows.

Theorem (Properties of congruences). If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$\begin{aligned} a + c &\equiv b + d \pmod{m} \\ a - c &\equiv b - d \pmod{m} \\ a \times c &\equiv b \times d \pmod{m} \\ a^k &\equiv b^k \pmod{m} \end{aligned}$$

This theorem is the basis for *modular arithmetic*. We do the regular operations of arithmetic (except division, which is a little trickier), but instead of using all the numbers, we reduce our operations to m equivalence classes. For example,

$$45 + 20 \equiv 3 + 5 \equiv 8 \equiv 1 \pmod{7}$$

All very pretty, but how is it of any use? One use, before the advent of the pocket calculator, was to check long calculations for errors using *digital sums*. Let's see how this works.

The digital sum of a number is obtained by adding all the digits of the number. If the result is larger than 9, the process is repeated until the result is between 1 and 9 inclusive. For example, to calculate the digital sum of 43 658 912, we would first calculate $4+3+6+5+8+9+1+2 = 38$; then, since the result is larger than 9, we would calculate $3 + 8 = 11$; and finally, $1 + 1 = 2$, which is the digital sum. You may be surprised to find out that if you calculate the remainder when 43 658 912 is divided by 9, the result is also 2.

To check a calculation, you can find the digital sums of the numbers involved. For example, if a friend of yours has calculated

$$23\,495 \times 103\,621 = 2\,433\,505\,395\,,$$

you would look at the digital sums of the two numbers being multiplied, as well as the digital sum of the answer. You would get 5, 4, and 3, respectively. But, $5 \times 4 = 20$, which has a digital sum of 2. Since that does not match the digital sum of the answer, the answer must be wrong.

This method does not always work. If the digital sums match, there may still be an error (try to come up with an example). On the other hand, if the digital sums do not match, you are certain that the answer is wrong.

For homework, try to determine why the method of the digital sums works. (*Hint:* it is related to doing arithmetic modulo 9). Next time we will look at how we can use modular arithmetic to develop divisibility rules.

Iterating Möbius Functions with Rational Coefficients, Part II

Kun-Chieh Wang

A Möbius function with rational coefficients is a function of the form $f(z) = \dfrac{az+b}{cz+d}$, where z is a complex variable and a, b, c, and d are rational numbers such that $ad \neq bc$. For such a function f, we consider the sequence $\{f_k\}_{k=0}^{\infty}$ of functions, where f_0 is the identity function, $f_1 = f$, and $f_k = f \circ f_{k-1}$ for $k \geq 2$. The sequence $\{f_k\}$ is said to be periodic if there exists a positive integer n such that $f_n = f_0$. The smallest such integer n is the period of the sequence.

In Part I, we found Möbius functions with rational coefficients that generate sequences with periods 1, 2, 3, 4, and 6, and we proved that periods 5, 8, and 12 are not possible. We then made a conjecture, which we now prove.

Theorem. Every periodic sequence $\{f_k\}$ generated by iterating a Möbius function f with rational coefficients has period 1, 2, 3, 4, or 6.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The determinant of A is $\det A = ad - bc$; the *trace* of A is $\operatorname{tr} A = a + d$. The *characteristic equation* of A is $\det(A - xI) = 0$, where I is the 2×2 identity matrix. The roots of the characteristic equation are the *eigenvalues* of A.

We will need to use some well-known results from linear algebra, which we state without proof. For any two 2×2 matrices A and B, we have $\det(AB) = \det(A)\det(B)$. For any 2×2 matrix A, the eigenvalues are λ and μ if and only if $\lambda + \mu = \operatorname{tr}(A)$ and $\lambda\mu = \det(A)$. If λ and μ are the eigenvalues of a 2×2 matrix A, then λ^k and μ^k are the eigenvalues of A^k, for any positive integer k.

We will also need the following results.

Lemma 1. If there is a sequence $\{f_k\}$ of period n obtained by iterating a Möbius function f with rational coefficients, then there is such a sequence of period m for every positive divisor m of n.

Proof: Let $n = m\ell$. Suppose $A^n = \alpha I$, where A is the coefficient matrix of f and α is some non-zero rational number. Let $B = A^{\ell}$. Then $B^m = \alpha I$. If $B^j = \beta I$ for some $j < m$ and some rational $\beta \neq 0$, then $A^{j\ell} = \beta I$. This is not possible, since $j\ell < n$ and n is the period of $\{f_k\}$. ■

Lemma 2. Let n be an odd positive integer. If there is a Möbius function f with rational coefficients such that iteration of f generates a sequence $\{f_k\}$ of period n, then there is such a function f with a coefficient matrix B such that $B^n = I$.

Proof: Let f be a Möbius function with rational coefficients such that iteration of f generates a sequence $\{f_k\}$ of period n, and let A be the coefficient matrix of f. Then $A^n = \alpha I$ for some rational number $\alpha \neq 0$. Note that $A^{2n} = \alpha^2 I$ and $\alpha^2 = \det(\alpha I) = \det(A^n) = (\det A)^n$. Hence, $\alpha^{2/n} = \det A$, which is a non-zero rational number. Let g be the Möbius function whose coefficient matrix is $B = \frac{1}{\alpha^{2/n}} A^2$, and let $\{g_k\}$ be the sequence obtained by iterating g. The entries of B are rational and $B^n = I$.

Now suppose that $B^k = \beta I$ for some positive divisor k of n and some rational number $\beta \neq 0$. Then $A^{2k} = \left(\alpha^{2/n}\right)^k \beta I$. Hence, $2k \geq n$. Since n is odd, $k > n/2$. Then $k = n$. We conclude that $\{g_k\}$ has period n. ∎

An important tool is the following trigonometric result.

Lemma 3. If both θ/π and $\cos\theta$ are rational, then $\cos\theta \in \{0, \pm 1, \pm\frac{1}{2}\}$.

Proof: Let $\theta/\pi = m/n$ where m is an integer and n is a positive integer. Then $\cos n\theta = \cos m\pi = (-1)^m$. Trigonometric identities yield

$$2\cos n\theta \;=\; (2\cos\theta)^n + a_1(2\cos\theta)^{n-1} + \cdots + a_{n-1}(2\cos\theta) + a_n\,,$$

where the coefficients a_i are integers. Now $2\cos\theta$ is a rational root of the monic polynomial $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients. Hence, $2\cos\theta$ must be an integer. It follows that $\cos\theta \in \{0, \pm 1, \pm\frac{1}{2}\}$.

Proof of Theorem: Let A be the coefficient matrix of f. Suppose $A^n = \alpha I$ for some odd positive integer n and some non-zero rational number α. By Lemma 2, we may take $\alpha = 1$. Then $(\det A)^n = \det(A^n) = \det I = 1$. Since n is odd, $\det A = 1$. Let λ and μ be the eigenvalues of A. Then λ^n and μ^n are the eigenvalues of $A^n = I$. Hence, $\lambda^n = 1$ and $\mu^n = 1$. Now $\lambda\mu = \det(A) = 1$. Since both λ and μ are n^{th} roots of unity, $\mu = \overline{\lambda}$. We also have $\lambda + \mu = \operatorname{tr} A$, which is rational.

Let $\lambda = \cos\theta + i\sin\theta$, where $\theta = 2t\pi/n$ for some integer t such that $0 \leq t < n$. Note that we have $\cos\theta = \frac{1}{2}\left(\lambda + \overline{\lambda}\right) = \frac{1}{2}\left(\lambda + \mu\right) = \frac{1}{2}\operatorname{tr} A$. This is rational, as is $\theta/\pi = 2t/n$. By Lemma 3, $\cos(2t\pi/n) \in \{0, \pm 1, \pm\frac{1}{2}\}$. Hence, the only possible odd values for n are 1 or 3.

By Lemma 1 and the result on odd values for n, the only possible even values for n must have the form 2^u or $2^u 3$ with $u \geq 1$. By Lemma 1 again, the elimination of 8 and 12 eliminates all but 2, 4, and 6 as possible even values for n. This completes the proof of the theorem. ∎

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