SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

2927★. [2004 : 172, 174] *Proposed by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina.*

Suppose that a, b and c are positive real numbers. Prove that

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2} \;\geq\; \frac{3(ab+bc+ca)}{a+b+c}\,.$$

Solution by Charles R. Diminnie, Angelo State University, San Angelo, TX, USA.

It is well known (and easy to prove) that $(a+b+c)^2 \geq 3(ab+bc+ca)$, with equality if and only if $a=b=c$. Thus, it suffices to prove the sharper inequality

$$\sum_{\text{cyclic}} \frac{a^3}{b^2-bc+c^2} \;\geq\; a+b+c\,, \tag{1}$$

which we can rewrite in the following equivalent forms:

$$\sum_{\text{cyclic}} \frac{a^3(b+c)}{b^3+c^3} \;\geq\; a+b+c\,,$$

$$\sum_{\text{cyclic}} a^3(b+c)(c^3+a^3)(a^3+b^3) \;\geq\; (a+b+c)\prod_{\text{cyclic}}(b^3+c^3)\,.$$

We have

$$\begin{aligned}
&\sum_{\text{cyclic}} a^3(b+c)(c^3+a^3)(a^3+b^3)-(a+b+c)\prod_{\text{cyclic}}(b^3+c^3)\\
&\qquad= \sum_{\text{cyclic}} a^7b(a^2-b^2)+\sum_{\text{cyclic}} ab^7(b^2-a^2)\\
&\qquad= \sum_{\text{cyclic}} (a^7b-ab^7)(a^2-b^2) \;=\; \sum_{\text{cyclic}} ab(a^6-b^6)(a^2-b^2)\\
&\qquad= \sum_{\text{cyclic}} ab(a^2-b^2)^2(a^4+a^2b^2+b^4) \;\geq\; 0\,,
\end{aligned}$$

with equality if and only if $a=b=c$. This completes the proof.

Also solved by ARKADY ALT, San Jose, CA, USA; GEORGE BALOGLOU, SUNY, Oswego, NY, USA; MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain;

BOGDAN IONIŢĂ, Bucharest, Romania, and TITU ZVONARU, Cománeşti, Romania; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; JASNA VILIĆ, Second High School Sarajevo, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Cománeşti, Romania; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

Many solvers used Schur's or Muirhead's Inequality. Most solvers proved the sharper inequality (1). *Cîrtoaje had published this inequality in the Romanian journal Gazeta Matematica, 1(2004), p. 43. Zvonaru and Ioniţă mentioned that the original problem was proposed by Milorad Stefanovic from Yugoslavia* (*but not used*) *at the Balkan Mathematical Olympiad 1990. They give a reference to Gazeta Matematica, 5(1991). Woo proved the generalization*

$$\sum_{\text{cyclic}} \frac{a^k(b+c)}{b^k+c^k} \geq a+b+c$$

for all real $k \geq 1$. *Janous supplied a chain of generalizations. First he proved that*

$$\sum_{\text{cyclic}} \frac{a^3(b+c)}{b^3+c^3} \geq 2\sum_{\text{cyclic}} \frac{a^3}{b^2+c^2} \geq a+b+c\,.$$

He then extended the sharper inequality $\sum_{\text{cyclic}} \frac{a^3}{b^2+c^2} \geq \frac{a+b+c}{2}$ *by replacing the left side by* $\sum_{\text{cyclic}} \frac{a^{\lambda+1}}{b^\lambda+c^\lambda}$, *where* $\lambda \geq 0$. *Finally, he suggested a generalization to* n *variables.*

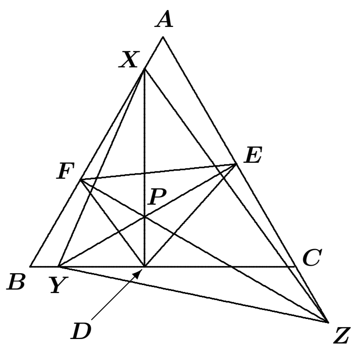
2928. [2004 : 172, 175] *Proposed by Christopher J. Bradley, Bristol, UK.*

Suppose that ABC is an equilateral triangle and that P is a point in the plane of $\triangle ABC$. The perpendicular from P to BC meets AB at X, the perpendicular from P to CA meets BC at Y, and the perpendicular from P to AB meets CA at Z.

1. If P is in the interior of $\triangle ABC$, prove that $[XYZ] \leq [ABC]$.
2. If P lies on the circumcircle of ABC, prove that X, Y, and Z are collinear.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

1. Let D, E, F be the points of intersection of PX with BC, PY with CA, PZ with AB, respectively. Since $XD \perp BC$ and $\angle ABC = 60^\circ$, we have $\angle PXF = \angle DXB = 30^\circ$. Thus, $PF = PX\sin 30^\circ = \frac{1}{2}PX$; that is, $PX = 2PF$. Similarly, $PY = 2PD$ and $PZ = 2PE$.

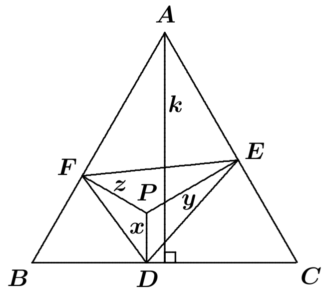


Since $\angle DPZ = \angle XPF = 60^\circ$, $\angle EPX = \angle YPD = 60^\circ$, and $\angle FPY = \angle ZPE = 60^\circ$, we have $\angle XPY = \angle FPD = 120^\circ$. Hence,

$$\frac{[XPY]}{[FPD]} = \frac{PX \cdot PY}{PF \cdot PD} = \frac{2PF \cdot 2PD}{PF \cdot PD} = 4\,.$$

Therefore, $[XPY] = 4[FPD]$. Similarly, we have $[YPZ] = 4[DPE]$ and $[ZPX] = 4[EPF]$. Thus,

$$\begin{aligned} [XYZ] &= [XPY] + [YPZ] + [ZPX] \\ &= 4\big([FPD] + [DPE] + [EPF]\big) \\ &= 4[DEF]\,. \end{aligned}$$



Now let $PD = x$, $PE = y$, and $PF = z$, and let k be the altitude of $\triangle ABC$. As is well known, we have $x + y + z = k$. Since

$$\angle FPD = \angle DPE = \angle EPF = 120^\circ\,,$$

we have

$$\begin{aligned} [DEF] &= [DPE] + [EPF] + [FPD] \\ &= \tfrac{1}{2}xy\sin 120^\circ + \tfrac{1}{2}yz\sin 120^\circ + \tfrac{1}{2}zx\sin 120^\circ \\ &= \tfrac{\sqrt{3}}{4}(xy + yz + zx)\,. \end{aligned}$$

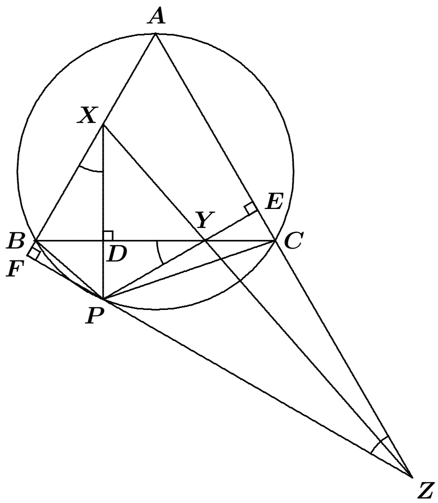
Since

$$(x+y+z)^2 - 3(xy+yz+zx) = x^2 + y^2 + z^2 - (xy+yz+zx) \;\geq\; 0\,,$$

we see that $xy + yz + zx \leq \frac{1}{3}k^2$. Thus,

$$[DEF] \;\leq\; \tfrac{\sqrt{3}}{4} \cdot \tfrac{1}{3}k^2 \;=\; \tfrac{\sqrt{3}}{12}k \cdot k \;=\; \tfrac{\sqrt{3}}{12}k \cdot \tfrac{\sqrt{3}}{2}BC \;=\; \tfrac{1}{4}[ABC]\,.$$

Therefore, $[XYZ] \leq [ABC]$.



2. Without loss of generality, we may assume that P is a point on the minor arc BC. Since $XD \perp BC$, we have $\angle PXB = \angle DXB = 30^\circ$. Similarly, $\angle PYB = \angle EYC = 30^\circ$ and $\angle PZC = \angle FZA = 30^\circ$. Since $\angle PXB = \angle PYB$, we see that B, P, Y, X are concyclic. Thus,

$$\angle PYX = \angle PBF\,. \qquad (1)$$

Since $\angle PYB = \angle PZC$, we see that P, Z, C, Y are concyclic. Thus,

$$\angle PYZ = \angle PCZ\,. \qquad (2)$$

Since A, B, P, C are concyclic, we have $\angle PBF = \angle PCA$. Hence, from (1), $\angle PYX = \angle PCA$. Using (2), we get

$$\angle PYX + \angle PYZ \;=\; \angle PCA + \angle PCZ \;=\; 180^\circ\,.$$

Therefore, X, Y, Z are collinear.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

1. Assume that $\triangle ABC$ has side-length 2, let G be the centroid of $\triangle ABC$, and let Ω be the disk centred at G with radius $2\sqrt{\frac{2}{3}}$. We will prove the stronger result that $[XYZ] \le [ABC]$ if and only if $P \in \Omega$.

Indeed, set up the coordinate system so that $A = (0, \sqrt{3})$, $B = (1, 0)$, and $C = (-1, 0)$. Suppose that $P = (u, v)$. Then it is routine to derive that

$$X \;=\; \left(u, \sqrt{3}(1-u)\right), \qquad Y \;=\; (u + \sqrt{3}v, 0)\,,$$
$$Z \;=\; \Bigl(-\tfrac{1}{2}(u - \sqrt{3}v + 3), -\tfrac{\sqrt{3}}{2}(u - \sqrt{3}v + 1)\Bigr)\,.$$

Thus,

$$[XYZ] \;=\; \tfrac{1}{2}\left|\det\begin{pmatrix} u & \sqrt{3}(1-u) & 1 \\ u+\sqrt{3}v & 0 & 1 \\ -\tfrac{1}{2}(u-\sqrt{3}v+3) & -\tfrac{\sqrt{3}}{2}(u-\sqrt{3}v+1) & 1 \end{pmatrix}\right|$$
$$\;=\; \tfrac{3\sqrt{3}}{4}\left|u^2 + \Bigl(v - \tfrac{1}{\sqrt{3}}\Bigr)^2 - \tfrac{4}{3}\right|\,.$$

Since $\partial\Omega$ (the boundary of Ω) has the equation $x^2 + \Bigl(y - \frac{1}{\sqrt{3}}\Bigr)^2 = \frac{8}{3}$, we see that $[XYZ] \le \sqrt{3} = [ABC]$ if and and only if $P \in \Omega$. Also, equality holds if and only if $P = G$ or $P \in \Omega$.

2. If P lies on the circumcircle of $\triangle ABC$, then $u^2 + \Bigl(v - \frac{1}{\sqrt{3}}\Bigr)^2 = \frac{4}{3}$. Hence, $[XYZ] = 0$; that is, X, Y, Z are collinear.

Also solved by MICHEL BATAILLE, Rouen, France; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2929. [2004 : 172, 175] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Suppose that $\triangle ABC$ has $\angle A = 90^\circ$ and $\angle B > \angle C$. Let H be the foot of the perpendicular from A to BC. The point B' lies on BC and is the mirror image of B in the line AH. Suppose that D is the foot of the perpendicular from B' to AC, that E is the foot of the perpendicular from D to BC, that F is the foot of the perpendicular from B to AB', and that G is the foot of the perpendicular from F to BC. Prove that $AH = DE + FG$.

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Since corresponding sides are parallel, we get $\triangle DB'C \sim \triangle ABC$; and since $\angle BB'A = \angle B'BA$, we also have $\triangle FB'B \sim \triangle ABC$. Since AH, DE, FG are corresponding altitudes of $\triangle ABC$, $\triangle DB'C$, $\triangle FB'B$, respectively, we have

$$\frac{DE}{AH} + \frac{FG}{AH} \;=\; \frac{B'C}{BC} + \frac{B'B}{BC} \;=\; 1\,,$$

and the result follows.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; D. KIPP JOHNSON, Beaverton, OR, USA; DOUG NEWMAN, Lancaster, CA, USA; JOEL SCHLOSBERG, Bayside, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; JASNA VILIĆ, Second High School Sarajevo, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2930. [2003 : 173, 175] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Suppose that a, b, and c are positive real numbers. Prove that

$$\begin{aligned}\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - 27\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}\right)^{-2} \\ \geq\; \frac{1}{3}\left[\left(\frac{1}{a} - \frac{1}{b}\right)^2 + \left(\frac{1}{b} - \frac{1}{c}\right)^2 + \left(\frac{1}{c} - \frac{1}{a}\right)^2\right].\end{aligned}$$

Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.

Expanding the right side and simplifying, we see that the inequality is equivalent to

$$\frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 \;\geq\; 27\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b}\right)^{-2}\;;$$

that is,

$$\left(\frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3}\right)^2 \left(\frac{\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b}}{3}\right)^2 \;\geq\; 1\,,$$

which is clearly true by the AM–GM Inequality.

Also solved by ARKADY ALT, San Jose, CA, USA; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; DIONNE T. BAILEY, ELSIE M. CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, *Beaverton, OR, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; JASNA VILIĆ, Second High School Sarajevo, Sarajevo, Bosnia and Herzegovina; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.*

Bencze poses the following generalization. If $x_k > 0$ for $k = 1, 2, \ldots, n$, then

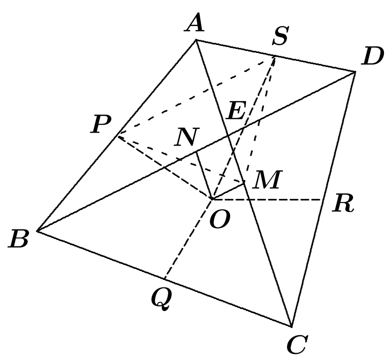
$$\left(\sum_{k=1}^{n} \frac{1}{x_k^{2p}}\right)^{m-2p} - n^m \left(\sum_{k=1}^{m} x_1 \cdots x_{k-1} x_k^{1-2p} x_{k+1} \cdots x_n\right)^{-2p} \geq \frac{1}{n} \sum_{1 \leq i < j \leq n} \left(\frac{1}{x_i} - \frac{1}{x_j}\right)^{2p}$$

2931. [2004 : 173, 175] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given quadrilateral $ABCD$, let P, Q, R, S, M and N be the midpoints of AB, BC, CD, DA, AC and BD, respectively. Suppose that the diagonals AC and BD intersect at E. Let O be the point such that quadrilateral $NEMO$ is a parallelogram.

Prove that $[OPAS] = [OQBP] = [ORCQ] = [OSDR]$ (where $[WXYZ]$ represents the area of quadrilateral $WXYZ$.)

Solution by Joel Schlosberg, Bayside, NY, USA. [Ed.: *Very similar solutions were submitted by D. Kipp Johnson, Beaverton, OR, USA; Toshio Seimiya, Kawasaki, Japan; Peter Y. Woo, Biola University, La Mirada, CA, USA; Li Zhou, Polk Community College, Winter Haven, FL, USA; and the proposer.*]



A dilation with centre A and factor 2 takes PS onto BD and $MPAS$ onto $CBAD$. It follows that $PS \parallel BD$ and $[CBAD] = 4[MPAS]$. Since $MO \parallel BD$ and $BD \parallel PS$, we get $MO \parallel PS$. Hence, $[OPS] = [MPS]$ and

$$[OPAS] = [MPAS] = \tfrac{1}{4}[ABCD].$$

The same reasoning shows that

$$[OQBP] = [NQBP] = \tfrac{1}{4}[ABCD],$$
$$[ORCQ] = [MRCQ] = \tfrac{1}{4}[ABCD],$$
$$[OSDR] = [NSDR] = \tfrac{1}{4}[ABCD],$$

giving the desired result.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DOUG NEWMAN, Lancaster, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; and TITU ZVONARU, Comănești, Romania.

2932. [2004 : 173, 175] *Proposed by Titu Zvonaru, Bucharest, Romania.*

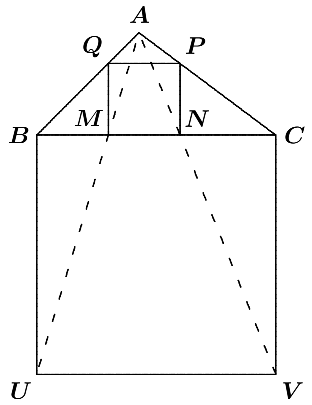
In $\triangle ABC$, suppose that the points M, N lie on the line segment BC, the point P lies on the line segment CA, and the point Q lies on the line segment AB, such that $MNPQ$ is a square. Suppose further that

$$\frac{AM}{AN} = \frac{AC+\sqrt{2}AB}{AB+\sqrt{2}AC}.$$

Characterize $\triangle ABC$.

Solution by Michel Bataille, Rouen, France; and Li Zhou, Polk Community College, Winter Haven, FL, USA.

We will prove that $\frac{AM}{AN} = \frac{AC+\sqrt{2}AB}{AB+\sqrt{2}AC}$ if and only if $AB = AC$ or $\angle BAC = \frac{3\pi}{4}$.



Let $a = BC$, $b = CA$, and $c = AB$. Consider the homothety $h\left(A, \frac{AB}{AQ}\right)$. Let U and V be the images of points M and N, respectively. Then $BUVC$ is the image of the square $QMNP$, and thus, it is also a square. Hence, $BU = CV = a$.

Using the Laws of Cosines and Sines, and the trigonometric identity

$$\sin A - \cos A = \sqrt{2}\sin\left(A - \frac{\pi}{4}\right),$$

we obtain

$$\begin{aligned}
AU^2 &= c^2 + a^2 - 2ac\cos\left(B + \frac{\pi}{2}\right) \\
&= c^2 + a^2 + 2ac\sin B \\
&= c^2 + a^2 + 2bc\sin A \\
&= c^2 + (b^2 + c^2 - 2bc\cos A) + 2bc\sin A \\
&= 2c^2 + b^2 + 2bc(\sin A - \cos A) \\
&= \left(b + \sqrt{2}c\right)^2 + 2\sqrt{2}bc\left(\sin\left(A - \frac{\pi}{4}\right) - 1\right).
\end{aligned}$$

Thus, $AU^2 = \left(b+\sqrt{2}c\right)^2 + m$, where $m = 2\sqrt{2}bc\left(\sin\left(A - \frac{\pi}{4}\right) - 1\right)$. Similarly, $AV^2 = \left(c+\sqrt{2}b\right)^2 + m$. Since the homothety implies that $\frac{AM}{AN} = \frac{AU}{AV}$, the given condition is equivalent to

$$\frac{\left(b+\sqrt{2}c\right)^2}{\left(c+\sqrt{2}b\right)^2} = \frac{\left(b+\sqrt{2}c\right)^2+m}{\left(c+\sqrt{2}b\right)^2+m}.$$

This simplifies to $m(b^2 - c^2) = 0$, which is true if and only if $b^2 = c^2$ or $\sin\left(A - \frac{\pi}{4}\right) = 1$; that is, $b = c$ or $A = \frac{3\pi}{4}$.

Also solved by CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; D.J. SMEENK, Zaltbommel, the Netherlands (2 solutions); and the proposer. There were also two incomplete solutions.

2933. [2004 : 173, 176] *Proposed by Titu Zvonaru, Bucharest, Romania.*

Prove, without the use of a calculator, that $\sin(40°) < \sqrt{\frac{3}{7}}$.

I. *Solution by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.*

Since

$$\begin{aligned}\sin(40°) &= 2\sin(20°)\cos(20°) < 2\sin(20°)\\ &= 2\sin(60° - 40°) = \sqrt{3}\cos(40°) - \sin(40°),\end{aligned}$$

we have $2\sin(40°) < \sqrt{3}\cos(40°)$.

Hence,

$$4\sin^2(40°) < 3\cos^2(40°) = 3(1 - \sin^2(40°)),$$

or $7\sin^2(40°) < 3$, from which $\sin(40°) < \sqrt{\frac{3}{7}}$ follows immediately.

II. *Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.*

We prove the slightly stronger inequality $\sin(40°) < \sqrt{\frac{5}{12}}$.

Note first that $\sin(40°) < \sqrt{\frac{5}{12}}$ is equivalent to $\frac{1}{2}\big(1 - \cos(80°)\big) < \frac{5}{12}$, or $\cos(80°) > \frac{1}{6}$, which is the same as $\sin(10°) > \frac{1}{6}$. Let $c = \sin(10°)$. Then $0 < c < 1$. From $\frac{1}{2} = \sin(30°) = 3\sin(10°) - 4\sin^3(10°) = 3c - 4c^3$, we obtain $8c^3 - 6c + 1 = 0$. Since $8c^3 > 0$, we must have $-6c + 1 < 0$. Hence, $c > \frac{1}{6}$, and we are done.

Also solved by SAMUEL ALEXANDER, student, University of Arizona, Tucson, AZ, USA; DIONNE BAILEY, ELSIE CAMPBELL, and CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Bristol, UK; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines, and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; KEE-WAI LAU, Hong Kong, China; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; NORVALD MIDTTUN, Royal Norwegian Naval Academy, Sjøkrigsskolen, Bergen, Norway; DOUG NEWMAN, Lancaster, CA, USA; MICHAEL PARMENTER, Memorial University of Newfoundland, St. John's, NL; D.J. SMEENK, Zaltbommel, the Netherlands; MIKE SPIVEY, Samford

University, Birmingam, AL, USA; Mª JESÚS VILLAR RUBIO, Santander, Spain; STAN WAGON, Macalester College, St. Paul, MN, USA; MICHAEL WATSON, student, University of Waterloo, Waterloo, ON; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Solvers who obtained the sharper bound given in Solution II above were Bailey et al., Beasley, Hess, and Janous.

Most of the solutions used calculus or concavity and Jensen's Inequality.

Wagon commented "it is pointless to pose a problem with the restriction 'without the use of a calculator' when calculations can be done by pencil and paper. The only meaningful interpretation of the problem is that it was meant to be done without numerical computations of any sort, but this would preclude using $1 + 1 = 2$. So, I have to say that I feel that such problems should not appear in Crux". We invite our readers to send us your opinions on whether you agree with his comments.

2934. [2004 : 173, 176] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

In $\triangle ABC$ with circumradius R, let AD, BE and CF be the altitudes. Let P be any interior point of the triangle. The line through P parallel to EF intersects the line AC at E_1 and the line AB at F_1. The line through P parallel to FD intersects the line AB at F_2 and the line BC at D_2. The line through P parallel to DE intersects the line BC at D_3 and the line AC at E_3.

Show that

$$E_1F_1 \cot A + F_2D_2 \cot B + D_3E_3 \cot C \;=\; 2R\,.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let L, M, and N be the feet of the perpendiculars from P to BC, CA, and AB, respectively. Note that

$$\angle PF_1F_2 \;=\; \angle EFA \;=\; \angle C \;=\; \angle DFB \;=\; \angle PF_2F_1\,;$$

thus, $PF_1 = PF_2$. Likewise, $PD_2 = PD_3$ and $PE_3 = PE_1$. Hence,

$$\begin{aligned}
&E_1F_1 \cot A + F_2D_2 \cot B + D_3E_3 \cot C \\
&\quad= (PF_1 + PE_1)\cot A + (PF_2 + PD_2)\cot B + (PE_3 + PD_3)\cot C \\
&\quad= (PF_1 + PE_3)\cot A + (PF_1 + PD_2)\cot B + (PE_3 + PD_2)\cot C \\
&\quad= PF_1(\cot A + \cot B) + PD_2(\cot B + \cot C) + PE_3(\cot C + \cot A) \\
&\quad= \frac{PF_1 \cdot AB}{CF} + \frac{PD_2 \cdot BC}{AD} + \frac{PE_3 \cdot CA}{BE} \\
&\quad= 2R\left(\frac{PF_1 \cdot \sin C}{CF} + \frac{PD_2 \cdot \sin A}{AD} + \frac{PE_3 \cdot \sin B}{BE}\right) \\
&\quad= 2R\left(\frac{PN}{CF} + \frac{PL}{AD} + \frac{PM}{BE}\right) \;=\; 2R\left(\frac{[APB]}{[ABC]} + \frac{[BPC]}{[ABC]} + \frac{[CPA]}{[ABC]}\right) \\
&\quad= 2R\,.
\end{aligned}$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; D. KIPP JOHNSON, Beaverton, OR, USA; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

2935. [2004 : 174] *Proposed by Titu Zvonaru, Bucharest, Romania.*

Suppose that a, b, and c are positive real numbers which satisfy $a^2+b^2+c^2=1$, and that $n>1$ is a positive integer. Prove that

$$\frac{a}{1-a^n}+\frac{b}{1-b^n}+\frac{c}{1-c^n} \;\geq\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

For $0<x<1$ we have, by the AM–GM Inequality,

$$\begin{aligned}\big(x(1-x^n)\big)^n &= \frac{nx^n(1-x^n)^n}{n} \;\leq\; \frac{1}{n}\left(\frac{nx^n+n(1-x^n)}{n+1}\right)^{n+1} \\ &= \frac{n^n}{(n+1)^{n+1}}\,,\end{aligned}$$

from which we see that $x(1-x^n)\leq\dfrac{n}{(n+1)^{1+\frac{1}{n}}}$. Hence,

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a}{1-a^n} &= \sum_{\text{cyclic}}\frac{a^2}{a(1-a^n)} \;\geq\; \sum_{\text{cyclic}}\frac{a^2(n+1)^{1+\frac{1}{n}}}{n} \\ &= \frac{(n+1)^{1+\frac{1}{n}}}{n}(a^2+b^2+c^2) \;=\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.\end{aligned}$$

Equality holds when $n=2$ and $a=b=c=1/\sqrt{3}$.

Also solved by MICHEL BATAILLE, Rouen, France; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; BABIS STERGIOU, Chalkida, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Both Howard and the proposer remarked that this problem is a generalization of Crux 2738 [2002 : 180; 2003 : 243]. *Heuver remarked that Crux 1445* [1989 : 148; 1990 : 216] *by the late Murray Klamkin and Andy Liu dealt with a more generalized version of this problem. Janous, noticing that the lower bound is not sharp, offered three conjectures, one of which is as follows:* Let x_1, x_2, …, x_n be positive real numbers satisfying $x_1^2+x_2^2+\cdots+x_n^2=1$. Then, for all $p>0$, we have $\displaystyle\sum_{j=1}^{n}\frac{x_j}{1-x_j^p}\geq\frac{n^{(p+1)/2}}{n^{p/2}-1}$.

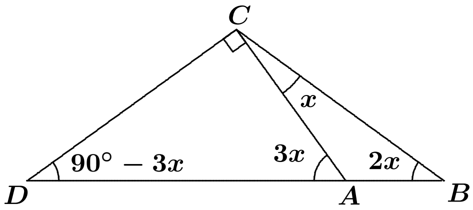
Though a few solvers stated that equality holds in the given inequality only if $n=2$ *and* $a=b=c=1/\sqrt{3}$, *no one actually gave a detailed proof (though this is not difficult). Indeed, from the proof given in the solution above we see that if equality holds, then we must have* $nx^n=1-x^n$, *or* $x=\dfrac{1}{\sqrt[n]{n+1}}$ *which implies that* $a=b=c=\dfrac{1}{(n+1)^{1/n}}$. *From* $a^2+b^2+c^2=1$, *we then get* $\dfrac{3}{(n+1)^{2/n}}=1$ *or* $(n+1)^2=3^n$, *which clearly holds when* $n=2$. *But by a simple induction, one can easily show that* $3^n>(n+1)^2$ *for all* $n\geq 3$, *and the conclusion follows.*

2936. [2004 : 174] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Consider $\triangle ABC$ with $\angle ABC = 2\angle ACB$ and $\angle BAC > 90^\circ$. Given that the perpendicular to AC through C meets AB at D, prove that

$$\frac{1}{AB} - \frac{1}{BD} = \frac{2}{BC}\,.$$

I. *Solution by Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

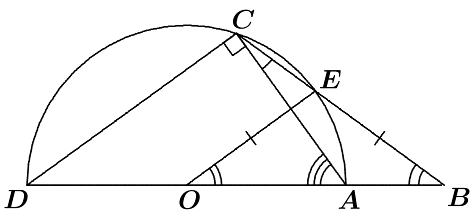


Let $\angle ACB = x$. Then $\angle ABC = 2x$, $\angle DAC = 3x$, $\angle BAC = \pi - 3x$, $\angle BCD = \frac{\pi}{2} + x$, and $\angle BDC = \frac{\pi}{2} - 3x$. Using the Sine Law on triangles ABC and BCD, we obtain

$$\begin{aligned}\frac{BC}{AB} - \frac{BC}{BD} &= \frac{\sin\angle BAC}{\sin\angle ACB} - \frac{\sin\angle BDC}{\sin\angle BCD} = \frac{\sin(\pi - 3x)}{\sin x} - \frac{\sin(\frac{\pi}{2} - 3x)}{\sin(\frac{\pi}{2} + x)} \\ &= \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = \frac{\sin 3x\cos x - \cos 3x\sin x}{\sin x\cos x} \\ &= \frac{\sin(3x - x)}{\frac{1}{2}\sin 2x} = 2\,,\end{aligned}$$

and the result follows.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*



Let O be the mid-point of the segment AD. Since $\angle ACD = 90^\circ$, the circumcentre of $\triangle ACD$ is at O. Suppose the circumcircle of $\triangle ACD$ intersects BC at E. Then $\angle AOE = 2\angle ACE = \angle ABE$. Hence, $BE = OE = \frac{1}{2}AD$. Therefore,

$$\frac{1}{2}AD \cdot BC = BE \cdot BC = BA \cdot BD\,,$$

so that

$$\frac{2AB}{BC} = \frac{AD}{BD} = \frac{BD - AB}{BD} = 1 - \frac{AB}{BD},$$

which gives

$$\frac{1}{AB} - \frac{1}{BD} = \frac{2}{BC},$$

as desired.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Bristol, UK; RICARDO BARROSO CAMPOS, Universidad de Sevilla, Sevilla, Spain; MARLON CAUMERAN, Philippine Science High School, Central Mindanao Campus, and I.J.L. GARCES, Ateneo de Manila University, The Phlippines; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; JOSÉMARÍA PEDRET, Barcelona, Spain; BOB SERKEY, Leonia, NJ, USA; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was also one incorrect solution submitted.

Zhou and Pedret were the only solvers who submitted a solution that did not use trigonometry.

2937. [2004 : 174, 176] *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n$ $(n \geq 2)$ are positive real numbers. Prove that

$$\left(x_1^2 + \cdots + x_n^2\right)\left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \geq \frac{n^2}{2}.$$

I. *Composite of essentially the same solution by Arkady Alt, San Jose, CA, USA; Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; John G. Heuver, Grande Prairie, AB; and Yufei Zhao, student, Don Mills Collegiate Institute, Toronto, ON.*

Since

$$\begin{aligned} &2(x_1^2 + x_2^2 + \cdots + x_n^2) - 2(x_1x_2 + x_2x_3 + \cdots + x_nx_1) \\ &\qquad = (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_n - x_1)^2 \geq 0, \end{aligned}$$

we have $x_1^2 + x_2^2 + \cdots + x_n^2 \geq x_1x_2 + x_2x_3 + \cdots + x_nx_1$. Hence,

$$\begin{aligned} &2(x_1^2 + x_2^2 + \cdots + x_n^2)\left(\frac{1}{x_1^2 + x_1x_2} + \frac{1}{x_2^2 + x_2x_3} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \\ &\qquad \geq \Big((x_1^2 + x_1x_2) + (x_2^2 + x_2x_3) + \cdots + (x_n^2 + x_nx_1)\Big) \cdot \\ &\qquad\qquad \cdot \left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right). \end{aligned}$$

The right side is at least n^2, by the AM–HM Inequality.

Clearly, equality holds if and only if all the x_i's are equal.

II. *Generalization by Mihály Bencze, Brasov, Romania, adapted by the editor.*

We show more generally that if $x_k > 0$, $k = 1, 2, \ldots, n$, where $n \geq 3$, then for all a, $b > 0$ and for all m with $2 \leq m \leq n-1$ we have

$$\left(\sum_{k=1}^{n} x_k^m\right) \sum_{k=1}^{n} \frac{1}{ax_k^m + bx_kx_{k+1}\cdots x_{k+m-1}} \geq \frac{n^2}{a+b}, \tag{1}$$

where all indices are reduced modulo n. The current problem is the special case when $m = 2$ and $a = b = 1$.

To prove (1), we first apply the Power–Mean Inequality to obtain

$$\begin{aligned}\sum_{k=1}^{n} ax_k^m + \sum_{k=1}^{n} bx_kx_{k+1}\cdots x_{k+m-1} & \\ \leq\ & a\sum_{k=1}^{n} x_k^m + b\sum_{k=1}^{n} \frac{x_k^m + x_{k+1}^m + \cdots + x_{k+m-1}^m}{m} \\ =\ & (a+b)\sum_{k=1}^{n} x_k^m\,. \end{aligned} \tag{2}$$

Next we have, by the AM–HM Inequality,

$$\left(\sum_{k=1}^{n}(ax_k^m + bx_kx_{k+1}\cdots x_{k+m-1})\right)\cdot \\ \cdot\sum_{k=1}^{n} \frac{1}{ax_k^m + bx_kx_{k+1}\cdots x_{k+m-1}} \geq n^2 \tag{3}$$

From (2) and (3), we see that (1) follows immediately.

Also solved by MICHEL BATAILLE, Rouen, France; VASILE CÎRTOAJE, University of Ploiesti, Romania; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer. There was one incomplete solution.

Most of the solutions use one or more of the following: AM–GM Inequality; AM–HM Inequality, Cauchy–Schwarz Inequality, Power–Mean Inequality, homogeneity, and convexity.

Cîrtoaje conjectured the following sharper inequality and proved it to be true for $n = 3$ and $n = 4$:

$$(x_1x_2 + x_2x_3 + \cdots + x_nx_1)\left(\frac{1}{x_1^2 + x_1x_2} + \cdots + \frac{1}{x_n^2 + x_nx_1}\right) \geq \frac{n^2}{2}\,.$$

Janous also obtained a generalization which is the special case when $a = b = 1$ of Bencze's generalization featured above.

2938. [2004 : 174, 176, 296, 298] *Proposed by Todor Mitev, University of Rousse, Rousse, Bulgaria.*

Suppose that $x_1, \ldots, x_n, \alpha$ are positive real numbers. Prove that

(a) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\geq\; \alpha + \sqrt[n]{x_1\cdots x_n}\,;$

(b) $\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} \;\leq\; \alpha + \dfrac{x_1+\cdots+x_n}{n}\,.$

Obervation by Michel Bataille, Rouen, France; Vedula N. Murty, Dover, PA, USA; and Eckard Specht, Otto-von-Guericke University, Magdeburg, Germany.

Part (a) is a special case of 2176 [1996 : 275; 1997 : 444] (which also appeared as 2730 [2002 : 117; 2003 : 186]) with $a_1 = \cdots = a_n = \alpha$ and $b_1 = x_1, \ldots, b_n = x_n$.

Part (b) follows immediately from the AM–GM Inequality applied to the positive real numbers $x_1+\alpha, \ldots, x_n+\alpha$.

Full solutions of the problem were submitted by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; KIN FUNG CHUNG, student, University of Toronto, Toronto, ON; CHIP CURTIS, Missouri Southern State College, Joplin, MO, USA; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RICHARD B. EDEN, Ateneo de Manila University, The Philippines; OVIDIU FURDUI, student, Western Michigan University, Kalamazoo, MI, USA; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; D. KIPP JOHNSON, Beaverton, OR, USA; VINAYAK MURALIDHAR, student, Corona del Sol High School, Tempe, AZ, USA; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PANOS E. TSAOUSSOGLOU, Athens, Greece; GEORGE TSAPAKIDIS, Agrinio, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; YUFEI ZHAO, student, Don Mills Collegiate Institute, Toronto, ON; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Comănești, Romania; and the proposer.

Furdui notes that the problem is an application of problem A2 on the 64th *Putnam Competition.*

Mihály Bencze, Brasov, Romania provided many extended results from this inequality. The interested reader is referred to "Mihály Bencze, About M. Bencze's Polynomial Inequalities, Octogon Mathematical Magazine, **9** *(2001) No. 1, pp. 263–272."*