The Diagonal Points of a Cyclic Quadrangle

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In what follows, $ABCD$ is a cyclic quadrangle in which there are no parallel sides, the circle $ABCD$ has centre O, the diagonals AC and BD meet at E, AB and CD meet at F, AD and BC meet at G, and M is the mid-point of FG. Then EFG is the diagonal-point triangle, which is self-conjugate.

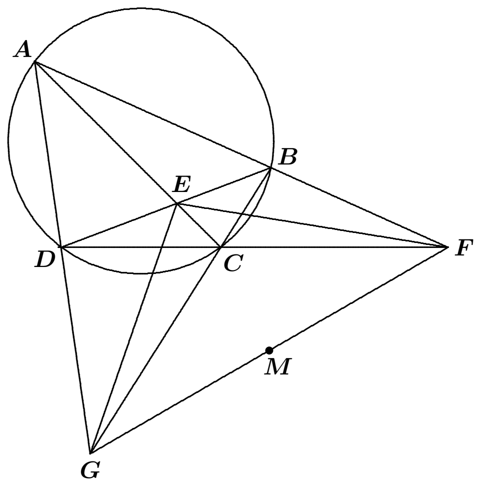

Figure 1

In this paper we are not concerned with the harmonic properties of the quadrangle, which are covered in many books, such as [1] and [2]. We do, however, set the stage for what follows by quoting, without proof, some well-known geometric properties of cyclic quadrangles. Proofs are indicated in a recent review [3].

1. *The internal angle bisectors of the angles at F and G meet at right angles.*
2. *If the tangents at A and C meet at T, and the tangents at B and D meet at U, then T, F, U, and G are collinear.*
3. *O is the orthocentre of the diagonal-point triangle.*
4. *The circles whose diameters are the sides of the diagonal-point triangle EFG are orthogonal to the circle $ABCD$.*

We refer to the set of four triangles AFG, BFG, CFG, DFG as 'Set 1' and the four triangles ACF, BDF, ACG, BDG as 'Set 2'.

Our first set of results is concerned with the triangles in Set 1.

Proposition 1. *If A', B', C', D' are the centroids of the triangles in Set 1, then $A'B'C'D'$ is homothetic with $ABCD$ and one-third the size.* (See Figure 2.)

This follows from the fact that $AA'M$, $BB'M$, $CC'M$, $DD'M$ are concurrent at M and $AA' = (2/3)AM$, etc. The circumcentres of the triangles in Set 1 are, of course, collinear on the perpendicular bisector of FG.

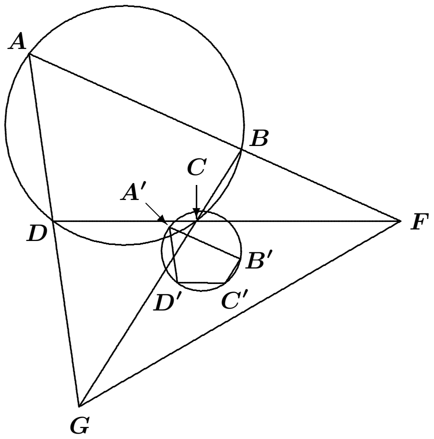

Figure 2

Proposition 2. *If A', B', C', D' are the respective orthocentres of the triangles in Set 1, then $A'B'C'D'$ is a cyclic quadrilateral with angles the same as those of $CDAB$; that is, $\angle B' = \angle D$, etc.* (See Figure 3.)

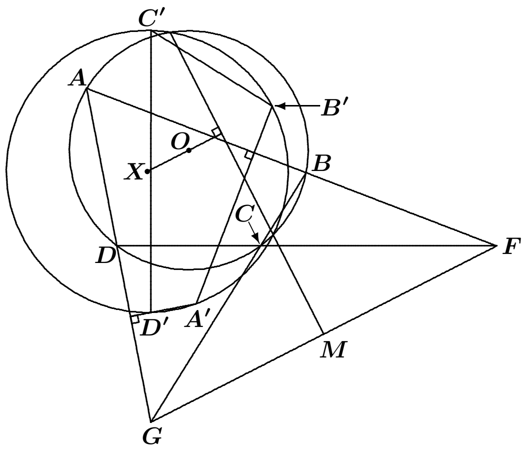

Figure 3

Since $A'B'$ is perpendicular to AB and $A'D'$ is perpendicular to AD, the angles at A and A' are supplementary, and the same is true for the other three pairs of angles. It follows that $A'B'C'D'$ is a cyclic quadrilateral. The reader is invited to show that the common chord of the two circles $ABCD$ and $A'B'C'D'$ is perpendicular to FG and passes through M. One proof uses Result 4 from the known properties of cyclic quadrangles listed above.

Conjecture. *If A', B', C', D' are the respective nine-point centres of the triangles in Set 1, then $A'D'B'C'$ is an isosceles trapezium and circle $A'D'B'C'$ passes through M.* (See Figure 4.)

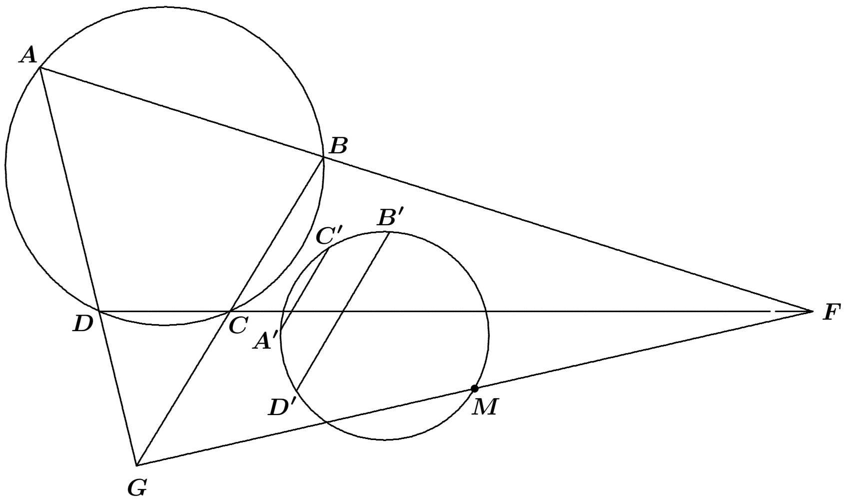

Figure 4

At the time of writing I have no proof of this conjecture. It was suggested by the software CABRI. We may be confident that the conjecture is true, such is the accuracy of the software.

Proposition 3. *If A', B', C', D' are the respective incentres of the triangles in Set 1, then $A'B'C'D'$ is a quadrilateral with opposite angles summing to an odd multiple of $\pi/2$ radians.* (See Figure 5.)

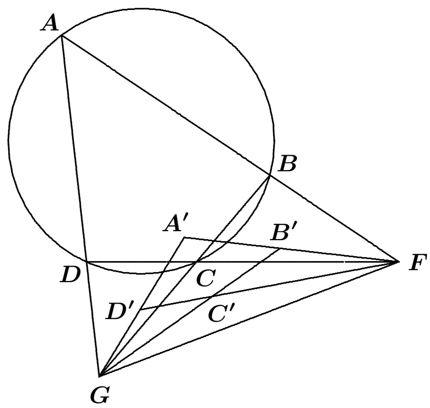

Figure 5

We use the simple notation $F = \angle AFG$, $G = \angle AGF$, $F' = \angle CFG$, and $G' = \angle CGF$. Then $F+G = C = 2(\pi - A')$ and $F' + G' = A = 2(\pi - C')$. Summing and dividing by 2, we obtain $(A+C) = \pi/2 = 2\pi - (A'+C')$. Therefore, $A' + C' = 3\pi/2$ and $B' + D' = \pi/2$.

Our next set of results is concerned with the triangles in Set 2.

Proposition 4. *If A', B', C', D' are the respective centroids of the triangles of Set 2, then $A'B'C'D'$ is a parallelogram.*

This result may be easily proved using vectors, and it turns out that $\overrightarrow{A'C'} = \overrightarrow{B'D'} = \frac{1}{3}\overrightarrow{FG}$. Note that the result is true whether or not $ABCD$ is cyclic. The configuration consisting of the circumcentres of the triangles of Set 2 appears to have no interesting properties.

Proposition 5. *If AC and BD are perpendicular and A', B', C', D' are the respective orthocentres of the triangles of Set 2, then $A'C'B'D'$ is a parallelogram, and $A'GEB'$ and $D'EC'F$ are straight lines. Furthermore, $B'F$, AC, $D'G$ are parallel, with $B'G = D'F$; and BD, FA', $C'G$ are parallel, with $FC' = GA'$. Also, AC, FG, $A'C'$ are concurrent, as are BD, FG, $B'D'$. Finally, E is the centre of circles $A'B'F$ and $C'D'G$.* (See Figure 6.)

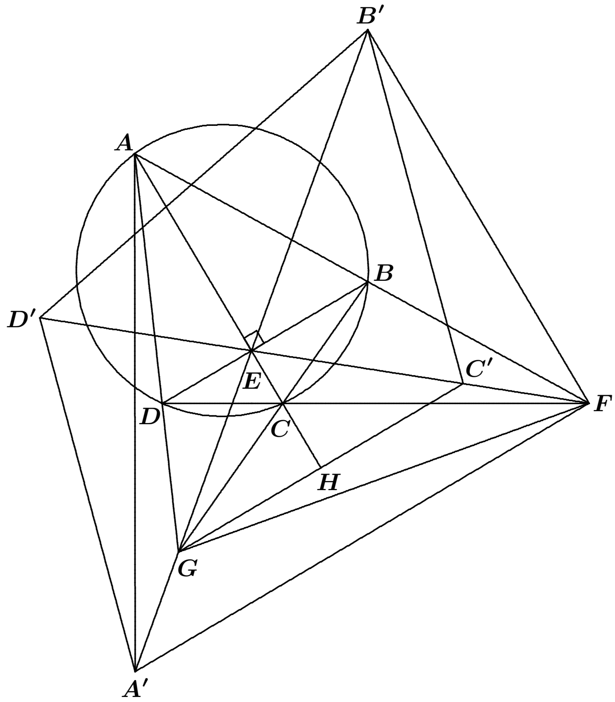

Figure 6

Two of the properties stated are easily proved. Since A' is the orthocentre of triangle ACF, it follows that FA' is perpendicular to AC and is therefore parallel to BD. Similarly, $C'G$ is parallel to BD, and both $B'F$ and $D'G$ are parallel to AC. To establish the remaining results, we use Cartesian coordinates.

Take E as the origin, and use the coordinates $A(-rs, 0)$, $B(0, pr)$, $C(pq, 0)$, and $D(0, -qs)$, where the coordinates are chosen to satisfy the intersecting chord theorem:

$$(AE)(EC) = (DE)(EB).$$

This ensures that A, B, C, D are concyclic and that AC and BD are perpendicular, with AC as the x–axis and DB as the y–axis. Details of the calculation are omitted, as it is straightforward, the difficult part being to find the coordinates of A', B', C', and D'. The points F and G turn out to have coordinates $F\big(ps(pr+qs)/(s^2-p^2), ps(pq+rs)/(s^2-p^2)\big)$ and $G\big(qr(pr+qs)/(r^2-q^2), -qr(pq+rs)/(r^2-q^2)\big)$. Note that $p \neq s$ and $r \neq q$, since AB is not parallel to CD and AD is not parallel to BC. It is worth noting that EF and EG are equally inclined to AC (a known result when AC and BD are prependicular).

After some algebra, we see that A' and B' are the reflections of F in AC and BD, respectively, and C' and D' are the reflections of G in AC and BD, respectively. Hence, there is no need to record their coordinates. From here it is easy to see that $A'B'C'D'$ is a parallelogram, that A' lies on EG, that $\angle A'FB' = \frac{\pi}{2}$, and that E is the centre of the circle $A'B'F$. All other results follow just as easily.

Proposition 6. *If AC and BD are perpendicular and A', B', C', D' are the respective nine-point centres of the triangles in Set 2, then A', D' lie on AC and C', D' lie on BD.*

Our proof of Proposition 6 relies on the following fact, whose proof is left to the reader: If XYZ is a triangle, then the nine-point centre of the triangle lies on XY if and only if $|\angle X - \angle Y| = \frac{\pi}{2}$. Using the angle properties of circle $ABCD$ and the fact that AC and BD are right angles, we have

$$\begin{aligned} \angle ACF - \angle FAC &= \angle BCA + \angle BCF - \angle FAC \\ &= \angle BCA + \angle BAD - \angle FAC \\ &= \angle BCA + \angle CAD \;=\; \angle BCA + \angle CBE \;=\; \tfrac{\pi}{2}\,. \end{aligned}$$

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References.

[1] E.A. Maxwell, *The Methods of Plane Projective Geometry based on the use of General Homogeneous Co-ordinates*, Cambridge University Press, 1957.

[2] C.V. Durell, *Modern Geometry*, Macmillan, London, 1946.

[3] C.J. Bradley, Cyclic Quadrilaterals, *Math. Gaz.* 88 (2004), 417–431.

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