THE OLYMPIAD CORNER

No. 245

R.E. Woodrow

This number starts with the problems of the XX Colombian Mathematical Olympiad, Higher Level, June 7–8, 2001. My thanks go to Chris Small, Canadian Team Leader to the 42nd IMO, for collecting them for our use.

XX COLOMBIAN MATHEMATICAL OLYMPIAD

Higher Level

June 7–8, 2001

1. [7 points] Let ABC be an isosceles triangle with $AB = AC$. Let M be the mid-point of side BC. The circle with diameter AB cuts side AC at point P. The parallelogram $MPDC$ is constructed so that $PD = MC$ and $PD \parallel MC$. Prove that triangles APD and APM are congruent.

2. [7 points] Find all positive integers z for which the equation

$$x(x+z) \;=\; y^2$$

has no solutions x, y that are positive integers.

3. [7 points] Let $n \geq 4$ be a fixed integer. Let $S = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane, no three of which are collinear and no four concyclic. Let a_t, $1 \leq t \leq n$, be the number of circles $P_iP_jP_k$ that contain P_t in the interior, and let

$$m(S) \;=\; a_1 + a_2 + \cdots + a_n\,.$$

Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

4. [7 points] Let x and y be any two real numbers. Prove that

$$3(x+y+1)^2 + 1 \;\geq\; 3xy\,.$$

Under what conditions does equality hold?

5. [7 points] Let b be an odd positive integer, and let $a = \dfrac{b^2-1}{4}$. For each positive integer $n > \sqrt{a}$, define the sequence n_0, n_1, $\ldots$, n_k, $\ldots$ in the following way: $n_0 = n$, and $n_i = n_{i-1}^2 - a$ for $i \geq 1$.

Determine all values of n for which there exists a positive integer k such that $n_k = n$.

6. Mr. Leonardo invited a group of children to go for a ride around a lake on his boat, in several turns. He later realized that the following things had happened:

- In each turn, there had been exactly three children on the boat.
- Each pair of children had been together on the boat in exactly one turn.

(a) [2 points] Prove that if Mr. Leonardo invited n children, then n must be a number of the form $6t+1$ or $6t+3$, where t is a non-negative integer.

(b) [5 points] Prove that, for any non-negative integer t, Mr. Leonardo can invite $6t+3$ children under the above conditions.

As a second problem set we give the 53th Polish Mathematical Olympiad 2001–02, Final Round, April 2002. Thanks go to Bill Sands, Chair of the IMO Committee of the Canadian Mathematical Society, for forwarding the collection for our use.

53th POLISH MATHEMATICAL OLYMPIAD 2001-02
Final Round
April 3-4, 2002

1. Determine all triples of positive integers a, b, c such that a^2+1 and b^2+1 are prime numbers satisfying $(a^2+1)(b^2+1) = c^2+1$.

2. On sides AC and BC of an acute-angled triangle ABC, rectangles $ACPQ$ and $BKCL$ are erected outwardly. Assuming that these rectangles have equal areas, show that the vertex C, the circumcentre of triangle ABC, and the mid-point of segment PL are collinear.

3. Three non-negative integers are written on a board. Two of them, k and m, are chosen. These two are erased and replaced by $k+m$ and $|k-m|$, while the third number remains unchanged. The same procedure is applied to the resulting triple of numbers, and so on. Determine whether it is always possible, given any initial triple of non-negative integers, to obtain a triple with at least two zeros.

4. Prove that, for every integer $n \geq 3$ and every sequence of positive numbers $x_1, x_2, \ldots, x_n$, at least one of the following two inequalities is satisfied:

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}+x_{i+2}} \geq \frac{n}{2}, \qquad \sum_{i=1}^{n} \frac{x_i}{x_{i-1}+x_{i-2}} \geq \frac{n}{2}.$$

(Note: Here $x_{n+1}=x_1$, $x_{n+2}=x_2$, $x_0=x_n$, and $x_{-1}=x_{n-1}$.)

5. In space we are given a triangle ABC and a sphere s disjoint from the plane ABC. Let K, L, M, and P be points on s such that AK, BL, CM are tangent to s, and

$$\frac{AK}{AP} = \frac{BL}{BP} = \frac{CM}{CP}.$$

Prove that the circumsphere of the tetrahedron $ABCP$ is tangent to s.

6. Let k be a fixed positive integer. The infinite sequence $\{a_n\}$ is defined by the formulae $a_1 = k + 1$ and $a_{n+1} = a_n^2 - ka_n + k$ for $n \geq 1$. Show that if $m \neq n$, then the numbers a_m and a_n are relatively prime.

As a third set for your puzzling pleasure, we give the 2nd Czeck-Polish-Slovak Mathematical Competition, June 17–18, 2002. Thanks again go to Bill Sands for obtaining these problems.

2nd CZECH-POLISH-SLOVAK MATHEMATICAL COMPETITION
Zwardoń, Poland
June 17–18, 2002

1. Let a and b be distinct real numbers, and let k and m be positive integers with $k + m = n \geq 3$, $k \leq 2m$, $m \leq 2k$. We consider sequences $x_1, \ldots, x_n$ with the following properties:

- k terms x_i are equal to a; in particular, $x_1 = a$;
- m terms x_i are equal to b; in particular, $x_n = b$;
- no three consecutive terms are equal.

Determine all possible values of the sum

$$x_n x_1 x_2 + x_1 x_2 x_3 + \cdots + x_{n-1} x_n x_1.$$

2. A given triangle ABC has area S and sidelengths $BC = a$, $CA = b$, and $AB = c$, where $a \leq b \leq c$. Determine the greatest number u and the least number v such that, for every point P inside triangle ABC, the inequality $u \leq PD + PE + PF \leq v$ holds, where D, E, F are the (respective) points of intersection of the rays AP, BP, CP with the opposite sides of the triangle. (The required values of u and v have to be expressed in terms of the given data a, b, c, S.)

3. Let n be a given positive integer, and let $S = \{1, 2, \ldots, n\}$. How many functions $f : S \to S$ are there such that $x + f^4(x) = n + 1$ for all $x \in S$? Note: The symbol f^4 denotes the fourth iterate: $f^4(x) = f(f(f(f(x))))$.

4. An integer $n > 1$ and a prime p are such that n divides $p - 1$, and p divides $n^3 - 1$. Show that $4p - 3$ is the square of an integer.

5. In an acute-angled triangle ABC with circumcentre O, points P and Q lying respectively on sides AC and BC are such that

$$\frac{AP}{PQ} = \frac{BC}{AB} \quad \text{and} \quad \frac{BQ}{PQ} = \frac{AC}{AB}.$$

Show that the points O, P, Q, and C are concyclic.

6. Let $n \geq 2$ be a fixed even integer. We consider polynomials of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$$

with real coefficients, having at least one real root. Determine the least possible value of the sum $a_1^2 + \cdots + a_{n-1}^2$.

Next we have a correction.

23. [2002 : 201–203; 2004 : 360] *St. Petersburg Contests 1965–1984*. The plane is divided into regions by n lines in general positions. Prove that at least $n - 2$ of the regions are triangles.

Correction by Pierre Bornsztein, Maisons-Laffitte, France.

My solution published in [2004 : 360] is totally erroneous. The 'non-destroying triangle' argument does not work at all. An excellent survey of this problem, its solution, and other erroneous attempts appeared in *Quantum*, March/April 2001, pp. 10–18.

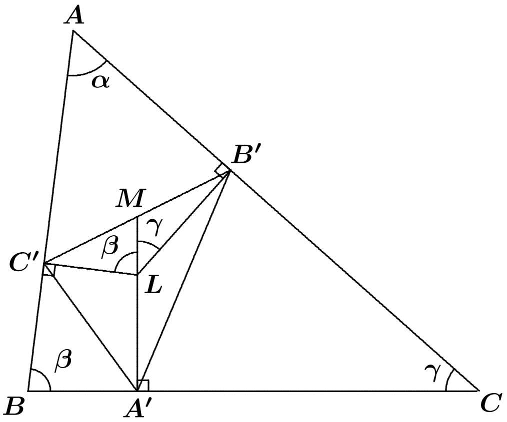
Now we look at solutions from the readers to problems of the 2000 Bulgarian Mathematical Olympiad given [2003 : 23–24].

2. Let ABC be an acute triangle.

(a) Prove that there exist unique points A', B', and C', on BC, CA, and AB, respectively, such that A' is the mid-point of the orthogonal projection of $B'C'$ onto BC, B' is the mid-point of the orthogonal projection of $C'A'$ onto CA, and C' is the mid-point of the orthogonal projection of $A'B'$ onto AB.

(b) Prove that $A'B'C'$ is similar to the triangle formed by the medians of ABC.

Solution by D.J. Smeenk, Zaltbommel, the Netherlands.



The points A', B', C' are the projections onto BC, CA, AB of the Lemoine Point (or symmedian point) L of $\triangle ABC$. We denote

$$\lambda = \frac{LA'}{a} = \frac{LB'}{b} = \frac{LC'}{c}.$$

(a) Let M be the point of intersection of $A'L$ and $B'C'$. Rectangle $BA'LC'$ is cyclic, which implies that $\angle C'LM = \beta$. Rectangle $CA'LB'$ is cyclic, implying that $\angle B'LM = \gamma$.

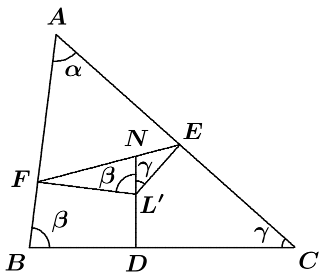
By the Sine Law, first in $\triangle B'C'L$ and then in $\triangle ABC$, we have

$$\frac{\sin\angle LC'B'}{\sin\angle LB'C'} = \frac{LB'}{LC'} = \frac{\lambda b}{\lambda c} = \frac{\sin\beta}{\sin\gamma} = \frac{\sin\angle C'LM}{\sin\angle B'LM}. \qquad (1)$$

Equation (1) is a necessary and sufficient condition for LM to be a median in $\triangle B'C'L$. It follows that A' is the mid-point of the orthogonal projection of $B'C'$ onto BC. By symmetry, analogous statements are true for B' and C'.

Now we will prove the uniqueness. Let L' be a point not on AL, and denote its projections onto BC, CA, and AB by D, E, and F respectively. Let DL' intersect EF at N.

Since $L'E : L'F \neq b : c$, equation (1) does not hold for $\triangle L'EF$, and N is not the mid-point of EF. Thus, A', B', C' are unique indeed.



(b) Let us denote the length of the median from A to BC by m_a. Applying the Law of Cosines, first in $\triangle B'C'L$ and then in $\triangle ABC$, we obtain

$$\begin{aligned}(B'C')^2 &= (LB')^2 + (LC')^2 - 2(LB')(LC')\cos(\beta+\gamma)\\ &= \lambda^2(b^2+c^2+2bc\cos\alpha)\\ &= 4\lambda^2\left(\tfrac{1}{2}(b^2+c^2) - \tfrac{1}{4}a^2\right)\\ &= 4\lambda^2 m_a^2\,;\end{aligned}$$

whence, $B'C' = \lambda m_a$. Similarly, $C'A' = \lambda m_b$ and $A'B' = \lambda m_c$, where m_b and m_c are the lengths of the medians (in $\triangle ABC$) to sides b and c, respectively. Therefore, $\triangle A'B'C'$ is similar to the triangle formed by the medians of $\triangle ABC$.

4. Find all polynomials $P(x)$ with real coefficients such that we have $P(x)P(x+1) = P(x^2)$ for all real x.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

If $P \equiv 0$ or $P \equiv 1$, then P is clearly a solution. These are the only constant polynomials which are solutions of the problem.

Now suppose that P is a non-constant polynomial which is a solution. Then $P(x)P(x+1) = P(x^2)$ for all real x. Hence, $P(z)P(z+1) = P(z^2)$ for all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be a root of P. Then, $P(z^2) = 0$ and $P\big((z-1)^2\big) = 0$.

Suppose that $0 < |z| < 1$. Define a sequence $\{z_n\}_{n=0}^{\infty}$ by $z_0 = z$ and $z_{n+1} = z_n^2$ for $n \geq 0$. Then, for each $n \geq 0$, we have $0 < |z_{n+1}| < |z_n| < 1$ and $P(z_n) = 0$. Thus, P has an infinite number of distinct roots, and then $P \equiv 0$, a contradiction.

Similarly, if $|z| > 1$, then $|z_{n+1}| > |z_n| > 1$ and $P(z_n) = 0$, which leads to the same contradiction.

Now suppose that $|z| = 1$ and $z \notin \{1, e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}}\}$. Let $z = e^{i\theta}$. Then $|(z-1)^2| = 2(1-\cos\theta) \in (0,1) \cup (1,4)$. Considering the sequence defined by $z_0 = (1-z)^2$ and $z_{n+1} = z_n^2$ for $n \geq 0$, we use an argument like the one above to get a contradiction.

Thus, the only possible roots of P are 0, 1, $e^{i\frac{\pi}{3}}$, and $e^{-i\frac{\pi}{3}}$. Since P has real coefficients, we have $P(e^{i\frac{\pi}{3}}) = 0$ if and only if $P(e^{-i\frac{\pi}{3}}) = 0$. If $e^{i\frac{\pi}{3}}$ is a root, then $P(e^{i\frac{\pi}{6}})P(e^{i\frac{\pi}{6}}+1) = P(e^{i\frac{\pi}{3}}) = 0$, and $e^{i\frac{\pi}{6}}$ or $e^{i\frac{\pi}{6}}+1$ is a root of P. Since these numbers do not belong to $\{0,\,1,\,e^{i\frac{\pi}{3}},\,e^{-i\frac{\pi}{3}}\}$, we have a contradiction.

Now the only possible roots are 0 and 1. Thus, $P(x) = ax^p(x-1)^q$, where a is a non-zero real number, and p and q are non-negative integers. Then $P(x)P(x+1) = P(x^2)$ is equivalent to

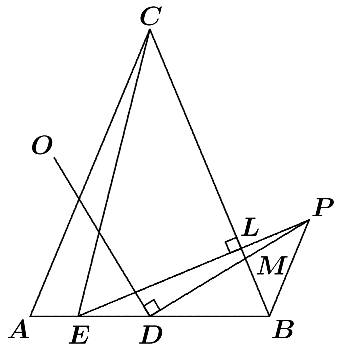
$$a^2x^{p+q}(x-1)^p(x+1)^q \;=\; ax^{2p}(x-1)^q(x+1)^q\,.$$

It follows that $a = 1$ and $p = q$. Thus, $P(x) = x^p(x-1)^p$. The integer p must now be positive, since P is supposed to be non-constant.

In conclusion, the solutions are $P(x) = 0$ and $P(x) = x^p(x-1)^p$, where p may be any non-negative integer.

5. In triangle ABC, we have $CA = CB$. Let D be the mid-point of AB and E an arbitrary point on AB. Let O be the circumcentre of $\triangle ACE$. Prove that the line through D perpendicular to DO, the line through E perpendicular to BC, and the line through B parallel to AC are concurrent.

Solution by Christopher J. Bradley, Bristol, UK.



Take rectangular Cartesian coordinates with $D(0,0)$, $A(-k,0)$, $B(k,0)$, $C(0,h)$, $E(t,0)$. The mid-point of AE is $\left(\frac{1}{2}(t-k),0\right)$; hence, the x–coordinate of O is $\frac{1}{2}(t-k)$.

The equation of AC is $ky - hx = hk$, and the mid-point of AC is $\left(-\frac{1}{2}k, \frac{1}{2}h\right)$. The perpendicular bisector of AC is then $hy + kx = \frac{1}{2}(h^2 - k^2)$. It follows that the coordinates of O are $\left(\frac{1}{2}(t-k), \frac{h^2-tk}{2h}\right)$. Therefore, the slope of OD is $\frac{h^2-tk}{h(t-k)}$.

The equation of the line through D perpendicular to OD is $(h^2-tk)y = h(k-t)x$. The line through B parallel to AC has equation $ky = h(x-k)$, and the line through E perpendicular to BC has equation $hy = k(x-t)$. It is easy to check that all three lines pass through the point P with coordinates

$$\left(\frac{k(h^2-kt)}{h^2-k^2}, \frac{hk(k-t)}{h^2-k^2}\right).$$

It appears that we must exclude the case when $h = \pm k$; that is, when $\angle ACB = 90^\circ$. However, in this case, the three lines of the problem are all perpendicular to BC and thus "meet at infinity".

6. Let $\mathcal{A}$ be the set of all binary sequences of length n, and let $0 \in \mathcal{A}$ be the sequence all terms of which are zeroes. The sequence $c = \langle c_1, c_2, \ldots, c_n\rangle$ is called the sum of $a = \langle a_1, a_2, \ldots, a_n\rangle$ and $b = \langle b_1, b_2, \ldots, b_n\rangle$ if $c_i = 0$ when $a_i = b_i$ and $c_i = 1$ when $a_i \neq b_i$. Let $f : \mathcal{A} \to \mathcal{A}$ be a function such that $f(0) = 0$ and if the sequences a and b differ in exactly k terms then the sequences $f(a)$ and $f(b)$ differ also exactly in k terms. Prove that if a, b, and c are sequences from $\mathcal{A}$ such that $a+b+c = 0$, then $f(a)+f(b)+f(c) = 0$.

Solved by Pierre Bornsztein, Maisons-Laffitte, France.

The set $\mathcal{A}$ is a vector space on $\mathbb{Z}/2\mathbb{Z}$ (the sum is the one defined in the statement, and let $0 \cdot a = 0$ and $1 \cdot a = a$) with dimension n and canonical basis $B_1 = (e_1, \ldots, e_n)$, where $e_1 = (1,0,\ldots,0)$, $e_2 = (0,1,0,\ldots,0)$, $\ldots$, $e_n = (0,\ldots,0,1)$. If $a = \langle a_1, a_2, \ldots, a_n\rangle$, then $a = \sum\limits_{i=1}^{n} a_i e_i$.

Let $d(a,b) = \sum\limits_{i=1}^{n} |a_i - b_i|$. Then $d(a,b)$ is simply the number of terms that differ in the sequences a and b. From the statement of the problem, $d\big(f(a), f(b)\big) = d(a,b)$. It follows trivially that f is injective.

Let $i \in \{1, \ldots, n\}$. Since $f(0) = 0$ and $d(0, e_i) = 1$, we deduce that $d\big(0, f(e_i)\big) = 1$. It follows that there exists $m \in \{1, \ldots, n\}$ such that $f(e_i) = e_m$. Let $f_i = f(e_i)$. Since f is injective, we deduce that

$B_2 = (f_1, \ldots, f_n)$ is a permutation of $(e_1, \ldots, e_n)$ and, therefore, is also a basis for $\mathcal{A}$.

Let $a = (a_1, a_2, \ldots, a_n)_{B_1}$ with $d(0, a) = k$. Then $d(0, f(a)) = k$. Let $i \in \{1, \ldots, n\}$.

- If $a_i = 1$, then $d(e_i, a) = k - 1$. Thus, $d\big(f_i, f(a)\big) = k - 1$; that is, the i^{th} coordinate of $f(a)$ in B_2 is equal to 1.
- If $a_i = 0$, then $d(e_i, a) = k$. Thus, $d\big(f_i, f(a)\big) = k$; that is, the i^{th} coordinate of $f(a)$ in B_2 is equal to 0.

It follows that, if $a = \sum\limits_{i=1}^{n} a_i e_i$, then $f(a) = \sum\limits_{i=1}^{n} a_i f_i = \sum\limits_{i=1}^{n} a_i f(e_i)$. Thus, f is linear.

Then, if $a + b + c = 0$, we have

$$f(a) + f(b) + f(c) \;=\; f(a + b + c) \;=\; f(0) \;=\; 0\,.$$

Next we turn to solutions from our readers to problems of the 2000 Belarusian Mathematical Olympiad given [2003 : 87–88].

1. Pete and Bill play the following game. At the beginning, Pete chooses a number a, then Bill chooses a number b, and then Pete chooses a number c. Can Pete choose his numbers in such a way that the three equations $x^3 + ax^2 + bx + c = 0$, $x^3 + bx^2 + cx + a = 0$, and $x^3 + cx^2 + ax + b = 0$ have a common

(a) real root?

(b) negative root?

Solved by Pierre Bornsztein, Maisons-Laffitte, France; Pavlos Maragoudakis, Lefkogia, Crete, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

(a) Yes. Pete simply chooses $a = -1$, and then, for any b chosen by Bill, Pete chooses $c = -b$. The three equations become $x^3 - x^2 + bx - b = 0$, $x^3 + bx^2 - bx - 1 = 0$, and $x^3 - bx^2 - x + b = 0$, which clearly have $x = 1$ as a common root.

(b) No. Suppose Bill chooses $b = 0$, and suppose r is a negative root common to all three equations. Then, in particular, we have

$$r^3 + cr + a \;=\; 0 \qquad (1)$$

$$\text{and} \quad r^3 + cr^2 + ar \;=\; 0\,. \qquad (2)$$

From (2) we get $r^2 + cr + a = 0$, or $cr + a = -r^2$. Substituting into (1), we then get $r^3 - r^2 = 0$, which yields $r = 0$ or 1, a contradiction.

2. How many pairs (n, q) satisfy $\{q^2\} = \left\{\frac{n!}{2000}\right\}$, where n is a positive integer and q is a non-integer rational number such that $0 < q < 2000$? [*Editor's comment:* $\{r\}$ means the "fractional part" of r.]

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We feature Bornsztein's solution, modified by the editor.

There are 2400 such pairs.

Let n be a positive integer and q a non-integer rational number such that $0 < q < 2000$. Let a and b be positive integers such that $q = a/b$ and $\gcd(a, b) = 1$. Then $b > 1$ (since q is not an integer). We say that (n, q) is a *good pair* if $\{q^2\} = \left\{\frac{n!}{2000}\right\}$. The problem is to find how many pairs (n, q) are good pairs.

For any real number x, let $\lfloor x \rfloor = x - \{x\}$ (the integer part of x). Then (n, q) is a good pair if and only if

$$\frac{a^2}{b^2} - \left\lfloor \frac{a^2}{b^2} \right\rfloor = \frac{n!}{2000} - \left\lfloor \frac{n!}{2000} \right\rfloor ; \tag{1}$$

that is,

$$2000a^2 = b^2 \left(2000 \lfloor q^2 \rfloor + n! - 2000 \left\lfloor \frac{n!}{2000} \right\rfloor \right) .$$

If this condition is satisfied, then, according to Gauss' Theorem, b^2 divides 2000, and it follows that $b \in \{2, 4, 5, 10, 20\}$.

Suppose that $b \neq 5$. Then $b = 2\tilde{b}$, where $\tilde{b} \in \{1, 2, 5, 10\}$, and a is odd, since $\gcd(a, b) = 1$. From (1), we have

$$a^2 = 4\tilde{b}^2 \lfloor q^2 \rfloor + \frac{n!\tilde{b}^2}{500} - 4\tilde{b}^2 \left\lfloor \frac{n!}{2000} \right\rfloor .$$

It follows that $\frac{n!\tilde{b}^2}{500}$ is an odd integer. This cannot be true if $n!$ is divisible by 8; therefore, $n \leq 3$. But then $\frac{n!\tilde{b}^2}{500}$ is not an integer, a contradiction.

Thus, we must have $b = 5$. Then a is not divisible by 5. From (1), we have

$$a^2 = 25 \lfloor q^2 \rfloor + \frac{n!}{80} - 25 \left\lfloor \frac{n!}{2000} \right\rfloor .$$

It follows that $\frac{n!}{80}$ is an integer not divisible by 5; that is, $n \in \{6, 7, 8, 9\}$.

We have reduced the problem to that of finding good pairs (n, q) of the form $\left(n, \frac{a}{5}\right)$, where $n \in \{6, 7, 8, 9\}$ and a is a positive integer not divisible by 5. Moreover, note that for any positive integer a,

$$\left\{ \left(\frac{25 \pm a}{5} \right)^2 \right\} = \left\{ 25 \pm 2a + \frac{a^2}{25} \right\} = \left\{ \left(\frac{a}{5} \right)^2 \right\} .$$

Therefore, if $\left(n, \frac{a}{5}\right)$ is a good pair, then $\left(n, \frac{25+a}{5}\right)$ is a good pair, and so is $\left(n, \frac{25-a}{5}\right)$ if $0 < a < 25$. Hence, we will be able to determine all good pairs $\left(n, \frac{a}{5}\right)$ by finding those pairs for which $1 \le a \le 12$ (and a is not divisible by 5).

Case 1. $n = 6$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{9}{25}$, and $\left(6, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{9}{25}$. It is easy to check that $a = 3$ is the only solution satisfying $1 \le a \le 12$. It follows that the good pairs of the form $(6, q)$ are all pairs of the form $\left(6, \frac{3+25k}{5}\right)$ and $\left(6, \frac{22+25k}{5}\right)$, for $k = 0, 1, \ldots, 399$. There are 800 such good pairs.

Case 2. $n = 7$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{13}{25}$, and $\left(7, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{13}{25}$. If this equation is satisfied, then

$$a^2 \;=\; 25\left\lfloor\frac{a^2}{25}\right\rfloor + 13 \;\equiv\; 3 \pmod{5}\ .$$

But a square is never congruent to $3 \pmod{5}$. Thus, there is no good pair in this case.

Case 3. $n = 8$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{4}{25}$, and $\left(8, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{4}{25}$. Here $a = 2$ is the only solution satisfying $1 \le a \le 12$. It follows that the good pairs of the form $(8, q)$ are all pairs of the form $\left(8, \frac{2+25k}{5}\right)$ and $\left(8, \frac{23+25k}{5}\right)$, for $k = 0, 1, \ldots, 399$. There are 800 such good pairs.

Case 4. $n = 9$.

Then $\left\{\frac{n!}{2000}\right\} = \frac{11}{25}$, and $\left(9, \frac{a}{5}\right)$ is a good pair if and only if $\left\{\frac{a^2}{25}\right\} = \frac{11}{25}$. Now $a = 6$ is the only solution satisfying $1 \le a \le 12$. It follows that the good pairs of the form $(9, q)$ are all pairs of the form $\left(9, \frac{6+25k}{5}\right)$ and $\left(9, \frac{19+25k}{5}\right)$, for $k = 0, 1, \ldots, 399$. There are 800 such good pairs, and we are done.

3. Given a fixed integer $N \ge 5$, and any sequence $e_1, e_2, \ldots, e_N$, where $e_i \in \{1, -1\}$ for $i = 1, 2, \ldots, N$, a move is made by choosing any five consecutive terms and changing their signs. Two such sequences are said to be similar if one of them can be obtained from the other in a finite number of moves. Find the maximal number of sequences no two of which are similar to each other.

Solved by Pierre Bornsztein, Maisons-Laffitte, France; and Pavlos Maragoudakis, Lefkogia, Crete, Greece. We give Bornsztein's solution.

The maximal number of sequences, no two of which are similar, is 16.

For $i \in \{1, 2, \ldots, N-4\}$, we define move number i to be the move that changes the signs of $e_i, \ldots, e_{i+4}$. A sequence of such moves will be represented by a sequence of move numbers, denoted by $U = (U_1, \ldots, U_p)$. The length of the sequence is $\ell(U) = p$. We also use the notation $\emptyset$ for the null sequence, which consists of no moves at all and has length $\ell(\emptyset) = 0$.

Consider an initial state $(e_1, \ldots, e_N)$ which is changed to a final state $(e'_1, \ldots, e'_N)$ by a finite sequence of moves $U = (U_1, \ldots, U_p)$. For each $i = 1, \ldots, N$, the value of e'_i differs from the value of e_i if and only if the parity of the total number of occurrences of $i-4, i-3, \ldots, i$ in the sequence U is odd. It follows that:

(a) the order of the moves has no importance;

(b) if the same move is used twice, then both applications of this move may be omitted without affecting the final state;

(c) we may suppose that U is ordered so that $U_1 < \cdots < U_p$.

We claim that whenever a sequence of moves changes an initial state $(e_1, \ldots, e_N)$ to a final state $(e'_1, \ldots, e'_N)$, there is a unique ordered sequence (as in (c) above) that produces this change.

To prove the claim, suppose that two different ordered sequences U and V produce the same change from some initial state $(e_1, \ldots, e_N)$ to a final state $(e'_1, \ldots, e'_N)$. Since U and V are different, they cannot both be null. We assume $U \neq \emptyset$.

Let $a = U_1$. Since $U_i > a$ for $i \geq 2$, the value of e_a is changed only by the first move in the sequence represented by U, while the value of e_i for any $i < a$ is not changed by any move in the sequence. Thus, $e'_a \neq e_a$ and $e'_i = e_i$ for $i < a$. In other words, $U_1 = a$ is the least index i such that $e'_i \neq e_i$. Since $e'_a \neq e_a$, the sequence V cannot be null. Applying to V the same argument that has just been applied to U, we deduce that $V_1 = a = U_1$.

Now we impose on the sequences U and V the additional condition that $\ell(U)$ is minimal. Let $p = \ell(U)$ and $q = \ell(V)$. Let $\tilde{U} = (U_1, \ldots, U_p, U_1)$ and $\tilde{V} = (V_1, \ldots, V_q, V_1)$. Then $\tilde{U}$ and $\tilde{V}$ have the same effect on the initial state $(e_1, \ldots, e_N)$ (because U and V have the same effect on this initial state and $U_1 = V_1$). Using (a) and (b), we get $\tilde{U} = (U_2, \ldots, U_p)$ and $\tilde{V} = (V_2, \ldots, V_p)$. Since $\ell(\tilde{U}) < \ell(U)$, the minimality of $\ell(U)$ implies that $\tilde{U} = \emptyset$ and, hence, $\tilde{V} = \emptyset$. Thus, $U = V$, which contradicts our hypothesis. Our claim has now been proved.

Let E denote the set of all states $(e_1, \ldots, e_N)$. Then $|E| = 2^N$. We define an equivalence relation $\sim$ on E by stating that two states are equivalent if and only if they are similar. The maximal number of sequences, no two of which are similar, is the number of equivalence classes with respect to this equivalence relation.

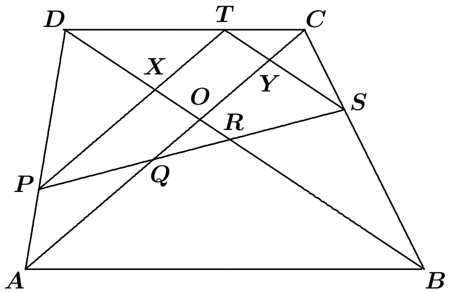
There are exactly 2^{N-4} ordered sequences of moves, including $\emptyset$ (because any such sequence either makes use of move number i or does not make use of it, for $i = 1, 2, \ldots, N-4$). By our uniqueness claim above, it follows that each equivalence class contains 2^{N-4} elements from E. Therefore, the number of classes is $\frac{2^N}{2^{N-4}} = 16$.

4. Let $ABCD$ be a quadrilateral with AB parallel to DC. A line ℓ intersects AD, AC, BD, and BC, forming three segments of equal lengths between consecutive points of intersection. Does it follow that ℓ is parallel to AB?

Solved by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL; and by Toshio Seimiya, Kawasaki, Japan. We give Seimiya's solution.

We cannot conclude that l is parallel to AB. Here is a counterexample.

Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $AB = 2CD$. Let P and S be interior points of the sides AD and BC, respectively, such that $\frac{AP}{PD} = \frac{CS}{SB} \neq 1$, which implies that PS is not parallel to AB. Let Q and R be the intersections of PS with AC and BD, respectively. We shall prove that $PQ = QR = RS$.



Let T be the point on the side CD such that

$$\frac{CT}{TD} = \frac{AP}{PD} = \frac{CS}{SB}.$$

Since $\frac{CT}{TD} = \frac{AP}{PD}$, we have $PT \parallel AC$. Since $\frac{CT}{TD} = \frac{CS}{SB}$, we have $TS \parallel DB$.

Let X and O be the intersections of BD with PT and AC, respectively, and let Y be the intersection of TS with AC.

Since $XR \parallel TS$, $PT \parallel AC$, and $AB \parallel CD$, we obtain

$$\frac{PR}{RS} = \frac{PX}{XT} = \frac{AO}{OC} = \frac{AB}{CD} = \frac{2}{1}; \quad \text{hence} \quad PR = 2RS. \qquad (1)$$

Since $QY \parallel PT$, $TS \parallel DB$, and $AB \parallel CD$, we have

$$\frac{PQ}{QS} = \frac{TY}{YS} = \frac{DO}{OB} = \frac{CD}{AB} = \frac{1}{2}; \quad \text{hence} \quad QS = 2PQ. \qquad (2)$$

It follows from (1) and (2) that $PQ = QR = RS$.

7. (a) Find all positive integers n such that $(a^a)^n = b^b$ has at least one solution in integers a and b, both exceeding 1.

(b) Find all positive integers a and b such that $(a^a)^5 = b^b$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

(a) All positive integers n except $n = 2$ are solutions.

If $n = 1$, just choose $a = b \geq 2$.

If $n \geq 3$, choose $a = (n-1)^{n-1}$ and $b = (n-1)^n$. Then $a, b \geq 2$, and

$$(a^a)^n = a^{an} = (n-1)^{n(n-1)^n} = \left((n-1)^n\right)^{(n-1)^n} = b^b .$$

Now suppose that there exist integers $a, b \geq 2$ such that

$$(a^a)^2 = b^b . \tag{1}$$

We cannot have $b \leq a$, because this gives $b^b \leq a^a < (a^a)^2$, since $a > 1$. We cannot have $2a \leq b$, because then $b^b \geq (2a)^{2a} = 2^{2a}(a^a)^2 > (a^a)^2$. Thus, we must have $a < b < 2a$. It follows that a does not divide b.

Let p be a prime number which divides a. Then, from (1), p divides b. Let α and β be the exponents of p in the prime decomposition of a and b, respectively. From (1), we have $2a\alpha = b\beta$. Then $\dfrac{\alpha}{\beta} = \dfrac{b}{2a} < 1$. Thus, $\alpha < \beta$. Since this is true for each prime p dividing a, it follows that a divides b, a contradiction.

We conclude that $n = 2$ is not a solution.

(b) Clearly, $(a,b) = (1,1)$ is a solution. Now suppose that $a > 1$ and b are positive integers such that

$$(a^a)^5 = b^b . \tag{2}$$

As in (a), where we proved that $a < b < 2a$, we now deduce that $a < b < 5a$.

Let p be a prime number which divides a. Then from (2), p divides b. Letting α and β be the exponents of p in the prime decomposition of a and b, respectively, we obtain $\alpha < \beta$, as in (a), from which we again deduce that a divides b. Then $b = ka$, where $k \in \{2, 3, 4\}$ (since $1 < a < b < 5a$). From (2), we have $a^{5a} = (ka)^{ka}$. Thus, $a^5 = (ka)^k$, which leads to $a^{5-k} = k^k$.

If $k = 2$, we must have $a^3 = 4$, which is impossible. If $k = 3$, then we need $a^2 = 27$, which is again impossible. If $k = 4$, our equation becomes $a = 4^4 = 256$, which leads to $b = 4^5 = 1024$. Conversely, we have seen in (a) that $(4^4, 4^5)$ is a solution of (2).

Thus, the solutions of (2) are $(1,1)$ and $(256, 1024)$.

Now we turn to solutions by our readers to problems of the 2000 Taiwanese Mathematical Olympiad given [2003 : 88].

1. Find all pairs (x, y) of positive integers such that $y^{x^2} = x^{y+2}$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Note that $(1, 1)$ and $(2, 2)$ are solutions. We prove that these are the only solutions.

Let x and y be positive integers such that

$$y^{x^2} = x^{y+2}. \qquad (1)$$

If either $x = 1$ or $y = 1$, then $x = y = 1$. Now assume that $x > 1$ and $y > 1$. From (1), we see that x and y have exactly the same prime divisors. Let $x = \prod p_i^{\alpha_i}$ and $y = \prod p_i^{\beta_i}$ be the prime decompositions of x and y. Then, from (1), for each j,

$$x^2\beta_j = (y+2)\alpha_j. \qquad (2)$$

Now suppose that one of the prime divisors of x and y, say p_i, is odd. Then $\gcd(p_i, y+2) = 1$. Since $p_i^{\alpha_i}$ divides x, we see that $p_i^{2\alpha_i}$ divides x^2. From (2), we deduce that $p_i^{2\alpha_i}$ divides α_j for each j. In particular, $p_i^{2\alpha_i}$ divides α_i. It follows that $9^{\alpha_i} < p_i^{2\alpha_i} \leq \alpha_i$. But an easy induction shows that $9^n > n$ for each positive integer n. We have arrived at a contradiction.

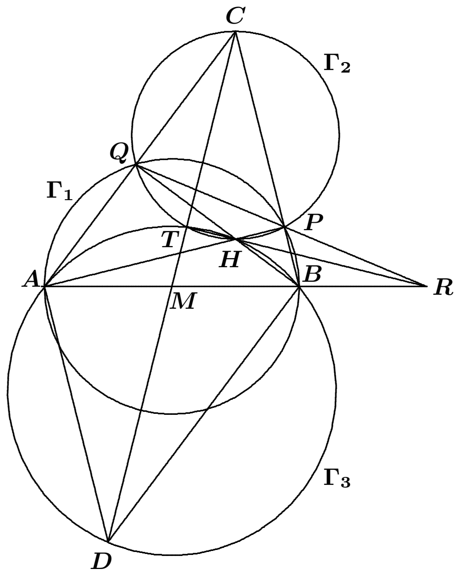
Thus, the unique prime divisor of x and y is 2. Let $x = 2^\alpha$ and $y = 2^\beta$, with α, $\beta \geq 1$. Equation (1) reduces to $2^{2\alpha-1}\beta = \alpha(2^{\beta-1}+1)$. If $\beta \geq 2$, then, since $2^{\beta-1}+1$ is odd, we see that $2^{2\alpha-1}$ divides α. But this is not possible, since $2^{2n-1} > n$ for each integer $n \geq 1$ (by an easy induction). Therefore, $\beta = 1$, and the equation reduces to $2^{2\alpha-2} = \alpha$. Since $2^{2n-2} > n$ for each integer $n \geq 2$ (by another easy induction), it follows that $\alpha = 1$. Thus, $x = y = 2$.

2. In an acute triangle ABC, $AC > BC$ and M is the mid-point of AB. Let AP be the altitude from A. Let BQ be the altitude from B meeting AP at H. Let the lines AB and PQ meet at R. Prove that the lines RH and CM are perpendicular to each other.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Lefkogia, Crete, Greece; Toshio Seimiya, Kawasaki, Japan; and Babis Stergiou, Chalkida, Greece. We first give Bataille's solution.

Since $\angle BQA = \angle BPA = 90^\circ$, the circle Γ with diameter AB passes through P and Q. Thus, the point H, as the intersection of the diagonals of $QPBA$, is on the polar of R with respect to Γ (since the sides QP and AB meet at R). Similarly, H is on the polar of C; whence, CR is the polar of H. Since M is the centre of Γ, it follows that CR is perpendicular to HM. Note that since H is the orthocentre of $\triangle ABC$, we also have $CH \perp RM$. As a result, H is the intersection of two altitudes in triangle CRM and, as such, H is the orthocentre of $\triangle CRM$. The result, $RH \perp CM$, follows.

We also give Seimiya's version.



Since $\angle APB = \angle AQB = 90^\circ$, the points A, B, P, and Q lie on a circle Γ_1. Let T be the foot of the perpendicular from H to CM. Since

$$\angle CPH = \angle CQH = \angle CTH = 90^\circ ,$$

we see that C, P, Q, T, and H all lie on a circle Γ_2.

On CM produced beyond M, let D be the point such that $CM = MD$. Then quadrilateral $CADB$ is a parallelogram. Since $BD \parallel CA$ and $BQ \perp AC$, we have $BQ \perp BD$. Similarly, $AD \perp AP$. Thus,

$$\angle HBD = \angle HAD = \angle HTD = 90^\circ .$$

Therefore A, D, B, H, and T all lie on a circle Γ_3.

Note that PQ is a common chord of Γ_1 and Γ_2, HT is a common chord of Γ_2 and Γ_3, and AB is a common chord of Γ_3 and Γ_1. It follows that PQ, HT, and AB are concurrent at R. Thus, T, H, and R are collinear. Therefore, $RH \perp CM$.

3. Let $S = \{1, 2, 3, \ldots, 100\}$, and let $\mathcal{P}$ denote the family of all subsets T of S with $|T| = 49$. For each set T in $\mathcal{P}$, we label it with a number chosen at random from $\{1, 2, \ldots, 100\}$. Prove that there exists a subset M of S with $|M| = 50$ such that for each $x \in M$, $M \setminus \{x\}$ is not labelled with x.

Solution by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON.

In general, for any positive integer n, let $S = \{1, 2, \ldots, 2n\}$, and let $\mathcal{P} = \{T \mid T \subset S, |T| = n - 1\}$. We label each set $T \in \mathcal{P}$ with a number $\ell(T)$ chosen at random from S.

Let $\mathcal{F} = \{M \mid M \subset S\,,\, |M| = n\}$. For each $T \in \mathcal{P}$, the set $T \cup \{\ell(T)\}$ is contained in $\mathcal{F}$ if and only if $\ell(T) \notin T$. The number of sets $M \in \mathcal{F}$ such that $M = T \cup \{\ell(T)\}$ for some $T \in \mathcal{P}$ is certainly no greater than the number of sets in $\mathcal{P}$, which is $\binom{2n}{n-1}$. On the other hand, the total number of sets in $\mathcal{F}$ is $\binom{2n}{n}$. Therefore, there must be at least $\binom{2n}{n} - \binom{2n}{n-1}$ sets $M \in \mathcal{F}$ such that $M \neq T \cup \{\ell(T)\}$ for any $T \in \mathcal{P}$. For any such M, if there is some $x \in M$ such that $M \setminus \{x\}$ is labelled with x, then $M = T \cup \{\ell(T)\}$ for some $T \in \mathcal{P}$, which is a contradiction.

We conclude that there are at least $\binom{2n}{n} - \binom{2n}{n-1}$ subsets M of S with $|M| = n$ such that for each $x \in M$, the set $M \setminus \{x\}$ is not labelled with x.

4. Let $\phi(k)$ denote the number of positive integers $n \leq k$ such that $\gcd(n,k) = 1$. Suppose that $\phi(5^m - 1) = 5^n - 1$ for some positive integers m and n. Prove that $\gcd(m,n) > 1$.

Solution by Pierre Bornsztein, Maisons-Laffitte, France.

Suppose, for the purpose of contradiction, that $\gcd(m,n) = 1$. Let $d = \gcd(5^m - 1, 5^n - 1)$. Then $d = 5^{\gcd(m,n)} - 1 = 4$. Now let the prime decomposition of $5^m - 1$ be $2^a p_1^{a_1} \ldots p_k^{a_k}$. Thus, $a \geq 2$ and

$$5^n - 1 \;=\; \varphi(5^m - 1) \;=\; 2^{a-1} p_1^{a_1 - 1} \cdots p_k^{a_k - 1} (p_1 - 1) \cdots (p_k - 1)\,.$$

Note that, if $5^m - 1$ is a power of 2, then $a > 3$ (otherwise, we do not have $\varphi(5^m - 1) \equiv 0 \pmod 4$), which leads to $5^n - 1 \equiv 0 \pmod 8$ and contradicts $d = 4$.

Thus, $5^m - 1$ has at least one odd prime divisor. Moreover, since $d = 4$, we must also have $a_i = 1$ for $i = 1, \ldots, k$.

Case 1. m is even.

Then $5^m - 1 \equiv 0 \pmod 8$, which implies that $a \geq 3$. But, as above, we then have $2^{a-1}(p_1 - 1) \equiv 0 \pmod 8$, which forces $5^n - 1 \equiv 0 \pmod 8$ and again contradicts $d = 4$.

Case 2. $m = 2k + 1$ is odd.

Thus, $a = 2$ and, from above, $5^m - 1 = 4p_1 \cdots p_k$. Obviously, none of the p_i is equal to 5. Hence, for each i, we have $5 \times 5^{2k} = 5^m \equiv 1 \pmod{p_i}$, from which we deduce that 5 is a quadratic residue $\pmod{p_i}$. From the Quadratic Reciprocity Law, it follows that p_i is a quadratic residue $\pmod 5$.

Since, for each integer x, we have $x^2 \equiv 0, 1$, or $-1 \pmod 5$, we deduce that $p_i \equiv 1$ or $-1 \pmod 5$ for each i.

Now assume that there exists i such that $p_i \equiv 1 \pmod 5$. Then 5 divides $p_i - 1$, which implies that 5 divides $5^n - 1$, which is absurd. Thus, $p_i \equiv -1 \pmod 5$ for each i, and $5^m - 1 \equiv 4(-1)^k \pmod 5$; whence, k is even.

Therefore, $5^n - 1 = 2(p_1 - 1)\cdots(p_k - 1) \equiv 2(-2)^k \pmod 5$, from which we see that $5^n - 1 \equiv 2$ or $-2 \pmod 5$, a contradiction.

We conclude that $\gcd(m, n) > 1$.

Comment. Bornsztein also points out that this is problem 10626 of the American Mathematical Monthly. A solution appears in the November 1999 issue of the Monthly, p. 869.

6. Let f be a function from the set of positive integers to the set of non-negative integers such that $f(1) = 0$ and

$$f(n) = \max\{f(j) + f(n-j) + j\}$$

for all $n \geq 2$. Determine $f(2000)$.

Solved by Michel Bataille, Rouen, France; and by Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's write-up.

In what follows, the maximum in the definition of $f(n)$ is considered over all j such that $1 \leq j \leq n-1$ (that is, all j for which $f(j)$ and $f(n-j)$ are defined). We will prove by induction that $f(n) = \dfrac{n(n-1)}{2}$ for all $n \geq 1$.

It is readily checked that $f(2) = f(1) + f(1) + 1 = 1$, that $f(3) = \max\{f(1) + f(2) + 1,\ f(2) + f(1) + 2\} = 3$, and that $f(4) = 6$. Assume that $n \geq 5$ and that $f(k) = \dfrac{k(k-1)}{2}$ for $1 \leq k < n$. Then

$$f(n-1) + f(1) + n - 1 = \frac{(n-2)(n-1)}{2} + 0 + n - 1 = \frac{n(n-1)}{2},$$

and for $1 \leq j \leq n-2$,

$$\begin{aligned} f(j) + f(n-j) + j &= \frac{j(j-1)}{2} + \frac{(n-j)(n-j-1)}{2} + j \\ &= \frac{n(n-1) - 2j(n-1-j)}{2} < \frac{n(n-1)}{2}. \end{aligned}$$

It follows that $f(n) = \max\{f(j) + f(n-j) + j\} = \dfrac{n(n-1)}{2}$. This concludes the induction.

Now, taking $n = 2000$ in the formula we have just proved, we find that $f(2000) = 1999000$.

Remark. Wang notes that $1 + f(n+1) = 1 + \binom{n+1}{2}$, which is well known to be the number of regions into which the plane is divided by n lines in general position (every pair of lines intersect and no three lines are concurrent).

That completes the *Corner* for this issue. Send me your nice solutions and generalizations as well as Olympiad contests!