SKOLIAD No. 85

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Please send your solutions to the problems in this edition by ***1 July, 2005***. A copy of **MATHEMATICAL MAYHEM Vol. 3** will be presented to one pre-university reader who sends in solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk (*) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our items this issue come from the 4th annual CNU Regional Mathematics Contest. Only the first 13 Questions have been included; the rest will appear in forthcoming Skoliads. Thanks go to R. Porsky, C.N.U., Newport News, VA.

4th Annual CNU Regional High School Mathematics Contest Saturday December 6, 2003

1. (*) If 64 is divided into three parts proportional to 2, 4, and 6, the smallest part is:

(A) $5\frac{1}{3}$ (B) 11 (C) $10\frac{2}{3}$ (D) none of these

2. (*) If, in applying the quadratic formula to a quadratic equation $f(x) = ax^2 + bx + c = 0$, it happens that $c = \frac{b^2}{4a}$, then the graph of $y = f(x)$ will certainly:

(A) have a maximum (B) have a minimum
(C) be tangent to the x–axis (D) be tangent to the y–axis

3. (*) Let $\{a_n\}$ be a geometric sequence. If $a_1 = 8$ and $a_7 = 5832$, then a_5 is:

(A) 648 (B) 832 (C) 1168 (D) 1944

4. (*) The area enclosed by $|x| + |y| = 1$ is:

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

5. ($*$) If the graph of $f(x) = \big||x-2|-a\big|-3$ has exactly three x–intercepts, then a equals:

(A) 3 (B) 4 (C) 0 (D) -3

6. ($*$) If $\frac{m}{n} = \frac{4}{3}$ and $\frac{r}{t} = \frac{9}{14}$, then the value of $\dfrac{3mr-nt}{4nt-7mr}$ is:

(A) $-5\frac{1}{2}$ (B) $-\frac{11}{14}$ (C) $-\frac{2}{3}$ (D) $-1\frac{1}{4}$

7. ($*$) Which functions satisfy $f\left(\dfrac{x+y}{2}\right) = \frac{1}{2}(f(x)+f(y))$?

(A) $\ln x$ (B) $\dfrac{1}{x}$ (C) $2x$ (D) 2^x

8. ($*$) Let x, $y > 0$, $x > y$, and $z \neq 0$. The inequality which is not always correct is:

(A) $x+z > y+z$ (B) $x-z > y-z$
(C) $xz > yz$ (D) $xz^2 > yz^2$

9. ($*$) If $a^x = c^q = b$ and $c^y = a^z = d$, then:

(A) $xy = qz$ (B) $x+y = q+z$ (C) $x-y = q-z$ (D) $x^y = q^z$

10. ($*$) The area of a square inscribed in a semicircle is to the area of the square inscribed in the entire circle as:

(A) $1:2$ (B) $2:3$ (C) $2:5$ (D) $3:4$

11. ($*$) If $0 < \alpha, \beta < \frac{\pi}{2}$ and $\alpha > \beta$, then:

(A) $\sin(\alpha-\beta) > \sin\alpha - \sin\beta$ (B) $\sin(\alpha-\beta) < \sin\alpha - \sin\beta$
(C) $\sin(\alpha-\beta) = \sin\alpha - \sin\beta$ (D) none of these

12. ($*$) Let $f(x) = 3^x + 5$. Then the domain of f^{-1} is:

(A) $(0,+\infty)$ (B) $(5,+\infty)$ (C) $(8,+\infty)$ (D) $(-\infty,+\infty)$

13. ($*$) A man has a pocket full of change, but can not make change for a dollar. What is the greatest value of coins he could have?

(A) $\$0.99$ (B) $\$1.09$ (C) $\$1.19$ (D) $\$1.29$

4^e Concours Annuel CNU Régional de Mathématique du Secondaire Samedi, le 6 Décembre 2003

1. $(*)$ Si 64 est divisé en 3 parties proportionnelles à 2, 4 et 6, la plus petite de ces parties est :

(A) $5\frac{1}{3}$ (B) 11 (C) $10\frac{2}{3}$ (D) aucune d'elles

2. $(*)$ Si, en appliquant la formule quadratique à l'équation $f(x) = ax^2 + bx + c = 0$, on obtient $c = \dfrac{b^2}{4a}$, donc le graphique de $y = f(x)$ va certainement :

(A) avoir un maximum (B) avoir un minimum
(C) être tangent a l'axe des x (D) être tangent a l'axe des y

3. $(*)$ Soit $\{a_n\}$ une suite géometrique. Si $a_1 = 8$ et $a_7 = 5832$, alors a_5 vaut :

(A) 648 (B) 832 (C) 1168 (D) 1944

4. $(*)$ L'aire comprise dans $|x| + |y| = 1$ est :

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

5. $(*)$ Si le graphique de $f(x) = \big||x-2| - a\big| - 3$ a 3 zéros exactement, alors a vaut :

(A) 3 (B) 4 (C) 0 (D) -3

6. $(*)$ Si $\dfrac{m}{n} = \dfrac{4}{3}$ et $\dfrac{r}{t} = \dfrac{9}{14}$, alors la valeur de $\dfrac{3mr - nt}{4nt - 7mr}$ est :

(A) $-5\frac{1}{2}$ (B) $-\frac{11}{14}$ (C) $-\frac{2}{3}$ (D) $-1\frac{1}{4}$

7. $(*)$ Quelles fonctions satisfont $f\left(\dfrac{x+y}{2}\right) = \dfrac{1}{2}(f(x) + f(y))$?

(A) $\ln x$ (B) $\dfrac{1}{x}$ (C) $2x$ (D) 2^x

8. $(*)$ Soit x, $y > 0$, $x > y$ et $z \neq 0$. L'inégalité qui n'est pas toujours correcte est :

(A) $x + z > y + z$ (B) $x - z > y - z$
(C) $xz > yz$ (D) $xz^2 > yz^2$

9. (∗) Si $a^x = c^q = b$ et $c^y = a^z = d$, alors :

(A) $xy = qz$ (B) $x + y = q + z$ (C) $x - y = q - z$ (D) $x^y = q^z$

10. (∗) L'aire d'un carré inscrit dans un demi-cercle est à l'aire du carré inscrit dans le cercle en proportion :

(A) $1:2$ (B) $2:3$ (C) $2:5$ (D) $3:4$

11. (∗) Si $0 < \alpha, \beta < \frac{\pi}{2}$ et $\alpha > \beta$, alors :

(A) $\sin(\alpha - \beta) > \sin\alpha - \sin\beta$ (B) $\sin(\alpha - \beta) < \sin\alpha - \sin\beta$
(C) $\sin(\alpha - \beta) = \sin\alpha - \sin\beta$ (D) aucune d'elles

12. (∗) Soit $f(x) = 3^x + 5$. Alors le domaine de f^{-1} est :

(A) $(0, +\infty)$ (B) $(5, +\infty)$ (C) $(8, +\infty)$ (D) $(-\infty, +\infty)$

13. (∗) Un homme a une poche pleine de change, mais ne peut pas faire de change pour un dollar. Quelle est la plus grande valeur possible de son change ?

(A) $0,99\$$ (B) $1,09\$$ (C) $1,19\$$ (D) $1,29\$$

Next we give the solutions to the 2004 BC Colleges High School Mathematics Contest, Final Round – Part B, for both Junior and Senior levels [2004 : 385–387].

2004 BC Colleges High School Mathematics Contest
Junior Final Round – Part B

1. The numbers greater than 1 are arranged in an array, in which the columns are numbered 1 to 5 from left to right, as shown:

(1)	(2)	(3)	(4)	(5)
	2	3	4	5
9	8	7	6	
	10	11	12	13
17	16	15	14	
⋮	⋮	⋮	⋮	⋮

(a) In which column will 2004 fall?

(b) In which column will 1999 fall?

(c) In which column(s) could $n^2 + 1$ fall, where n is a positive integer?

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Modulo 8, we see that residue 1 belongs to column (1); residues 0 and 2 belong to (2); 3 and 7 belong to (3); 4 and 6 belong to (4); and 5 belongs to (5). Now $2004 \equiv 4 \pmod{8}$ and $1999 \equiv 7 \pmod{8}$. Thus, 2004 and 1999 will fall in columns (4) and (3), respectively. Now n^2 is congruent modulo 8 to one of 0, 1, or 4. Hence, $n^2 + 1$ could appear only in columns (1), (2), or (5).

2. How many sets of two or more consecutive positive integers have a sum of 105?

Official solution.

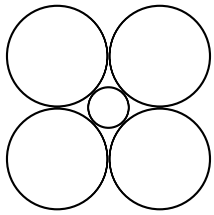
If we have an odd number of consecutive integers with a sum of 105, such as $34 + 35 + 36 = 105$, we see that the middle integer is their average: $\frac{34+35+36}{3} = \frac{105}{3} = 35$. This means that the sum (105) divided by the odd number (of consecutive integers) must be an integer. Since $105 = 3 \times 5 \times 7$, the divisors of 105 are 1, 3, 5, 7, 15, 21, 35, and 105 (all of which are odd). We eliminate 1 because we need at least two consecutive integers. On the other hand, if we had 15 consecutive integers, then, since $\frac{105}{15} = 7$, the 15 consecutive integers would have to be 0, 1, ..., 14. But not all of these are positive. In this way we eliminate the cases of 15, 21, 35, and 105 consecutive integers, leaving 3, 5, and 7.

If we have the sum of an even number of consecutive integers, such as $52 + 53 = 105$, we see that the average is midway between the middle two terms. In other words, the average of the integers must be an integer plus $\frac{1}{2}$. Consequently, 105 divided by half the number of terms must be an integer. We have the same divisors as before, namely 1, 3, 5, 7, 15, 21, 35, and 105. Thus, we could have 2, 6, 10, 14, 30, 42, 70, or 210 consecutive integers adding up to 105. The requirement that all the integers be positive leads us to eliminate anything greater than 14.

Therefore, in total there are seven ways to write 105 as a sum of at least two consecutive, positive integers:

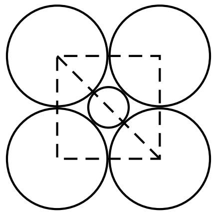
number	average	sum
2	52.5	$52 + 53$
3	35	$34 + 35 + 36$
5	21	$19 + 20 + 21 + 22 + 23$
6	17.5	$15 + 16 + 17 + 18 + 19 + 20$
7	15	$12 + 13 + 14 + 15 + 16 + 17 + 18$
10	10.5	$6 + 7 + \cdots + 10 + 11 + \cdots + 14 + 15$
14	7.5	$1 + 2 + \cdots + 6 + 7 + 8 + 9 + \cdots + 13 + 14$

3. The centres of four circles of radius 12 form a square. Each circle is tangent to the two circles whose centres are the vertices of the square that are adjacent to the centre of the circle. A smaller circle, with centre at the intersection of the diagonals of the square, is tangent to each of the four larger circles. Find the radius of the smaller circle.



Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Clearly, the main diagonal of the square formed by the centres of the four circles of radius 12 has length $24\sqrt{2}$. But it is also twice the radius of the small circle plus twice the radius of a large circle, because the points of tangency of the circles must be on the main diagonal (by symmetry). Thus, $r = 12(\sqrt{2} - 1)$ immediately.



4. The Fibonacci Sequence begins: 1, 1, 2, 3, 5, 8, 13, 21, ... (Each number beyond the second number is the sum of the previous two numbers.) The notation f_n means the n^{th} number; for example, $f_4 = 3$ and $f_7 = 13$.

(a) Which of the following terms in the Fibonacci Sequence are odd? Explain your conclusions.

$$f_{38}\,,\quad f_{51}\,,\quad f_{150}\,,\quad f_{200}\,,\quad f_{300}$$

(b) Which of the following terms in the Fibonacci Sequence are divisible by 3? Explain your conclusions.

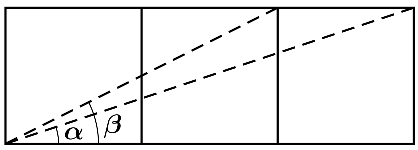
$$f_{48}\,,\quad f_{75}\,,\quad f_{196}\,,\quad f_{379}\,,\quad f_{1000}$$

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

(a) Modulo 2, the Fibonnaci Sequence is 1, 1, 0, 1, 1, 0, $\overline{1, 1, 0}$, ... Clearly, this pattern goes on forever, since each residue is determined by the two residues before it. Hence, every 3rd term is even, namely f_{51}, f_{150}, and f_{300} (and the rest are odd, f_{38} and f_{200}).

(b) Modulo 3, the sequence is $\overline{1, 1, 2, 0, 2, 2, 1, 0}$, ... Thus, every 4th term is divisible by 3, namely f_{48}, f_{196}, f_{1000}.

5. The diagram shows three squares. Find the measure of the angle $\alpha + \beta$.



I. *First solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

We have $\tan\alpha = 1/3$ and $\tan\beta = 1/2$. Thus,

$$\tan(\alpha+\beta) \;=\; \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \;=\; \frac{1/3 + 1/2}{1 - 1/6} \;=\; \frac{5/6}{5/6} = 1\,.$$

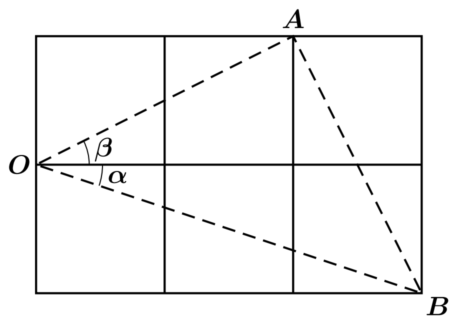
Since α and β belong to $[0, \pi/2]$, and since $\tan$ is a single-valued function on the interval $[0, \pi]$ (excluding $\pi/2$, of course), it follows that $\alpha + \beta$ is $\pi/4$.

II. *Second solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Let $O = (0,0)$, $A = (2,1)$, and $B = (3,-1)$. [*Ed:* See the diagram in the official solution below.] One can see that $AO = AB$, because both are diagonals of a 2×1 rectangle. A rotation of 90° clockwise centred at A maps B to O. Thus, $\angle OAB = 90^\circ$. Now, $\angle AOB$ must be 45° (since $\triangle AOB$ is isoceles); that is, $\alpha + \beta = 45^\circ$.

III. *Official solution.*

If we flip the squares as shown in the diagram, we get a triangle with $\alpha + \beta = \angle AOB$. We can use the Pythagorean Theorem to calculate $OA = AB = \sqrt{5}$ and $OB = \sqrt{10}$. Then $OB^2 = OA^2 + AB^2$, implying that $\triangle AOB$ is a right triangle with $\angle OAB = 90^\circ$. Since the triangle is also isosceles, we see that



$$\angle AOB \;=\; \alpha + \beta \;=\; 45^\circ\,.$$

2004 BC Colleges High School Mathematics Contest
Senior Final Round – Part B

1. Find the number of different 7–digit numbers that can be made by rearranging the digits in the number 3053345.

Official solution.

Suppose that we give the three 3s and two 5s different colours so that we can distinguish them. Since a valid number cannot start with a 0, there are 6(6!) ways to arrange the seven digits into a valid number. In each of these, there are 3! ways to arrange the three 3s and 2! ways to arrange the two 5s. Since the 3s and 5s are indistinguishable without the colours, the number of 7–digit numbers that can be made is $\frac{6(6!)}{3!2!} = 6 \cdot 5 \cdot 4 \cdot 3 = 360$.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON. There was one incorrect solution submitted.

2. The number 2004 has only 12 integer factors, including 1 and 2004.

(a) How many distinct factors does 2004^4 have?

(b) If the product of the factors in part (a) is written as 2004^N, find the value of N?

Solution to part (a) *by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.*

Since $2004 = 167 \cdot 3 \cdot 2^2$, we see that $2004^4 = 167^4 \cdot 3^4 \cdot 2^8$. Thus, 2004^4 has $(4+1)(4+1)(8+1) = 225$ distinct factors.

Official solution to part (b).

The product of the 225 factors from part (a) will be of the form $2^\alpha \times 3^\beta \times 167^\gamma$, for some positive integers α, β, and γ. We need to find these integers.

Each factor of 2004^4 will include exactly one of 2^0, 2^1, $\ldots$, 2^8. Multiplying these together gives $2^{(0+1+2+\cdots+8)} = 2^{36}$. Each of these possible factors occurs once for each possible combination of the powers on 3 and 167. Since there are 5 possible powers for both 5 and 167, this gives 25 combinations. Thus, the product of the factors of 2004^4 contains $2^{(36\times 25)}$; that is, $\alpha = 36 \times 25 = 900$.

Similarly, multiplying the possible powers of 3 gives $3^{(0+1+\cdots+4)} = 3^{10}$ and, with the 9 possible powers for 2 and the 5 possible for 167, there are 45 possible combinations giving this factor. Hence, 2004^4 contains $3^{(10\times 45)}$; that is, $\beta = 10 \times 45 = 450$. The same argument shows that $\gamma = 450$. Thus, the product of all of the factors of 2004^4 is $2^{900} \times 3^{45} \times 167^{450} = 2004^{450}$.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON. Part (b) was also solved by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

3. You are given two parallel panes of glass. Each pane will transmit 70%, reflect 20%, and absorb 10% of the light that falls on it. For example, for the portion of a beam of light incident on the pane on the left that follows the path in Figure 1, the fraction transmitted is $0.7 \times 0.7 = 0.49$, but for the portion of the beam following the path shown in Figure 2, the fraction transmitted is $0.7 \times 0.2 \times 0.2 \times 0.7 = 0.0196$. If a light source is placed on one side of the two panes, find the total fraction of light that passes through to the other side.

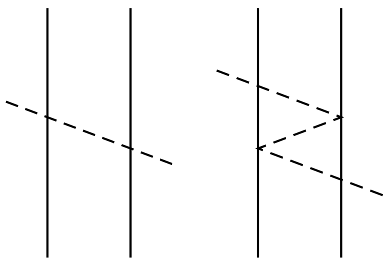


Figure 1 Figure 2

Official solution.

If light is seen through the two panes of glass, it will have been either unreflected, or reflected twice, four times, six times, etc. Since 70% of the light is transmitted through each pane and 20% is reflected, the total fraction of light that passes through to the other side is

$$0.7 \times 0.7 + 0.7 \times 0.2^2 \times 0.7 + 0.7 \times 0.2^4 \times 0.7 + \cdots$$
$$= 0.7^2 \times (1 + 0.2^2 + 0.2^4 + \cdots) = 0.49 \times \frac{1}{1 - 0.04} = \frac{49}{96}.$$

There was one incorrect solution submitted.

4. Let f be a function whose domain is all real numbers. If

$$f(x) + 2f\left(\frac{x + 2001}{x - 1}\right) = 4013 - x$$

for all x not equal to 1, find the value of $f(2003)$.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

Consider $x = 2$ and $x = 2003$ in the given equation. We now have

$$\begin{aligned} f(2) + 2f(2003) &= 4011\,, \\ f(2003) + 2f(2) &= 2010\,. \end{aligned}$$

Solving this system yields $f(2) = 3$ and $f(2003) = 2004$.

Also solved by Alan Guo, grade 10 student, O'Neill Collegiate and Vocational Institute, Oshawa, ON.

5. An ant is crawling at a rate of 48 centimetres per minute along a strip of rubber which can be infinitely and uniformly stretched. The strip is initially one metre long and one centimetre wide and is stretched an additional one metre at the end of each minute. Assume that when the strip is stretched, the ratio of the distances from each end of the strip remains the same before and after the stretch. If the ant starts at one end of the strip of rubber, find the number of minutes until it reaches the other end.

Solution by Alex Wice, grade 11 student, Leaside High School, Toronto, ON.

At the end of the first minute, the ant has traveled 48 percent of the way. At the end of the second minute the ant has travelled $48 + 48/2$ percent of the way. After the 3^{rd} minute, $48(1 + 1/2 + 1/3)$ percent; after the 4^{th} minute, $48(1 + 1/2 + 1/3 + 1/4)$ percent, which is exactly 100 percent. Clearly, the function mapping minutes to percentage is strictly increasing. Thus, after only 4 minutes, the ant reaches the end.

That brings us to the end of another issue. This months winner of a past volume of Mayhem is Alex Wice. Congratulations, Alex! Continue sending in your contests and solutions.