Pólya's Paragon

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Last September, I began this column with a short discussion about how mathematics is beautiful and how this beauty can arise from the variety of ways of proving mathematical statements. I will start this new school year with a cute proof that not only are numbers beautiful, but they are also interesting!

Theorem 1. All positive integers are interesting.

Proof. First, we suppose for a contradiction that there exists at least one uninteresting positive integer. Then, among all such integers, one of them must be the smallest. Now this integer is very interesting indeed, since it is the smallest one with no other interesting properties! This contradicts our assumption. We must conclude that all positive integers are interesting. □

We have not really proved anything here, because the term 'interesting' has not been defined. Nevertheless, this is a wonderful example of how a "proof by contradiction" works. Basically, you assume the negation of what you are trying to prove and show that this assumption leads to a contradiction. This shows that the assumption is false, which then implies that the original statement is true. More generally, this is known as an "indirect proof".

The next theorem I would like to consider involves a number which I find quite interesting. The proof is one of the most famous examples of proof by contradiction.

Theorem. $\sqrt{2}$ is irrational.

Proof. Assume for a contradiction that $\sqrt{2}$ is rational. Then, we can write $\sqrt{2}$ as a/b where a, b are integers with no common divisor greater than 1 and $b \neq 0$ (this is the definition of a rational number). Now we do a little algebra:

$$\begin{aligned} \sqrt{2} &= \frac{a}{b} \\ 2 &= \frac{a^2}{b^2} \\ 2b^2 &= a^2 \end{aligned}$$

This shows that a^2 is even. Now, since an odd number squared is again odd, we must conclude that a is even; that is, $a = 2n$ for some other integer n. Substituting for a above, we get the following:

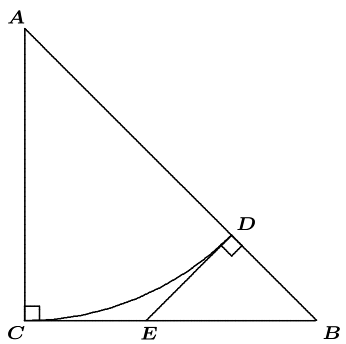
$$\begin{aligned} 2b^2 &= (2n)^2 \\ 2b^2 &= 4n^2 \\ b^2 &= 2n^2 \end{aligned}$$

This means that b^2 is also an even number which, by the same argument as above, shows that b is even. We have now shown that a and b are both even and hence, both divisible by 2. This contradicts the assumption that a and b have no common divisor greater than 1. Therefore, without any further work we must conclude that $\sqrt{2}$ is not a rational number. This, of course, implies that it must be irrational, as required. □

The fact that $\sqrt{2}$ is irrational was known to the Greeks many years before mathematicians used variables to represent unknown quantities. We might wonder how they ever went about proving such a result without expressing the results algebraically. Here is an alternate geometric proof that does not need variables. It is again a proof by contradiction.

Proof. We know by the Pythagorean Theorem that we can construct a right angled triangle with side lengths 1, 1 and $\sqrt{2}$. If we assume that $\sqrt{2}$ is rational, then we can find a similar triangle that has all integer side lengths (you can multiply all the sides by the denominator b if $\sqrt{2} = a/b$). In particular, we choose the smallest such triangle ABC with integer lengths which is similar to the original.

Now we make the following constructions: Draw a circle centred at A with radius equal to one of the shorter sides. This intersects the hypotenuse at D. Construct a line from D perpendicular to the hypotenuse which meets BC at E.



We need only make arguments about which sides in the diagram must have integer lengths. First, $AD = AC$; thus, AD is an integer. Then we know that $DB = AB - AD$ is also an integer, since it is the difference of two integers. Since $\angle DBE = 45°$ and $\angle EDB = 90°$, we know $\angle DEB = 45°$, which means that $DB = DE$. Therefore, DE has integral length as well. Also, DE and CE are tangents to the circle centred at A; hence, they are the same length. Thus, CE has integral length. Finally, we can now see that $EB = CB - CE$ is an integer, since its length can be expressed as the difference of two integers. Hence, $\triangle BDE$ is similar to $\triangle ACB$. We have shown that $\triangle BDE$ has integer side lengths. This contradicts our assumption that $\triangle ABC$ is the smallest such triangle. We must therefore conclude that $\sqrt{2}$ is irrational. $\square$

Clearly, this is not quite as elegant as the first proof. Nevertheless, it is nice to know that such things can be proven without the algebraic tools that we have come to depend on.

Problems for you to try:

There are many other problems that can be solved using an indirect proof. Here are three more problems whose proofs lend themselves to this technique.

1. Prove that if r is an irrational number, and if a and b are rational numbers with $b \neq 0$, then the expression $a + br$ represents an irrational number.

2. Prove that there are infinitely many primes. This is another famous use of indirect proof. [*Hint:* Begin by assuming that there are only finitely many. What can you say about the number you get when you multiply them all together and add 1?]

3. Prove that e is an irrational number. (Note: this is quite hard. You may use the fact that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.)