Mayhem Solutions

M51. *Proposed by the Mayhem Staff.*

You have a deck with cards numbered 1 through 25. You perform the following operations on the deck:

- you place the top card on the bottom of the deck.
- you place the new top card on the bottom of the deck.
- you flip the new top card face up on the table.

You continue this process until all cards are face up on the table. Find the order of the cards in the deck if, when the process is performed, the cards get laid out on the table in the order $1, 2, 3, \ldots, 25$.

Solution by Robert Bilinski, Outremont, QC.

First, let a, b, c, d, e, f, $\ldots$, y, be the sequence of 25 cards, in their original positions. In the following table, we put the first few steps of the stated algorithm.

Step	Undistributed cards	Found sequence
0	$a, b, c, d, e, f, \ldots, y$	
1	$d, e, f, g, \ldots, y, a, b$	c
2	$g, h, \ldots, y, a, b, d, e$	c, f

We thus obtain the following sequence:

$$c,\ f,\ i,\ l,\ o,\ r,\ u,\ x,\ b,\ g,\ k,\ p,\ t,\ y,\ e,\ m,\ s,\ a,\ j,\ v,\ h,\ w,\ q,\ d,\ n,$$

which equals $1, 2, 3, \ldots, 25$.

Thus, the original order of the cards was:

$$18,\ 9,\ 1,\ 24,\ 15,\ 2,\ 10,\ 21,\ 3,\ 19,$$
$$11,\ 4,\ 16,\ 25,\ 5,\ 12,\ 23,\ 6,\ 17,\ 13,\ 7,\ 20,\ 22,\ 8,\ 14.$$

Also solved by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

M52. *Proposé par J. Walter Lynch, Athens, GA, USA.*

On a deux pièces de monnaie. L'une est une pièce d'un dollar normale et l'autre une fausse pièce d'un dollar, avec deux faces. On jette au hasard chacune des pièces dans deux tiroirs différents. Quelqu'un entre dans la chambre et ouvre un des tiroirs et aperçoît une pièce, côté face. Quelle est la probabilité que cette pièce soit celle à deux faces ?

Solution de Robert Bilinski, Outremont, QC.

Si on regarde les 2 pièces de monnaie, il y a au total 3 faces. Sur ces trois faces, il y en a deux qui appartiennent à la pièce truquée. Ainsi, par la définition de probabilité, on a :

$$P(\text{pièce truquée}) = \frac{\#\text{ cas favorable}}{\#\text{ total}} = \frac{2}{3}.$$

Solutioné aussi par Alexandre Ortan, écolier, École Joseph-François-Perrault, Montréal, QC.

M53. *Proposed by the Mayhem Staff.*

A circular path is surrounded by 17 stepping stones numbered 0, 1, 2, ..., 16. Sally starts on stone 0 and moves 1 step to stone 1, then 4 steps to stone 5, then 9 steps to step 14 and continues in the following pattern until at last she moves 2002^2 steps and stops (to rest). What stone is Sally standing on while she rests?

Solution by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

Let us first find how many steps Sally has made:

$$1^2 + 2^2 + 3^2 + \cdots + 2002^2 = \frac{2002 \times 2003 \times 4005}{6} = 2\,676\,679\,005$$

steps. Since her walk is cyclical, the residue of the number of steps on division by 17 will give us the position where Sally ends her walk.

Since $2\,676\,679\,005 = 17 \times 157\,451\,706 + 3$, Sophie ended up on the third step.

Also solved by Robert Bilinski, Outremont, QC.

M54. *Proposed by Gary Tupper, Pedagoguery Software Inc., Terrace, BC.*

An ellipse with major axis AB and foci F and F' is inscribed in a circle with diameter AB and centre C. P is a point on the ellipse and D is a point on the circle so that radius CD bisects FP. Show that line DP is tangent to the ellipse.

Solution by Andrei Ismail, student, 11^{th} Grade, Vasile Alecsandri National College, Galati, Romania.

Let the ellipse have equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then its foci are

$$F(\sqrt{a^2 - b^2}, 0) \qquad \text{and} \qquad F'(-\sqrt{a^2 - b^2}, 0).$$

Because point P is located on the ellipse, its coordinates can be expressed as $P(a\cos t, b\sin t)$. Also, the points A and B have coordinates $A(-a, 0)$ and $B(a, 0)$. The circle with diameter AB has the equation $x^2 + y^2 = a^2$. Let us consider M, the mid-point of FP. It has coordinates

$$M\left(\frac{a\cos t + \sqrt{a^2 - b^2}}{2}, \frac{b\sin t}{2}\right).$$

The condition that CD bisects FP is equivalent to the statement that D is the intersection of the line CM with the circle $x^2 + y^2 = a^2$. Now, it is easy to see that D has coordinates

$$D\left(a \cdot \frac{a\cos t + \sqrt{a^2-b^2}}{\sqrt{(a\cos t+\sqrt{a^2-b^2})^2+(b\sin t)^2}},\ a \cdot \frac{b\sin t}{\sqrt{(a\cos t+\sqrt{a^2-b^2})^2+(b\sin t)^2}}\right).$$

Without loss of generality, we assume that P is in the half-plane $y > 0$ (and therefore, so is D). Simplifying the denominators yields

$$D\left(a \cdot \frac{a\cos t + \sqrt{a^2-b^2}}{A},\ a \cdot \frac{b\sin t}{A}\right),$$

where $A = a + \cos t\sqrt{a^2-b^2}$.

All we have to show now is that the line DP is tangent to the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is well-known that the slope of the tangent to the ellipse at the point $P(a\cos t, b\sin t)$ is $-\frac{b\cos t}{a\sin t}$. Thus, it will suffice to show that:

$$\frac{a \cdot \dfrac{b\sin t}{A} - b\sin t}{a \cdot \dfrac{a\cos t + \sqrt{a^2-b^2}}{A} - a\cos t} = -\frac{b\cos t}{a\sin t},$$

or, equivalently,

$$\frac{\dfrac{a\cos t + \sqrt{a^2-b^2}}{A} - \cos t}{\dfrac{a\sin t}{A} - \sin t} = -\frac{\sin t}{\cos t}. \qquad (1)$$

Now, the left-hand side of (1) can be simplified as

$$\begin{aligned}\frac{\dfrac{a\cos t + \sqrt{a^2-b^2}}{A} - \cos t}{\dfrac{a\sin t}{A} - \sin t} &= \frac{a\cos t + \sqrt{a^2-b^2} - A\cos t}{a\sin t - A\sin t}\\ &= \frac{a\cos t + \sqrt{a^2-b^2} - a\cos t - \cos^2 t\cdot\sqrt{a^2-b^2}}{a\sin t - a\sin t - \cos t\sin t\cdot\sqrt{a^2-b^2}}\\ &= \frac{1-\cos^2 t}{-\cos t\sin t}\\ &= -\frac{\sin t}{\cos t},\end{aligned}$$

which is the right-hand side of (1). Therefore, DP is indeed tangent to the ellipse.

Also solved by D.J. Smeenk, Zaltbommel, the Netherlands.

M55. *Proposé par l'équipe de Mayhem*. Trouver la somme des 2002 premiers termes de la suite suivante

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ \ldots .$$

Solution by Robert Bilinski, Outremont, QC.

Il faut en premier déterminer avec la répétition de quel nombre s'arrête la suite lorsqu'elle est à la position 2002. On remarque que chaque nombre est répété sa valeur de fois. On cherche le plus grand n tel que $\sum\limits_{k=1}^{n} k \le 2002$ ou bien que $\frac{n(n+1)}{2} \le 2002$ qui équivaut à $n^2+n-4004 \le 0$. On obtient $n = 62$ qui donne $\frac{n(n+1)}{2} = 1953$. Ce qui veut dire que dans les 2002 premiers nombres, les nombres de 1 à 62 apparaissent "au complet" et qu'il y aura ensuite 49 fois la valeur 63. La somme des 2002 premiers nombres a donc la valeur

$$\sum_{k=1}^{62} k^2 + 49 \cdot 63 \;=\; \frac{62 \cdot 63 \cdot 125}{6} + 49 \cdot 63 \;=\; 84\,462\,.$$

Solutioné aussi par Alexandre Ortan, écolier, École Joseph-François-Perrault, Montréal, QC.

M56. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Prove the identity

$$\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 \;=\; 4 \cos A \cos B \cos C\,,$$

where the sums are cyclic and $A + B + C = \pi$.

Solution by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.

We claim that, if $A + B + C = \pi$, then

$$\cos^2 A \;=\; 1 - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C\,. \qquad (1)$$

In fact, since $B + C = \pi - A$, we have

$$\begin{aligned}
\cos^2 A &= \cos^2(B+C) = (\cos B \cos C - \sin B \sin C)^2 \\
&= \cos^2 B \cos^2 C + (1-\cos^2 B)(1-\cos^2 C) \\
&\qquad\qquad -2\sin B \sin C \cos B \cos C \\
&= 1 - \cos^2 B - \cos^2 C \\
&\qquad\qquad +2\cos B \cos C\,(\cos B \cos C - \sin B \sin C) \\
&= 1 - \cos^2 B - \cos^2 C + 2\cos B \cos C \cos(B+C) \\
&= 1 - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C\,,
\end{aligned}$$

as claimed.

Now we prove the identity proposed. Taking into account (1) and the fact that $\cos A = \sin B \sin C - \cos B \cos C$ (cyclic), we have

$$\begin{aligned}\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 &= \sum \sin^2 A - \left(1 + \sum \cos^2 A\right) \\ &= \sum (\sin^2 A - 1) + 2 - \sum \cos^2 A \\ &= 2\left(1 - \sum \cos^2 A\right) \\ &= 4 \cos A \cos B \cos C\,,\end{aligned}$$

where again all the above sums are cyclic.

Also solved by Robert Bilinski, Outremont, QC.

This month, Andrei Ismail wins a copy of GrafEq from Pedagoguery Software. Congratulations Andrei! Keep sending your problems and solutions.