MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

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Mayhem Problems

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Please include in all correspondence your name, school, grade, city, province or state, and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 March 2004*. Solutions received after this time will be considered only if there is time before publication of the solutions.

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

M101. *Proposed by the Mayhem Staff.*

Find the smallest value of k such that $k!$ ends with 100 zeros. [*Note:* $k! = k(k-1)(k-2)\cdots(3)(2)(1)$.]

. . . .

Trouver la plus petite valeur de k telle que $k!$ finisse avec 100 zéros. [*Note :* $k! = k(k-1)(k-2)\cdots(3)(2)(1)$.]

M102. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that $ABCD$ is a parallelogram and that G_A, G_B, G_C, and G_D are the centroids of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively.

Prove that:

1. $G_AG_BG_CG_D$ is a parallelogram;
2. $[G_AG_BG_CG_D] = \frac{1}{9}[ABCD]$, where $[ABCD]$ is the area of $ABCD$.

. . . .

Dans un parallélogramme $ABCD$ on suppose que G_A, G_B, G_C et G_D sont les centres de gravité respectifs des triangles BCD, ACD, ABD et ABC.

Montrer que :

1. $G_AG_BG_CG_D$ est un parallélogramme ;
2. $[G_AG_BG_CG_D] = \frac{1}{9}[ABCD]$, où $[ABCD]$ désigne l'aire de $ABCD$.

M103. *Proposed by the Mayhem Staff.*

Solve for n:

$$100^{1/n} \times 100^{2/n} \times 100^{3/n} \times \cdots \times 100^{2003/n} = 1000\,.$$

. . . .

Résoudre par rapport à n :

$$100^{1/n} \times 100^{2/n} \times 100^{3/n} \times \cdots \times 100^{2003/n} = 1000\,.$$

M104. *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

Suppose that $ABCD$ is a parallelogram and that O_A, O_B, O_C, and O_D are the circumcentres of $\triangle BCD$, $\triangle ACD$, $\triangle ABD$, and $\triangle ABC$, respectively.

Prove that:

1. $O_AO_BO_CO_D$ is a parallelogram;
2. parallelograms $ABCD$ and $O_AO_BO_CO_D$ are similar;
3. AO_BCO_D is a parallelogram;
4. O_ABO_CD is a parallelogram;
5. parallelograms AO_BCO_D and O_ABO_CD are similar.

Dans un parallélogramme $ABCD$ on suppose respectivement que O_A, O_B, O_C and O_D sont les centres des cercles circonscrits des triangles BCD, ACD, ABD and ABC.

Montrer que :

1. $O_AO_BO_CO_D$ est un parallélogramme ;
2. les parallélogrammes $ABCD$ et $O_AO_BO_CO_D$ sont semblables ;
3. AO_BCO_D est un parallélogramme ;
4. O_ABO_CD est un parallélogramme ;
5. les parallélogrammes AO_BCO_D et O_ABO_CD sont semblables.

M105. *Proposed by Andrew Critch, Clarenville High School, Clarenville, NL.*

Suppose that the roots of $P(x) = x^3 - 2kx^2 - 3x^2 + hx - 4$ are distinct, and that $P(k) = P(k+1) = 0$. Determine the value of h.

. . . .

On suppose que les racines de $P(x) = x^3 - 2kx^2 - 3x^2 + hx - 4$ sont distinctes, et que $P(k) = P(k+1) = 0$. Trouver la valeur de h.

M106. *Proposed by the Mayhem Staff.*

A 4 by 4 square has an area of 16 square units and a perimeter of 16 units. That is, the area and perimeter are numerically equivalent (ignoring units of measurement). Are there any other rectangles with integral dimensions that share this property? If possible, show that you have found all such examples.

. . . .

Un carré de 4 par 4 a une aire de 16 unités carrées et un périmètre de 16 unités. Autrement dit, l'aire et le périmètre sont numériquement équivalents (si on laisse tomber les unités). Y a-t-il d'autres rectangles de dimensions entières possédant cette propriété ? Si possible, montrez que vous les avez tous trouvés.

Mayhem Solutions

M51. *Proposed by the Mayhem Staff.*

You have a deck with cards numbered 1 through 25. You perform the following operations on the deck:

- you place the top card on the bottom of the deck.
- you place the new top card on the bottom of the deck.

- you flip the new top card face up on the table.

You continue this process until all cards are face up on the table. Find the order of the cards in the deck if, when the process is performed, the cards get laid out on the table in the order $1, 2, 3, \ldots, 25$.

Solution by Robert Bilinski, Outremont, QC.

First, let a, b, c, d, e, f, $\ldots$, y, be the sequence of 25 cards, in their original positions. In the following table, we put the first few steps of the stated algorithm.

Step	Undistributed cards	Found sequence
0	$a, b, c, d, e, f, \ldots, y$	
1	$d, e, f, g, \ldots, y, a, b$	c
2	$g, h, \ldots, y, a, b, d, e$	c, f

We thus obtain the following sequence:

$$c,\ f,\ i,\ l,\ o,\ r,\ u,\ x,\ b,\ g,\ k,\ p,\ t,\ y,\ e,\ m,\ s,\ a,\ j,\ v,\ h,\ w,\ q,\ d,\ n,$$

which equals $1, 2, 3, \ldots, 25$.

Thus, the original order of the cards was:

$$\begin{array}{l} 18,\ 9,\ 1,\ 24,\ 15,\ 2,\ 10,\ 21,\ 3,\ 19, \\ \quad 11,\ 4,\ 16,\ 25,\ 5,\ 12,\ 23,\ 6,\ 17,\ 13,\ 7,\ 20,\ 22,\ 8,\ 14. \end{array}$$

Also solved by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

M52. *Proposé par J. Walter Lynch, Athens, GA, USA.*

On a deux pièces de monnaie. L'une est une pièce d'un dollar normale et l'autre une fausse pièce d'un dollar, avec deux faces. On jette au hasard chacune des pièces dans deux tiroirs différents. Quelqu'un entre dans la chambre et ouvre un des tiroirs et aperçoît une pièce, côté face. Quelle est la probabilité que cette pièce soit celle à deux faces ?

Solution de Robert Bilinski, Outremont, QC.

Si on regarde les 2 pièces de monnaie, il y a au total 3 faces. Sur ces trois faces, il y en a deux qui appartiennent à la pièce truquée. Ainsi, par la définition de probabilité, on a :

$$P(\text{pièce truquée}) = \frac{\text{\# cas favorable}}{\text{\# total}} = \frac{2}{3}.$$

Solutioné aussi par Alexandre Ortan, écolier, École Joseph-François-Perrault, Montréal, QC.

M53. *Proposed by the Mayhem Staff.*

A circular path is surrounded by 17 stepping stones numbered 0, 1, 2, ..., 16. Sally starts on stone 0 and moves 1 step to stone 1, then 4 steps to stone 5, then 9 steps to step 14 and continues in the following pattern until at last she moves 2002^2 steps and stops (to rest). What stone is Sally standing on while she rests?

Solution by Alexandre Ortan, student, École Joseph-François-Perrault, Montréal, QC.

Let us first find how many steps Sally has made:

$$1^2+2^2+3^2+\cdots+2002^2 \;=\; \frac{2002\times 2003\times 4005}{6} \;=\; 2\,676\,679\,005$$

steps. Since her walk is cyclical, the residue of the number of steps on division by 17 will give us the position where Sally ends her walk.

Since $2\,676\,679\,005 = 17\times 157\,451\,706+3$, Sophie ended up on the third step.

Also solved by Robert Bilinski, Outremont, QC.

M54. *Proposed by Gary Tupper, Pedagoguery Software Inc., Terrace, BC.*

An ellipse with major axis AB and foci F and F' is inscribed in a circle with diameter AB and centre C. P is a point on the ellipse and D is a point on the circle so that radius CD bisects FP. Show that line DP is tangent to the ellipse.

Solution by Andrei Ismail, student, 11^{th} Grade, Vasile Alecsandri National College, Galati, Romania.

Let the ellipse have equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then its foci are

$$F(\sqrt{a^2-b^2},0) \qquad \text{and} \qquad F'(-\sqrt{a^2-b^2},0)\,.$$

Because point P is located on the ellipse, its coordinates can be expressed as $P(a\cos t, b\sin t)$. Also, the points A and B have coordinates $A(-a,0)$ and $B(a,0)$. The circle with diameter AB has the equation $x^2+y^2=a^2$. Let us consider M, the mid-point of FP. It has coordinates

$$M\left(\frac{a\cos t+\sqrt{a^2-b^2}}{2},\,\frac{b\sin t}{2}\right)\,.$$

The condition that CD bisects FP is equivalent to the statement that D is the intersection of the line CM with the circle $x^2+y^2=a^2$. Now, it is easy to see that D has coordinates

$$D\left(a\cdot\frac{a\cos t+\sqrt{a^2-b^2}}{\sqrt{(a\cos t+\sqrt{a^2-b^2})^2+(b\sin t)^2}},\,a\cdot\frac{b\sin t}{\sqrt{(a\cos t+\sqrt{a^2-b^2})^2+(b\sin t)^2}}\right)\,.$$

Without loss of generality, we assume that P is in the half-plane $y > 0$ (and therefore, so is D). Simplifying the denominators yields

$$D\left(a\cdot\frac{a\cos t+\sqrt{a^2-b^2}}{A},\ a\cdot\frac{b\sin t}{A}\right)\,,$$

where $A = a + \cos t\sqrt{a^2-b^2}$.

All we have to show now is that the line DP is tangent to the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. It is well-known that the slope of the tangent to the ellipse at the point $P(a\cos t, b\sin t)$ is $-\frac{b\cos t}{a\sin t}$. Thus, it will suffice to show that:

$$\frac{a\cdot\dfrac{b\sin t}{A}-b\sin t}{a\cdot\dfrac{a\cos t+\sqrt{a^2-b^2}}{A}-a\cos t} = -\frac{b\cos t}{a\sin t}\,,$$

or, equivalently,

$$\frac{\dfrac{a\cos t+\sqrt{a^2-b^2}}{A}-\cos t}{\dfrac{a\sin t}{A}-\sin t} = -\frac{\sin t}{\cos t}\,. \tag{1}$$

Now, the left-hand side of (1) can be simplified as

$$\begin{aligned}\frac{\dfrac{a\cos t+\sqrt{a^2-b^2}}{A}-\cos t}{\dfrac{a\sin t}{A}-\sin t} &= \frac{a\cos t+\sqrt{a^2-b^2}-A\cos t}{a\sin t-A\sin t}\\ &= \frac{a\cos t+\sqrt{a^2-b^2}-a\cos t-\cos^2 t\cdot\sqrt{a^2-b^2}}{a\sin t-a\sin t-\cos t\sin t\cdot\sqrt{a^2-b^2}}\\ &= \frac{1-\cos^2 t}{-\cos t\sin t}\\ &= -\frac{\sin t}{\cos t}\,,\end{aligned}$$

which is the right-hand side of (1). Therefore, DP is indeed tangent to the ellipse.

Also solved by D.J. Smeenk, Zaltbommel, the Netherlands.

M55. *Proposé par l'équipe de Mayhem.* Trouver la somme des 2002 premiers termes de la suite suivante

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ \ldots .$$

Solution by Robert Bilinski, Outremont, QC.

Il faut en premier déterminer avec la répétition de quel nombre s'arrête la suite lorsqu'elle est à la position 2002. On remarque que chaque nombre est répété sa valeur de fois. On cherche le plus grand n tel que $\sum\limits_{k=1}^{n} k \leq 2002$ ou bien que $\frac{n(n+1)}{2} \leq 2002$ qui équivaut à $n^2+n-4004 \leq 0$. On obtient $n = 62$ qui donne $\frac{n(n+1)}{2} = 1953$. Ce qui veut dire que dans les 2002 premiers nombres, les nombres de 1 à 62 apparaissent "au complet" et qu'il y aura ensuite 49 fois la valeur 63. La somme des 2002 premiers nombres a donc la valeur

$$\sum_{k=1}^{62} k^2 + 49 \cdot 63 \;=\; \frac{62 \cdot 63 \cdot 125}{6} + 49 \cdot 63 \;=\; 84\,462\,.$$

Solutioné aussi par Alexandre Ortan, écolier, École Joseph-François-Perrault, Montréal, QC.

M56. *Proposed by Vedula N. Murty, Dover, PA, USA.*

Prove the identity

$$\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 \;=\; 4 \cos A \cos B \cos C\,,$$

where the sums are cyclic and $A+B+C=\pi$.

Solution by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.

We claim that, if $A+B+C=\pi$, then

$$\cos^2 A \;=\; 1 - \cos^2 B - \cos^2 C - 2\cos A \cos B \cos C\,. \qquad (1)$$

In fact, since $B+C = \pi - A$, we have

$$\begin{aligned}
\cos^2 A &= \cos^2(B+C) = (\cos B \cos C - \sin B \sin C)^2 \\
&= \cos^2 B \cos^2 C + (1-\cos^2 B)(1-\cos^2 C) \\
&\qquad\qquad -2 \sin B \sin C \cos B \cos C \\
&= 1 - \cos^2 B - \cos^2 C \\
&\qquad\qquad +2\cos B \cos C\,(\cos B \cos C - \sin B \sin C) \\
&= 1 - \cos^2 B - \cos^2 C + 2 \cos B \cos C \cos(B+C) \\
&= 1 - \cos^2 B - \cos^2 C - 2 \cos A \cos B \cos C\,,
\end{aligned}$$

as claimed.

Now we prove the identity proposed. Taking into account (1) and the fact that $\cos A = \sin B \sin C - \cos B \cos C$ (cyclic), we have

$$\begin{aligned} \left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 &= \sum \sin^2 A - \left(1 + \sum \cos^2 A\right) \\ &= \sum (\sin^2 A - 1) + 2 - \sum \cos^2 A \\ &= 2\left(1 - \sum \cos^2 A\right) \\ &= 4 \cos A \cos B \cos C\,, \end{aligned}$$

where again all the above sums are cyclic.

Also solved by Robert Bilinski, Outremont, QC.

This month, Andrei Ismail wins a copy of GrafEq from Pedagoguery Software. Congratulations Andrei! Keep sending your problems and solutions.

Pólya's Paragon

Paul Ottaway

Last September, I began this column with a short discussion about how mathematics is beautiful and how this beauty can arise from the variety of ways of proving mathematical statements. I will start this new school year with a cute proof that not only are numbers beautiful, but they are also interesting!

Theorem 1. All positive integers are interesting.

Proof. First, we suppose for a contradiction that there exists at least one uninteresting positive integer. Then, among all such integers, one of them must be the smallest. Now this integer is very interesting indeed, since it is the smallest one with no other interesting properties! This contradicts our assumption. We must conclude that all positive integers are interesting. □

We have not really proved anything here, because the term 'interesting' has not been defined. Nevertheless, this is a wonderful example of how a "proof by contradiction" works. Basically, you assume the negation of what you are trying to prove and show that this assumption leads to a contradiction. This shows that the assumption is false, which then implies that the original statement is true. More generally, this is known as an "indirect proof".

The next theorem I would like to consider involves a number which I find quite interesting. The proof is one of the most famous examples of proof by contradiction.

Theorem. $\sqrt{2}$ is irrational.

Proof. Assume for a contradiction that $\sqrt{2}$ is rational. Then, we can write $\sqrt{2}$ as a/b where a, b are integers with no common divisor greater than 1 and $b \neq 0$ (this is the definition of a rational number). Now we do a little algebra:

$$\begin{aligned} \sqrt{2} &= \frac{a}{b} \\ 2 &= \frac{a^2}{b^2} \\ 2b^2 &= a^2 \end{aligned}$$

This shows that a^2 is even. Now, since an odd number squared is again odd, we must conclude that a is even; that is, $a = 2n$ for some other integer n. Substituting for a above, we get the following:

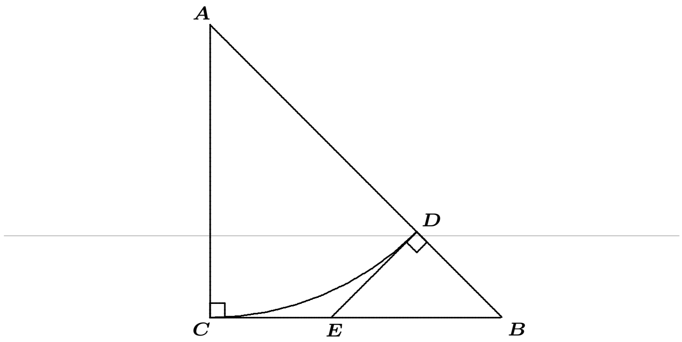
$$\begin{aligned} 2b^2 &= (2n)^2 \\ 2b^2 &= 4n^2 \\ b^2 &= 2n^2 \end{aligned}$$

This means that b^2 is also an even number which, by the same argument as above, shows that b is even. We have now shown that a and b are both even and hence, both divisible by 2. This contradicts the assumption that a and b have no common divisor greater than 1. Therefore, without any further work we must conclude that $\sqrt{2}$ is not a rational number. This, of course, implies that it must be irrational, as required. □

The fact that $\sqrt{2}$ is irrational was known to the Greeks many years before mathematicians used variables to represent unknown quantities. We might wonder how they ever went about proving such a result without expressing the results algebraically. Here is an alternate geometric proof that does not need variables. It is again a proof by contradiction.

Proof. We know by the Pythagorean Theorem that we can construct a right angled triangle with side lengths 1, 1 and $\sqrt{2}$. If we assume that $\sqrt{2}$ is rational, then we can find a similar triangle that has all integer side lengths (you can multiply all the sides by the denominator b if $\sqrt{2} = a/b$). In particular, we choose the smallest such triangle ABC with integer lengths which is similar to the original.

Now we make the following constructions: Draw a circle centred at A with radius equal to one of the shorter sides. This intersects the hypotenuse at D. Construct a line from D perpendicular to the hypotenuse which meets BC at E.



We need only make arguments about which sides in the diagram must have integer lengths. First, $AD = AC$; thus, AD is an integer. Then we know that $DB = AB - AD$ is also an integer, since it is the difference of two integers. Since $\angle DBE = 45°$ and $\angle EDB = 90°$, we know $\angle DEB = 45°$, which means that $DB = DE$. Therefore, DE has integral length as well. Also, DE and CE are tangents to the circle centred at A; hence, they are the same length. Thus, CE has integral length. Finally, we can now see that $EB = CB - CE$ is an integer, since its length can be expressed as the difference of two integers. Hence, $\triangle BDE$ is similar to $\triangle ACB$. We have shown that $\triangle BDE$ has integer side lengths. This contradicts our assumption that $\triangle ABC$ is the smallest such triangle. We must therefore conclude that $\sqrt{2}$ is irrational. □

Clearly, this is not quite as elegant as the first proof. Nevertheless, it is nice to know that such things can be proven without the algebraic tools that we have come to depend on.

Problems for you to try:

There are many other problems that can be solved using an indirect proof. Here are three more problems whose proofs lend themselves to this technique.

1. Prove that if r is an irrational number, and if a and b are rational numbers with $b \neq 0$, then the expression $a + br$ represents an irrational number.

2. Prove that there are infinitely many primes. This is another famous use of indirect proof. [*Hint:* Begin by assuming that there are only finitely many. What can you say about the number you get when you multiply them all together and add 1?]

3. Prove that e is an irrational number. (Note: this is quite hard. You may use the fact that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.)

Binomial Inversion: Two Proofs and an Application to Derangements

Heba Hathout

There is an old problem that goes by many names but generally runs something like this:

> A group of n men enter a restaurant and check their hats. The hat-checker is absent-minded and distributes the hats back to the men at random when they leave. What is the probability, P_n, that no man gets his own hat back, and how does P_n behave as n approaches infinity?

The answer is that

$$P_n \;=\; 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \;=\; \sum_{j=0}^{n} (-1)^j \frac{1}{j!}\,, \tag{1}$$

which approaches $1/e$ as n approaches infinity.

This problem is usually tackled using the inclusion-exclusion principle. It can also be solved by developing a recursive relationship among P_n, P_{n-1}, and P_{n-2}. This paper introduces a different approach, using a technique called *binomial inversion*.

If a sequence of numbers b_0, b_1, b_2, ... is defined in terms of another sequence of numbers a_0, a_1, a_2, ... by the formula

$$b_k \;=\; \sum_{i=0}^{k} \binom{k}{i} a_i\,, \tag{2}$$

then this relationship can be inverted and the numbers a_i retrieved by the following formula, called the *Binomial Inversion Formula*:

$$a_k \;=\; \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} b_i\,. \tag{3}$$

We will first use the formula (3) to solve our problem. Then we will derive the formula by two different methods, the first using infinite series and the second using linear algebra.

Solution using Binomial Inversion

The set of all possible permutations of the hats can be divided into subsets as follows: a subset S_0 comprised of the permutations where none of the men gets his own hat back, a subset S_1 consisting of the permutations where just one of the men gets his own hat, and so on, up to a subset S_n consisting of permutations where all of the men get their own hats.

Consider the subset S_2, for example. If the two men who get the correct hats are man # 1 and man # 2, then the number of possible arrangements is D_{n-2}, the number of derangements for the other $n-2$ hats. Since there are $\binom{n}{2}$ possibilities for the pair of men who get their own hats, the number of permutations in the set S_2 is $|S_2| = \binom{n}{2}D_{n-2}$.

Applying the same logic to each subset S_i, we obtain $|S_i| = \binom{n}{i}D_{n-i}$. The total number of permutations of the hats is

$$\begin{aligned} n! &= |S_0| + |S_1| + |S_2| + \cdots + |S_n| \\ &= \sum_{i=0}^{n} \binom{n}{i} D_{n-i} = \sum_{j=0}^{n} \binom{n}{n-j} D_j = \sum_{j=0}^{n} \binom{n}{j} D_j \,, \end{aligned}$$

where we have changed the index of summation from i to $j = n - i$ and then used the fact that $\binom{n}{n-j} = \dfrac{n!}{j!(n-j)!} = \binom{n}{j}$.

Now we use binomial inversion (with $a_n = D_n$ and $b_n = n!$):

$$\begin{aligned} D_n &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i! = \sum_{i=0}^{n} (-1)^{n-i} \frac{n!}{i!(n-i)!}\, i! \\ &= n! \sum_{i=0}^{n} (-1)^{n-i} \frac{1}{(n-i)!} = n! \sum_{j=0}^{n} (-1)^j \frac{1}{j!} \,. \end{aligned}$$

Finally, since $P_n = D_n/n!$, we obtain (1).

Binomial Inversion using Infinite Series

We will derive the inversion formula (3) from formula (2) using infinite series. However, since the issue of convergence of the series will be ignored, the argument that we will give here is not completely rigorous.

We introduce exponential generating functions as follows:

$$A(x) = \sum_{k=0}^{\infty} \frac{a_k}{k!} x^k \,, \qquad B(x) = \sum_{k=0}^{\infty} \frac{b_k}{k!} x^k \,.$$

Substituting for b_k from (2), we get

$$B(x) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \binom{k}{i} a_i \right) \frac{1}{k!} x^k = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{a_i}{i!(k-i)!} x^k \,.$$

Interchanging the order of summation and simplifying,

$$\begin{aligned} B(x) &= \sum_{i=0}^{\infty}\sum_{k=i}^{\infty} \frac{a_i}{i!(k-i)!}x^k = \sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\left(\frac{a_i x^i}{i!}\right)\left(\frac{x^{k-i}}{(k-i)!}\right) \\ &= \sum_{i=0}^{\infty}\frac{a_i x^i}{i!}\left(\sum_{k=i}^{\infty}\frac{x^{k-i}}{(k-i)!}\right) = \left[\sum_{i=0}^{\infty}\frac{a_i}{i!}x^i\right]\left[\sum_{j=0}^{\infty}\frac{x^j}{j!}\right]. \end{aligned}$$

We recognize the sums in the last expression above as $A(x)$ and e^x, respectively. Therefore, we can now write $B(x) = A(x)e^x$, from which we get $A(x) = e^{-x}B(x)$. Then

$$A(x) = \left[\sum_{j=0}^{\infty}\frac{(-1)^j x^j}{j!}\right]\left[\sum_{i=0}^{\infty}\frac{b_i}{i!}x^i\right] = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{(-1)^j b_i}{j!i!}x^{i+j}.$$

We can re-index (letting $k = i + j$) to get

$$A(x) = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{(-1)^{k-i}b_i}{(k-i)!i!}x^k = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}b_i\right)\frac{1}{k!}x^k.$$

The coefficient of x_k in this formula must be the same as the coefficient $a_k/k!$ in the initial formula defining $A(x)$. Thus, we obtain (3).

Binomial Inversion using Linear Algebra

We now assume that the sequences related by the formula (2) are *finite* sequences $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$. (This case is all that we needed for our problem of the hats.) Then (2) can be recast into matrix form as

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

On the right-hand side is an $n+1$ by $n+1$ matrix T whose rows are the first $n+1$ rows of Pascal's triangle, filled in with zeroes. The entry in row i and column j is $T(i,j) = \binom{i}{j}$, where i and j go from 0 to n (rather than the conventional numbering starting at 1). The inversion formula (3) states that the matrix T has an inverse X whose entries are $X(i,j) = \binom{i}{j}(-1)^{i-j}$. This is what we will prove.

Let $M = TX$. The goal is to show that $M = I$, the identity matrix. We must show that the diagonal elements of M are 1 and the off-diagonal elements are 0.

The entry in row a and column b of M is

$$M(a,b) \;=\; \sum_{i=0}^{n} T(a,i)X(i,b) \;=\; \sum_{i=0}^{n} \binom{a}{i}\binom{i}{b}(-1)^{i-b}\,. \tag{4}$$

If $i > a$, we have $\binom{a}{i} = 0$, while if $i < b$, then $\binom{i}{b} = 0$. Thus, some terms in the above sum are zero, in general. In particular, we see that $M(a,b) = 0$ when $a < b$.

The diagonal entries of M occur when $a = b$. In this case, the only non-zero term in the sum in (4) is for $i = a$. We have

$$M(a,a) \;=\; \binom{a}{a}\binom{a}{a}(-1)^{a-a} \;=\; 1\,.$$

Thus, the diagonal elements of M are 1, as desired.

Now consider $a > b$. Then (4) becomes

$$M(a,b) \;=\; \sum_{i=b}^{a} \binom{a}{i}\binom{i}{b}(-1)^{i-b} \;=\; \sum_{i=b}^{a} \frac{a!}{(a-i)!(i-b)!b!}(-1)^{i-b}\,,$$

which is the same as

$$\frac{b!}{a!}M(a,b) \;=\; \sum_{i=b}^{a} \frac{1}{(a-i)!(i-b)!}(-1)^{i-b}\,.$$

Now, we let $j = i - b$ (and hence $i = j + b$). Then our equation is

$$\frac{b!}{a!}M(a,b) \;=\; \sum_{j=0}^{a-b} \frac{1}{(a-b-j)!j!}(-1)^{j}\,.$$

Finally, we let $m = a - b$ and multiply both sides by $m!$ to get

$$m!\frac{b!}{a!}M(a,b) \;=\; \sum_{j=0}^{m} \frac{m!}{(m-j)!j!}(-1)^{j} \;=\; \sum_{j=0}^{m} \binom{m}{j}(-1)^{j}\,.$$

The right-hand side equals 0 by a well-known combinatorics identity, because it represents $(x+y)^m$, when $x = 1$ and $y = -1$. Therefore, $M(a,b)$ is zero, and the proof is complete.

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