SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologise for omitting the name of ANDREI SIMION, student, Cooper Union for Advancement of Science and Art, New York, NY, USA from the list of solvers of 2717 and 2718, and the name of MICHEL BATAILLE, Rouen, France from the list of solvers of 2729.

2289*. [1997 : 501] *Proposed by Clark Kimberling, Evansville, IN, USA.*

Use any sequence, $\{c_k\}$, of 0's and 1's to define a *repetition-resistant sequence* $s = \{s_k\}$ inductively as follows:

1. $s_1 = c_1$, $s_2 = 1 - s_1$;
2. for $n \geq 2$, let

$$\begin{aligned} L &= \max\{i \geq 1 : (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad = (s_{n-i+2}, \ldots, s_n, 0) \quad \text{for some } m < n\}, \\ L' &= \max\{i \geq 1 : (s_{m-i+2}, \ldots, s_m, s_{m+1}) \\ &\qquad = (s_{n-i+2}, \ldots, s_n, 1) \quad \text{for some } m < n\}. \end{aligned}$$

(so that L is the maximal length of the tail-sequence of $(s_1, s_2, \ldots, s_n, 0)$ that already occurs in $(s_1, s_2, \ldots, s_n)$, and similarly for L'), and

$$s_{n+1} = \begin{cases} 0 & \text{if } L < L', \\ 1 & \text{if } L > L', \\ c_n & \text{if } L = L'. \end{cases}$$

(For example, if $c_i = 0$ for all i, then

$$s = (0,1,0,0,0,1,1,1,0,1,0,1,1,1,0,0,1,0,0,$$
$$1,1,1,1,0,1,1,0,0,0,0,0,1,0,1,0,\ldots)\ )$$

Prove or disprove that s contains every binary word.

Solution by Alejandro Dau, Universidad de Buenos Aires, Buenos Aires, Argentina, modified slightly by the editors.

We will prove that every binary word occurs infinitely many times in s. For any symbol $\sigma \in \{0, 1\}$, we write $\overline{\sigma}$ for the opposite symbol.

We first show that the words of length 1 appear infinitely many times; that is, the symbols 0 and 1 each appear infinitely often in s. Each symbol appears at least once, because, by definition, s starts with 01 or 10. Suppose the symbol σ appears just finitely many times, the last time being at s_k. Then $s_{k+i} = \overline{\sigma}$ for all $i \geq 1$. In particular, $s_{2k+2} = \overline{\sigma}$. But this contradicts the definition of s, because the string $s_{k+2} \ldots s_{2k+2} = \overline{\sigma}^{k+1}$ already occurs in $s_1 \ldots s_{2k+1}$ as $s_{k+1} \ldots s_{2k+1}$, whereas the word $\overline{\sigma}^k \sigma$ does not occur as a substring of $s_1 \ldots s_{2k+1}$. We conclude that each word of length 1 appears infinitely often in s.

Now suppose there is some word of length greater than 1 that appears only finitely often. Then there is a shortest length $n \geq 2$ for such words. Choose some word of length n that appears only finitely often, say $t = d_1 \ldots d_n$, and let $k \geq 0$ be the number of times that t appears. The word $d_1 \ldots d_{n-1}$ appears infinitely often, because its length is less than n. Letting $t^* = d_1 \ldots d_{n-1}\overline{d_n}$, we see that t^* must appear infinitely often, since t appears only finitely often.

Suppose $k = 0$. Then the word t does not appear at all in s. Let the first two appearances of t^* end at positions i and j, respectively. Thus, $s_{i-n+1} \ldots s_i = t^*$ (the first appearance) and $s_{j-n+1} \ldots s_j = t^*$ (the second appearance). The fact that $s_j = \overline{d_n}$ contradicts the definition of s, because the word t^* already occurs in $s_1 \ldots s_{j-1}$ as $s_{i-n+1} \ldots s_i$, whereas the word t does not occur in $s_1 \ldots s_{j-1}$.

Suppose $k \geq 1$. Then the word t appears at least once in s. Let the last appearance of t end at position p. Thus, $s_{p-n+1} \ldots s_p = t$, and this is the last appearance of t in s. Consequently, there are no words in s of length $p+1$ that end with the string t. On the other hand, there must be some word of length $p+1$ ending in t^* that occurs more than once in s, because there are infinitely many appearances of t^* in s and only finitely many words of length $p+1$. Thus, there must exist two strings $s_{i-p} \ldots s_i$ and $s_{j-p} \ldots s_j$, with $i < j$, that represent the same word w ending in t^*. We have $s_j = \overline{d_n}$. But the definition of s requires that $s_j = d_n$, because the tail-sequence w ending in t^* already occurs in $s_1 \ldots s_{j-1}$ as $s_{i-p} \ldots s_i$, whereas there are no words in s of length $p+1$ ending in t. We have a contradiction.

A contradiction comes from assuming that a word appears in s only a finite number of times. Therefore, every word appears infinitely many times.

2664. [2001 : 403] *Remark: In his solution of a generalization of this problem, Walther Janous proved in* [2002 : 410] *the following result:*

Let a, b, c be non-negative reals and let $r \in \mathbb{R}$ with $r \geq 2$.

Then

$$a^r(b+c) + b^r(c+a) + c^r(a+b) \;\geq\; \frac{2}{3^{\frac{r-1}{2}}}(ab+bc+ca)^{\frac{r+1}{2}}\,. \qquad (1)$$

At the end of his solution he made the remark:

Clearly equality holds in our theorem if $r = 1$. This leads to the natural question: What happens for $r \in (1,2)$?

The following is a solution to this question.

Solution by Vasile Cirtoaje, University of Ploiesti, Romania.

We will prove using two different methods that (1) also holds for all $r \in (1,2)$.

Method I. We will prove (1) for $r \geq 1$ by showing that the function

$$f(x) \;=\; \frac{1}{x+1}\ln\left[\frac{a^x(b+c)+b^x(c+a)+c^x(a+b)}{6}\right]$$

is increasing on $[1,\infty)$; that is, by showing that $f'(x) \geq 0$ on $[1,\infty)$. The required inequality then follows from $f(r) \geq f(1)$.

The inequality $f'(x) \geq 0$ is equivalent to $A \geq C$, where

$$\begin{aligned} A &= a^{(b+c)a^x(x+1)}b^{(c+a)b^x(x+1)}c^{(a+b)c^x(x+1)}\,, \\ C &= \left[\frac{a^x(b+c)+b^x(c+a)+c^x(a+b)}{6}\right]^{a^x(b+c)+b^x(c+a)+c^x(a+b)}\,. \end{aligned}$$

We will show this by proving that $A \geq B \geq C$, where

$$B \;=\; a^{(b+c)a^x x+a(b^x+c^x)}b^{(c+a)b^x x+b(c^x+a^x)}c^{(a+b)c^x x+c(a^x+b^x)}\,.$$

The inequality $A \geq B$ is equivalent to $\ln A \geq \ln B$. To prove the latter true we must show that $g(x) \geq 0$ for all $x \geq 1$, where

$$\begin{aligned} g(x) \;&=\; [(b+c)a^x-a(b^x+c^x)]\ln a+[(c+a)b^x-b(c^x+a^x)]\ln b \\ &\quad+\;[(a+b)c^x-c(a^x+b^x)]\ln c\,. \end{aligned}$$

We have

$$\begin{aligned} g'(x) \;&=\; (b+c)a^x\ln^2 a+(c+a)b^x\ln^2 b+(a+b)c^x\ln^2 c \\ &\quad-(a^xb+ab^x)\ln a\ln b-(b^xc+bc^x)\ln b\ln c \\ &\quad-(c^xa+ca^x)\ln c\ln a \\ &=\; ab(\ln a-\ln b)(a^{x-1}\ln a-b^{x-1}\ln b) \\ &\quad+bc(\ln b-\ln c)(b^{x-1}\ln b-c^{x-1}\ln c) \\ &\quad+ca(\ln c-\ln a)(c^{x-1}\ln c-a^{x-1}\ln a)\,. \end{aligned}$$

For $x \geq 1$, it easily follows that $g'(x) \geq 0$. Consequently, $g(x)$ is increasing on $[1,\infty)$; that is, $g(x) \geq g(1) = 0$.

To prove the inequality $B \geq C$, we rewrite this inequality as follows:

$$\begin{aligned} &x_1^{x_1}x_2^{x_2}x_3^{x_3}x_4^{x_4}x_5^{x_5}x_6^{x_6} \\ &\quad\geq\;\left(\frac{x_1+x_2+x_3+x_4+x_5+x_6}{6}\right)^{x_1+x_2+x_3+x_4+x_5+x_6}\,, \end{aligned}$$

where $x_1 = a^xb$, $x_2 = ab^x$, $x_3 = b^xc$, $x_4 = bc^x$, $x_5 = c^xa$, $x_6 = ca^x$. This known inequality follows from Jensen's Inequality applied to the convex function $h(x) = x\ln x$.

Method II. Since (1) is homogeneous, we can reformulate the problem (without loss of generality) as follows:

> If a, b, c are positive integers such that $ab + bc + ca = 3$ and $r \in (1, 2)$, then
> $$a^r(b+c) + b^r(c+a) + c^r(a+b) \;\geq\; 6\,. \qquad (2)$$

From $ab+bc+ca = 3$, we may write $a(b+c) = 3 - bc$, $b(c+a) = 3 - ca$, $c(a+b) = 3 - ab$, and (2) becomes

$$a^{r-1}(3-bc) + b^{r-1}(3-ca) + c^{r-1}(3-ab) \;\geq\; 6\,,$$

or

$$a^{r-1} + b^{r-1} + c^{r-1} \;\geq\; a^{r-1}b^{r-1}c^{r-1} \cdot \frac{(ab)^{2-r} + (bc)^{2-r} + (ca)^{2-r}}{3} + 2\,.$$

Since $0 < 2-r < 1$, the function $f(x) = x^{2-r}$ is concave. Thus, by Jensen's Inequality, we have

$$\frac{(ab)^{2-r} + (bc)^{2-r} + (ca)^{2-r}}{3} \;\leq\; \left(\frac{ab+bc+ca}{3}\right)^{2-r} \;=\; 1\,,$$

and it suffices to prove that

$$a^{r-1} + b^{r-1} + c^{r-1} \;\geq\; a^{r-1}b^{r-1}c^{r-1} + 2\,. \qquad (3)$$

Because the inequality is symmetice, we may assume that $a \geq b \geq c$, without any loss of generality. Now we write (3) as

$$a^{r-1} + b^{r-1} - 2 \;\geq\; (a^{r-1}b^{r-1} - 1)\left(\frac{3-ab}{a+b}\right)^{r-1}\,.$$

Setting $x = \sqrt{ab}$, it follows from the AM-GM Inequality that $a + b \geq 2x$ and $a^{r-1} + b^{r-1} \geq 2x^{r-1}$. Since $a \geq b \geq c$ and $ab + bc + ca = 3$ imply $1 \leq x < \sqrt{3}$, it suffices to prove that

$$2(x^{r-1} - 1) \;\geq\; (x^{2r-2} - 1)\left(\frac{3-x^2}{2x}\right)^{r-1}\,.$$

Divinding by the non-negative factor $x^{r-1} - 1$, this last inequality becomes

$$2 \;\geq\; (x^{r-1} + 1)\left(\frac{3-x^2}{2x}\right)^{r-1}\,,$$

or

$$2 \;\geq\; \left(\frac{3-x^2}{2}\right)^{r-1} + \left(\frac{3-x^2}{2x}\right)^{r-1}\,.$$

This inequality is true because $1 \geq (3-x^2)/2 \geq (3-x^2)/(2x)$ for $x \geq 1$.

2682. [2001 : 461] and [2002 : 478] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

The sequence of functions, $\{J(n) = J(n,w)\}$, $n = 0, 1, \ldots$, is defined as follows:

$$J(0) = a\,, \qquad J(1) = w + b\,,$$
$$J(n+1) = \frac{J(n)\ (J(n)\ (w\,J(n) - 1) - J(n-1))}{J(n-1)\ (w\,J(n) + 1) + J(n)} \qquad \text{for } n > 0.$$

(a) Show that, if $a = 0$, then the sequence consists of polynomials.

(b) Show that there exists a pair (a, b) of non-zero integers such that all the $J(n)$ are polynomials with integer coefficients.

Solution by Wangwei, student, and Jun-hua Huang, the Middle School attached to Hunan Normal University, Changsha, China.

The given recurrence is equivalent to

$$J(n+1)J(n-1)wJ(n) + J(n+1)J(n-1) + J(n+1)J(n) = w\big(J(n)\big)^3 - \big(J(n)\big)^2 - J(n)J(n-1)\,,$$

or $$wJ(n)\left(\big(J(n)\big)^2 - J(n+1)J(n-1)\right) = \big(J(n+1) + J(n)\big)\big(J(n) + J(n-1)\big)\,.$$

Setting $a_n = wJ(n)$, we obtain

$$a_n(a_n^2 - a_{n+1}a_{n-1}) = (a_{n+1} + a_n)(a_n + a_{n-1})\,. \tag{1}$$

(a) Assume that $a = 0$. Let $x = wJ(1) = w(w+b)$.

Lemma: Let $\{b_n\}$ be the sequence defined by $b_0 = 0$, $b_1 = x$, and for all $n \geq 1$,

$$b_{n+1} = (x-2)b_n - b_{n-1} + x\,. \tag{2}$$

Then, for all $n \geq 1$,

$$b_n^2 - b_{n+1}b_{n-1} = xb_n\,. \tag{3}$$

Proof: The proof is by induction on n. When $n = 1$, equation (3) is true, since $b_0 = 0$ and $b_1 = x$. Now, suppose equation (3) is true for $n = k \geq 1$. Then

$$\begin{aligned} b_{k+1}^2 - b_{k+2}b_k &= \big((x-2)b_k + x - b_{k-1}\big)b_{k+1} \\ &\qquad - \big((x-2)b_{k+1} + x - b_k\big)b_k \\ &= (x-2)b_kb_{k+1} + xb_{k+1} - b_{k-1}b_{k+1} \\ &\qquad - (x-2)b_kb_{k+1} - xb_k + b_k^2 \\ &= xb_{k+1} + b_k^2 - b_{k-1}b_{k+1} - xb_k \;=\; xb_{k+1}\,. \end{aligned}$$

Therefore, the lemma holds by induction.

From (2) and (3), one can verify that

$$b_n(b_n^2 - b_{n+1}b_{n-1}) = (b_{n+1}+b_n)(b_n+b_{n-1}) .$$

Comparing this to equation (1) and noting that

$$a_0 = wJ(0) = wa = 0 \qquad \text{and} \qquad a_1 = wJ(1) = w(w+b) = x ,$$

we conclude that $a_n = b_n$ for all $n \geq 0$. Hence,

$$a_{n+1} = (w(w+b)-2)a_n - a_{n-1} + w(w+b) .$$

It is easy to show by induction that a_n is a polynomial in w. Therefore, $J(n)$ is a polynomial in w, since w is a factor of a_n for all n.

(b) Taking $(a,b) = (1,-1)$, we have $J(0) = 1$ and $J(1) = w-1$. Then $a_0 = w$ and $a_1 = w(w-1)$. Let $\{c_n\}$ be the sequence defined by $c_0 = w$, $c_1 = w(w-1)$, and for all $n \geq 1$,

$$c_{n+1} = (w-2)c_n - c_{n-1} + w .$$

Using the same inductive proof as in the lemma above we can establish

$$c_n^2 - c_{n+1}c_{n-1} = wc_n ,$$

for all $n \geq 1$ (the only difference in the proofs is in the initial step). Therefore, $a_n = c_n$ for all $n \geq 0$, which means that, for all $n \geq 1$, we have

$$a_{n+1} = (w-2)a_n + w - a_{n-1} .$$

It is easy to show that a_n is a polynomial with integer coefficients for all $n \geq 0$. Since w is a factor of a_n for all $n \geq 0$, we see that $J(n) = J(n,w)$ is a polynomial with integer coefficients for all $n \geq 1$.

2741. [2002 : 245] *Proposed by Victor Oxman, University of Haifa, Haifa, Israel.*

In the plane are given an ellipse with its two foci M and N, and two points, A and B, on it, so that $AB \parallel MN$.

With only an unmarked straight-edge, construct a diameter of the circle $ABNM$.

I. *Solution by Michel Bataille, Rouen, France.*

We shall construct the diameter along the minor axis l of the ellipse (which is the perpendicular bisector of both MN and AB). If AM and BN meet the ellipse again at C and D, respectively, l passes through the intersection points of AN with BM, of DM with CN, and of AM with BN (at most one of which could fail to exist due to parallel defining lines). Note that by symmetry, l passes through the centre of the circle $ABNM$ whose diameter we are to construct.

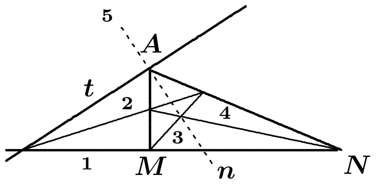


Figure 1

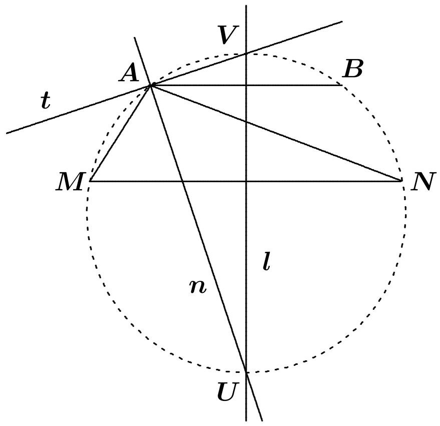


Figure 2

Using only a straight-edge, we now construct the tangent t and normal n to the conic at A. The tangent comes from an application of Pascal's Theorem (as described in problem 2740 [2003 : 246]), while n is the harmonic conjugate of t with respect to AM and AN, whose construction is shown in Figure 1, where the lines are numbered in the order of their appearance in the construction. Let t and n meet l at V and U, respectively. (See Figure 2.) The desired diameter is UV. Indeed, since n is the internal angle bisector of $\angle MAN$, it meets the circle at the mid-point of the arc MN not containing A. This point is also on l. Furthermore t, being perpendicular to n at A, meets the circle at the point diametrically opposite U. This point is V, since it is also on l.

II. *Solution by Václav Konečný, Big Rapids, MI, USA.*

We construct the diameter NI through the focus N. To do this we make use of the following straight-edge constructions.

(a) *Construct the tangent to a given conic at a given point.* This construction was described in problem 2740.

(b) *Construct the line passing through a given point and parallel to a given segment whose mid-point is also given.* This construction was used in problem 2695 [2002 : 553–554]; it can also be found in *A Survey of Geometry* by Howard Eves, p. 175, where there is a discussion of straight-edge constructions. In Figure 3 we are given segment XY with mid-point Z and a point P not on the line XY, and we construct the line through P parallel to XY.

(c) *Construct the line through a focus that is perpendicular to a chord through that focus, given the conic, focus, and corresponding directrix.* This is problem #2 of the J.I.R. McKnight Problems Contest 1982, [1998 : 232]. [Begin by extending the given chord BN beyond N to the point R where it meets the directrix. In Figure 4, RS is the directrix and SN is the desired perpendicular to BN. Perpendicularity is a consequence of the projective property that conjugate lines through a focus are perpendicular. Alternatively, Konečný verified that $BN \perp NS$ using coordinates.]

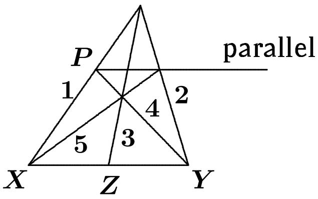


Figure 3

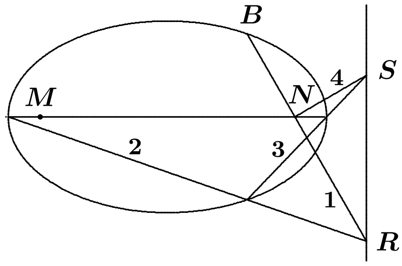


Figure 4

Here is the construction. (See Figure 5.)

1. Draw MN (the major axis), then construct [as in solution I] the minor axis l of the ellipse (the line that is perpendicular to the major axis MN at the centre O of the ellipse).

2. Using (b), construct the line through the focus N parallel to the minor axis (whose mid-point is O). Denote by N' one of the points where this line meets the conic.

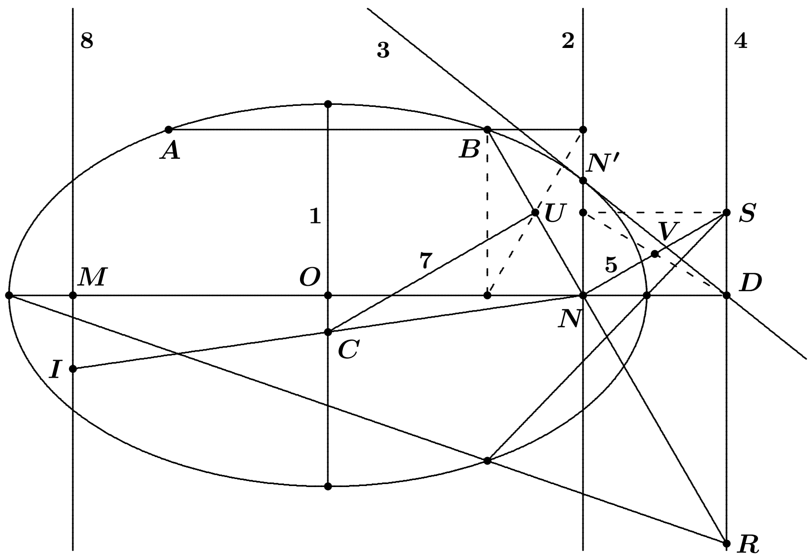


Figure 5

3. By (a), construct the tangent at N'. Call D the point where this tangent intersects the extension of the major axis. Then D lies on the directrix, since it is conjugate to N.

4. By (b), construct the line through D parallel to the minor axis. This line is the directrix corresponding to N.

5. By (c), draw the perpendicular NS to BN at N, where S is on the directrix.

6. We need to construct the mid-points U of BN and V of NS. Since O is the mid-point of the major axis, we can construct the parallel to the major axis through S. Since O is the mid-point of the minor axis, we can construct the parallel to the minor axis through B. These lines complete a pair of rectangles, one with corners B and N whose diagonals intersect at U, the other with corners N and S whose diagonals intersect at V.

7. Since V is the mid-point of NS, we can construct the parallel to NS through U. This is the perpendicular bisector of NB. It therefore passes through the centre of circle $ABNM$, which must be the point C where it intersects the minor axis.

8. Draw the parallel through M to the minor axis, and let I be the point where this line intersects NC. By symmetry, NI is a diameter of circle $ABNM$.

Also solved by VÁCLAV KONEČNÝ, Big Rapids, MI, USA (a second solution); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

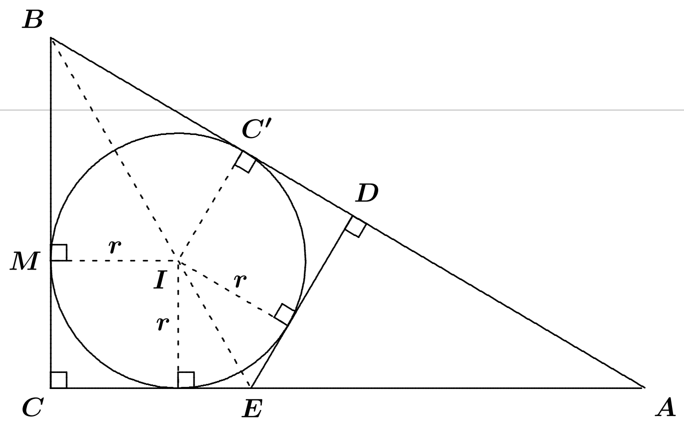
2745. [2002 : 246] *Proposed by K.R.S. Sastry, Bangalore, India.*

Let ABC be a primitive Pythagorean triangle (that is, the gcd of the sides is 1) in which $\angle ACB$ is the right angle. Let D be a point in AB and E a point in AC such that DE is perpendicular to AB and also tangent to the incircle of $\triangle ABC$.

Prove that BE has rational length if and only if the length of AB is the square of an integer.

Combination of solutions by Šefket Arslanagić, University of Sarajevo, Sarajevo, Bosnia and Herzegovina and Li Zhou, Polk Community College, Winter Haven, FL, USA.

Let a, b, and c be the lengths of the sides BC, CA, and AB, respectively. We show that the problem is correct under the extra assumption that b is even.



First, we show that BE is the bisector of $\angle ABC$. Let C' and M be the points of contact of the incircle of $\triangle ABC$ and the lines AB and BC. Then $BM = BC'$. If r is the radius of the incircle, then

$$BC = BM + MC = BM + r = BC' + r = BC' + C'D = BD\,.$$

Similarly, $EC = ED$. Hence, BE is the bisector of $\angle ABC$.

It is well-known that

$$BE = \frac{2\sqrt{ac\,s(s-b)}}{a+c}\,,$$

where s is the semiperimeter of $\triangle ABC$. We have

$$\begin{aligned} BE &= \frac{\sqrt{ac(a+b+c)(a-b+c)}}{a+c} = \frac{\sqrt{ac[(a+c)^2-b^2]}}{a+c} \\ &= \frac{\sqrt{ac[(a+c)^2-(c^2-a^2)]}}{a+c} = \frac{\sqrt{2a^2c(a+c)}}{a+c} = a\sqrt{\frac{2c}{a+c}}\,. \end{aligned}$$

If b is even, then $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, where m and n are positive integers with $m > n$ and $\gcd(m,n) = 1$, so that $BE = \frac{(m^2-n^2)\sqrt{c}}{m}$ is rational if and only if c is a square. If b is odd, the problem statement is incorrect. For example, if $a = 24$ and $b = 7$, then $c = 25$, and $BE = \frac{120\sqrt{2}}{7}$ is not rational.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2751. [2002 : 328] *Proposed by Bill Sands, University of Calgary, Calgary, AB.*

On each side of $\triangle ABC$, draw squares outwards to create six new points, D, E, F, G, H, and I. Characterise those triangles such that the points D, E, F, G, H, and I are concyclic.

I. *Solution by Toshio Seimiya, Kawasaki, Japan.*

We assume that D, E, F, G, H, and I are concyclic and labelled so that the squares are $ABHI$, $BCDE$, and $CAFG$. Since F, G, H, and I are concyclic, we have $\angle HIG = \angle HFG$. Since $\angle HIA = \angle AFG = 90^\circ$, we get $\angle AIG = \angle AFH$. If these angles are not zero, we have $\triangle AIG \sim \triangle AFH$ (because $\angle IAG = \angle FAH$). This implies $AI : AG = AF : AH$; that is, $AB : \sqrt{2}AC = AC : \sqrt{2}AB$, which then implies that $AB = AC$. On the other hand, if $\angle AIG = \angle AFH = 0$, then I, A, G are collinear, as are F, A, H. In this case, $\angle BAC = 45^\circ$.

Similarly, since D, E, F, G concyclic, we have either $AC = BC$ or $\angle ACB = 45^\circ$. Combining these conditions we get the following four cases.

Case 1. $AB = AC$ and $AC = BC$. Thus, $\triangle ABC$ is equilateral.

Case 2. $AB = AC$ and $\angle ACB = 45^\circ$. Then $\triangle ABC$ is an isosceles right triangle with right angle at A.

Case 3. $\angle BAC = 45^\circ$ and $AC = BC$. Then, as in Case 2, $\triangle ABC$ is an isosceles right triangle with right angle at C.

Case 4. $\angle BAC = 45^\circ$ and $\angle ACB = 45^\circ$. Then $\triangle ABC$ is an isosceles right triangle with right angle at B.

We conclude that $\triangle ABC$ is either equilateral or it is an isosceles right triangle. The converse, if $\triangle ABC$ is either an equilateral triangle or an isosceles right triangle then D, E, F, G, H, and I are concyclic, is straightforward.

Editor's comment: Seimiya provided the details for the converse, but we leave them for the reader in order to make room for an alternative solution.

II. *Combination of solutions by Christopher J. Bradley, Clifton College, Bristol, UK and D.J. Smeenk, Zaltbommel, the Netherlands.*

If D, E, F, G, H, I are concyclic, then the centre of circle $DEFGHI$ must lie on the perpendicular bisector of DE, and hence, on the perpendicular bisector of BC. Using the same argument for FG (with AC), we conclude that the centre must be the circumcentre O of $\triangle ABC$. Thus, the proposed problem is equivalent to characterizing those triangles such that

$$OD = OF = OH . \qquad (1)$$

Let M be the mid-point of BC and M' the mid-point of DE. Let R

be the radius of the circumscribed circle of $\triangle ABC$. We have

$$\begin{aligned} OD^2 &= OM'^2 + M'D^2 \;=\; (OM + MM')^2 + M'D^2 \\ &= (R\cos A + 2R\sin A)^2 + (R\sin A)^2 \\ &= R^2(3 + 2\sin 2A - 2\cos 2A)\,, \end{aligned}$$

With similar calculations for OF and OH, we can rewrite (1) as

$$\sin 2A - \cos 2A \;=\; \sin 2B - \cos 2B \;=\; \sin 2C - \cos 2C\,. \qquad (2)$$

One possibility for equality is $\angle A = \angle B = \angle C$.

Suppose that $\angle A \neq \angle B$. Since $\sin 2A - \cos 2A = \sin 2B - \cos 2B$ can be written as $\sin(2A - 45^\circ) = \sin(2B - 45^\circ)$, we conclude that $(2A - 45^\circ) + (2B - 45^\circ) = 180^\circ$. In this case $C = 45^\circ$. Now consider angles A and C. Either $A = C\,(= 45^\circ)$ or $A \neq C$, in which case the same argument as above using (2) gives us $B = 45^\circ$. In either case, $\triangle ABC$ is an isosceles right triangle.

We conclude that the remote vertices of the squares are concyclic if and only if $\triangle ABC$ is either an equilateral triangle or an isosceles right triangle.

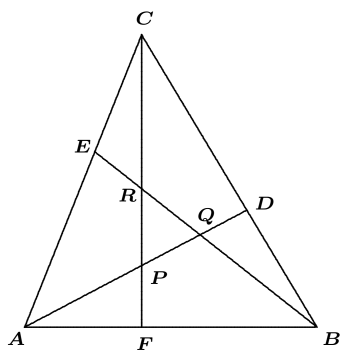
Also solved by MICHEL BATAILLE, Rouen, France; ROBERT BILINSKI, Outremont, QC; IAN JUNE L. GARCES and WINFER C. TABARES, Manila, The Philippines; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; BRUCE SHAWYER, Memorial University of Newfoundland, St. John's, NL; PAUL YIU, Florida Atlantic University, Boca Raton, FL, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Loeffler, using an argument as in Solution II, explored what would happen if the squares were constructed inwards. He easily found that there would be no non-trivial solutions. With one inward and two outward squares there are various degenerate triangles, but also the isosceles right triangle with an inward square on the hypotenuse (in which case E coincides with I, and D coincides with F). Finally, with two squares constructed inward and one outward there are again various trivial solutions along with a solution triangle having angles 30°, 75°, and 75°.

2752. [2002 : 329] *Proposed by Bruce Shawyer, Memorial University of Newfoundland, St. John's, NL.*

A generalization of Putnam 2001, question A4.

Suppose that $\dfrac{AF}{FB} = i$, $\dfrac{BD}{DC} = g$ and $\dfrac{CE}{EA} = h$. Determine the area of $\triangle PQR$ as a proportion of the area of $\triangle ABC$.



Solution by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.

The content of this problem is a very famous one; indeed, its answer is given by the following:

Routh's Theorem. *With the notation of the problem, we have:*

$$\frac{[PQR]}{[ABC]} = \frac{(ghi-1)^2}{(gh+g+1)(hi+h+1)(ig+i+1)}$$

and

$$\frac{[DEF]}{[ABC]} = \frac{ghi+1}{(g+1)(h+1)(i+1)}\,.$$

where $[XYZ]$ represents the area of the figure XYZ.

Remark. As two consequences of this theorem, we have:

- **Ceva's Theorem**. The transversals AD, BE, and CF are concurrent if and only if $ghi = 1$.
- **Menelaus' Theorem**. The three points D, E, and F on the (extended) sides of $\triangle ABC$ are collinear if and only if $ghi = -1$.

Reference.

[1] Z.A. Melzak, *Introduction to Geometry*, John Wiley and Sons, New York, 1983.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PIERRE BORNSZTEIN, Pontoise, France; PAUL BRACKEN, CRM, Université de Montréal, Montréal, QC; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; ZELJKO HANJŠ, University of Zagreb, Zagreb, Croatia; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; VÁCLAV KONEČNÝ, Big Rapids, MI, USA; RICK MABRY, LSU, Shreveport, LA, USA; DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; STAN WAGON, Macalester College, St. Paul, MN, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer.

About half of the solvers actually gave a proof, sometimes with a reference, and the rest gave multiple references to the literature. Francisco Bellot Rosado is very interested in obtaining a copy of L.A. Graham, *Ingenious Mathematical Problems and Methods,* Dover Publications, NY, 1950, *where the problem is treated in several ways. Can anyone help him? Contact* fbellot@hotmail.com.

2753. [2002 : 329] *Proposed by Mikhail Kotchetov, Memorial University of Newfoundland, St. John's, NL.*

Consider two circles, Γ_1 and Γ_2, centres O_1 and O_2, respectively, of different radii.

The two common tangents, t_1 and t_2, that do not intersect the line segment O_1O_2 meet at Q. A common tangent, t_c that does intersect the line segment O_1O_2 meets the tangents t_1 and t_2 at E_1 and E_2, respectively.

Let P be the mid-point of the line segment O_1O_2.

Prove that P, Q, E_1, and E_2 are concyclic.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Suppose, without loss of generality, that Γ_2 has smaller radius than Γ_1. Consider $\triangle E_1E_1Q$. Its incircle is Γ_2 and its excircle opposite Q is Γ_1.

It is well known that the mid-point of the line segment joining the incentre to an excentre lies on the circumcircle of the triangle. Thus, P, the mid-point of O_1O_2, lies on the circumcircle of $\triangle E_1E_2Q$.

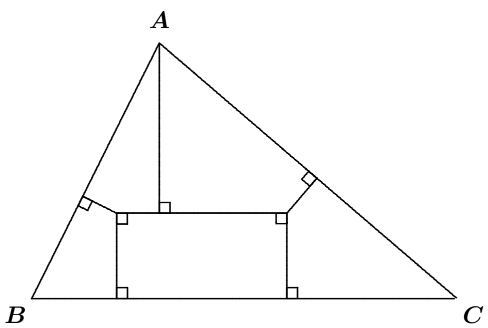
Also solved by MICHEL BATAILLE, Rouen, France; P. BAUTISTA and IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2754. [2002 : 330] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Divide a triangle into five concyclic quadrilaterals.

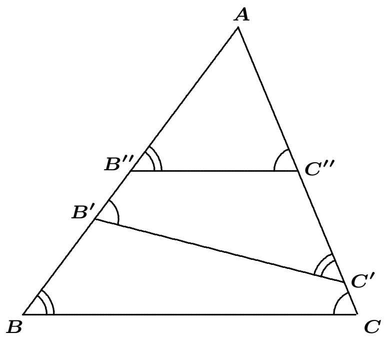
I. *Virtually identical solutions by Walther Janous, Ursulinengymnasium, Innsbruck, Austria and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let the triangle be ABC; assume that $\angle A$ is the largest angle. Then one class of solutions is given in the diagram:



II. *Combination of solutions by John G. Heuver, Grande Prairie, AB, David Loeffler, student, Trinity College, Cambridge, UK, Peter Y. Woo, Biola University, La Mirada, CA, USA, and Titu Zvonaru, Bucharest, Romania.*

We will again let the triangle be ABC, and assume that $\angle A$ is the largest angle. Let P be any interior point (for example, the incentre of $\triangle ABC$) such that the perpendiculars dropped from P to the sides of $\triangle ABC$ intersect the sides at an interior point of each side. The three quadrilaterals into which these perpendiculars subdivide $\triangle ABC$ are all clearly concyclic.



Now construct any line $B'C'$ where B' lies on AB and C' lies on AC such that $\angle AB'C' = \angle ACB$ and $\angle AC'B' = \angle ABC$. Also construct the line $B''C''$ parallel to BC where B'' lies on AB' and C'' lies on AC'. (See the diagram above.) Clearly, the quadrilaterals $B'C'CB$ and $B''C''C'B'$ are both concyclic. Furthermore, using the argument in the first paragraph we can subdivide $\triangle AB''C''$ into three concyclic quadrilaterals. This gives us a class of solutions to the problem.

Note that the number five in the problem statement can be replaced by any integer greater than or equal to 3.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; TOSHIO SEIMIYA, Kawasaki, Japan; and the proposer.

Woo remarks: "For acute triangles, it seems possible to draw the quadrilaterals so that they have the same area. I have not found out the condition on ABC for which this is possible." *Perhaps a reader can help to answer this question.*

2755. [2002 : 180] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Evaluate

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{f_{n+1}^2}{1 + f_n f_{n+1}^2 f_{n+2}}\right)$$

where f_n is the n^{th} Fibonacci number (that is, $f_0 = 0$, $f_1 = 1$ and, for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$).

Solution by D. Kipp Johnson, Beaverton, OR, USA.
We have

$$f_{n+1}^2 = (f_{n+2} - f_n)f_{n+1} = f_{n+2}f_{n+1} - f_{n+1}f_n \,.$$

Using the trigonometric identity

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1} x - \tan^{-1} y \,,$$

we may write

$$\begin{aligned} \tan^{-1}\left(\frac{f_{n+1}^2}{1+f_n f_{n+1}^2 f_{n+2}}\right) &= \tan^{-1}\left(\frac{f_{n+2}f_{n+1} - f_{n+1}f_n}{1+f_{n+2}f_{n+1}\cdot f_{n+1}f_n}\right) \\ &= \tan^{-1}(f_{n+2}f_{n+1}) - \tan^{-1}(f_{n+1}f_n) \,. \end{aligned}$$

Let S be the sum we seek. Then

$$\begin{aligned} S &= \lim_{k\to\infty} \sum_{n=1}^{k} \left(\tan^{-1}(f_{n+2}f_{n+1}) - \tan^{-1}(f_{n+1}f_n)\right) \\ &= \lim_{k\to\infty} \left(\tan^{-1}(f_{k+2}f_{k+1}) - \tan^{-1}(f_2 f_1)\right) \,, \end{aligned}$$

since the sum telescopes. Now the product $f_{k+2}f_{k+1}$ increases without bound and $f_2 f_1 = 1$. Hence,

$$S = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; OVIDIU FURDUI, Kalamazoo, MI, USA; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Howard points out that the problem can be solved by an application of the results in L. Bragg, Arctangent Sums, *College Math. J.* **32**(4) (2001), *pp. 255–56.*

2756. [2002 : 330] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Circles $\Gamma_1(O,R)$ and $\Gamma_2(I,r)$ touch line t at D, where $R > r$ and O and I lie on the same side of t. The point A is any point on Γ_1. The tangents to Γ_2 through A intersect t at B and C, respectively. Denote the inradii of $\triangle ABD$ and $\triangle ACD$ by r_1 and r_2, respectively.

Show that $r_1 + r_2$ is constant as A varies on Γ_1.

Comment.

Several solvers noticed that this problem and its solution have appeared before in Crux as Problem 2320 [1998 : 108; 1999 : 126–127].

Solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Janous has noted that the claim is true only if the point A moves along the arc of the circle Γ_2 lying above the tangent to the circle Γ_1 parallel to the common tangent at D.

2757★. [2002 : 331] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let A, B, and C be the angles of a triangle. Show that

$$\sum_{\text{cyclic}} \frac{1}{\tan\left(\frac{A}{2}\right) + 8\tan\left(\frac{\pi-A}{4}\right)^3} \;\le\; \frac{9\sqrt{3}}{11}\,.$$

Solution by Manuel Benito, Oscar Ciaurri and Emilio Fernández, Logroño, Spain (modified by the editor).

Define the function f on $[0,\pi]$ by $f(\pi) = 0$, and for $0 \le x < \pi$,

$$f(x) \;=\; \frac{1}{\tan\left(\dfrac{x}{2}\right) + 8\tan^3\left(\dfrac{\pi-x}{4}\right)}$$

Then clearly $f(x) > 0$ for all $x \in [0,\pi]$. We shall prove that the maximum of the function $J(x,y,z) = f(x) + f(y) + f(z)$ over the compact set

$$T \;=\; \{(x,y,z) \in \mathbb{R}^3 : 0 \le x,y,z \le \pi\,;\;\; x+y+z = \pi\}$$

is equal to $\dfrac{9\sqrt{3}}{11} \approx 1.41713$. Our proof consists of five parts.

I. Variation of $f(x)$ on $[0,\pi]$.

By direct computation, we find that

$$\begin{aligned} f'(x) &= -\big(f(x)\big)^2 \left[\frac{1}{2}\sec^2\left(\frac{x}{2}\right) + 24\tan^2\left(\frac{\pi-x}{4}\right)\sec^2\left(\frac{\pi-x}{4}\right)\left(-\frac{1}{4}\right)\right] \\ &= \frac{1}{2}\big(f(x)\big)^2\left[12\tan^2\left(\frac{\pi-x}{4}\right)\left(1+\tan^2\left(\frac{\pi-x}{4}\right)\right) - 1 - \tan^2\left(\frac{x}{2}\right)\right] \,. \quad (1) \end{aligned}$$

Set $r = \tan\left(\dfrac{x}{2}\right)$, $s = \tan\left(\dfrac{\pi-x}{4}\right)$ and $t = \tan\left(\dfrac{x}{4}\right)$. Then for $x \ne \pi$ we have $r = \dfrac{2t}{1-t^2}$, and $s = \dfrac{1-t}{1+t}$. Eliminating t, we get $r = \dfrac{1-s^2}{2s}$. Since $f(x) > 0$ for $x \in [0,\pi)$, setting $f'(x) = 0$ yields

$$\begin{aligned} 12s^2(1+s^2) \;&=\; 1 + \left(\frac{1-s^2}{2s}\right)^2 \;=\; \frac{(1+s^2)^2}{4s^2} \\ \text{or}\quad (1+s^2)(48s^4 - s^2 - 1) \;&=\; 0\,. \end{aligned}$$

Solving $48s^4 - s^2 - 1 = 0$, we get $s^2 = \dfrac{1+\sqrt{193}}{96}$. Thus, the only positive root is $s_0 = \sqrt{\dfrac{1+\sqrt{193}}{96}}$. Hence, $\xi_0 = \pi - 4\tan^{-1}(s_0) \approx 1.64076$ is the only critical value of $f(x)$ on $(0,\pi)$. Since $f(0) = 0.125$, $f(\pi) = 0$, and $f(\xi_0) \approx 0.640475$, we conclude that

$$\max\{f(x) : 0 \le x \le \pi\} \;=\; f(\xi_0) = M\,.$$

II. Values of $J(x,y,z)$ on ∂T, the boundary of T.
At the vertices of ∂T, we have

$$J(\pi,0,0) \;=\; J(0,\pi,0) \;=\; J(0,0,\pi) \;=\; 0.25\,.$$

In the interior of ∂T, one of the three coordinates is zero, and therefore, for all $\xi \in (0,\pi)$, we have

$$J(x,y,z) \;=\; f(0) + f(\xi) + f(\pi-\xi) \;\le\; 0.125 + 2M \;<\; 1.406 \;<\; \frac{9\sqrt{3}}{11}\,.$$

III. Variation of $f'(x)$ on $[0,\pi]$.
Differentiating (1) we obtain, after some simplifications,

$$f''(x) \;=\; f(x)\left[f'(x)A(x) + \tfrac{1}{2}f(x)B(x)\right]\,,$$

where

$$A(x) \;=\; 12\tan^4\left(\frac{\pi-x}{4}\right) + 12\tan^2\left(\frac{\pi-x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\,,$$

and

$$\begin{aligned} B(x) \;=\; A'(x) \;=\; & -12\tan^5\left(\frac{\pi-x}{4}\right) - 18\tan^3\left(\frac{\pi-x}{4}\right) \\ & - 6\tan\left(\frac{\pi-x}{4}\right) - \tan^3\left(\frac{x}{2}\right) - \tan\left(\frac{x}{2}\right)\,. \end{aligned}$$

We set $f''(x) = 0$ to search for solutions in $(0,\pi)$. Using the definition of $f(x)$, the fact that $f(x) > 0$, and the right hand side of (1) to substitute for $f'(x)$, we are led to the equation:

$$\begin{aligned} & \left[12\tan^4\left(\frac{\pi-x}{4}\right) + 12\tan^2\left(\frac{\pi-x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\right]^2 \\ & = \left[8\tan^3\left(\frac{\pi-x}{4}\right) + \tan\left(\frac{x}{2}\right)\right]\left[12\tan^5\left(\frac{\pi-x}{4}\right) + 18\tan^3\left(\frac{\pi-x}{4}\right)\right. \\ & \qquad \left. + 6\tan\left(\frac{\pi-x}{4}\right) + \tan^3\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right)\right]\,. \end{aligned} \qquad (2)$$

In terms of the variables r, $s \in (0,1)$ introduced earlier, equation (2) becomes

$$(12s^4 + 12s^2 - r^2 - 1)^2 \;=\; (8s^3 + r)(12s^5 + 18s^3 + 6s + r^3 + r)\,. \qquad (3)$$

Substituting $r = \dfrac{1-s^2}{2s}$ and $1+r^2 = \dfrac{(1+s^2)^2}{4s^2}$ into (3), we then have

$$\left(12s^4 + 12s^2 - \frac{\left(1+s^2\right)^2}{4s^2}\right)^2 = \left(8s^3 + \frac{1-s^2}{2s}\right)\left(12s^5 + 18s^3 + 6s + \frac{\left(1-s^2\right)\left(1+s^2\right)^2}{8s^3}\right)$$

or $$\left(1+s^2\right)^2\left(48s^4 - s^2 - 1\right)^2 = \left(16s^4 - s^2 + 1\right)\left[8s^3\left(12s^5 + 18s^3 + 6s\right) + \left(1-s^2\right)\left(1+s^2\right)^2\right].$$

Simplifying, we obtain

$$4s^2(s^2+1)(192s^8 + 388s^6 - 60s^4 - 39s^2 + 1) = 0. \qquad (4)$$

[*Ed:* This can be checked and verified easily using MAPLE or some other computer algebra system.]

Equation (4) has exactly two solutions in $(0,1)$, namely: $s_1 \approx 0.595805$ and $s_2 \approx 0.157625$, yielding the following two critical values of $f'(x)$: $\xi_1 = \pi - 4\tan^{-1}(s_1) \approx 0.992276$ and $\xi_2 = \pi - 4\tan^{-1}(s_2) \approx 2.51623$, respectively. Also, for the argument to be used in Part IV below, we need to find the value ξ_3 in $(0,\pi)$ such that $f'(\xi_3) = f'(0) = \frac{23}{128}$. This condition leads to the following equation in s:

$$(s^2+1)(48s^4 - s^2 - 1) = \frac{23}{64}(16s^4 - s^2 + 1)^2$$

or $$(1-s^2)(5888s^6 + 2080s^4 - 169s^2 - 87) = 0.$$

The second factor in the equation above has a *unique* positive root $s_3 \approx 0.439150$ which yields a unique $\xi_3 = \pi - 4\tan^{-1}(s_3) \approx 1.48641$. Incorporating all the information obtained above, we can summarize the variation of $f'(x)$ on $[0,\pi]$ in the chart below:

x	0	$\cdots$	ξ_1 ≈ 0.99	$\cdots$	ξ_3 ≈ 1.49	$\cdots$	ξ_0 ≈ 1.64	$\cdots$	ξ_2 ≈ 2.51	$\cdots$	π
$f'(x)$	$\frac{23}{128}$	$\nearrow$	≈ 0.449	$\searrow$	$\frac{23}{128}$	$\searrow$	0	$\searrow$	≈ -0.526	$\nearrow$	-0.5

IV. A necessary condition for the relative extrema of $J(x,y,z)$ to occur in the interior of T.

Suppose $J(x,y,z)$ attains a relative extremum at the interior point (x_0,y_0,z_0) of T. Then the method of Lagrange multipliers assures that there exists a $\lambda_0 \in \mathbb{R}$ such that

$$L_x(x_0,y_0,z_0,\lambda_0) = L_y(x_0,y_0,z_0,\lambda_0) = L_z(x_0,y_0,z_0,\lambda_0) = 0,$$

where $L(x,y,z,\lambda) = f(x) + f(y) + f(z) - \lambda(x+y+z-\pi)$. Thus, we have $f'(x_0) = f'(y_0) = f'(z_0) = \lambda_0$ with $0 \le x_0,\ y_0,\ z_0 \le \pi$ such that $x_0 + y_0 + z_0 = \pi$.

We explore the possibilities for these conditions to be satisfied by using the information on the variation of $f'(x)$ to look at the possible location of the value λ_0 in the range of $f'(x)$.

If $f'(\xi_2) \le \lambda_0 < \frac{23}{128} = f'(\xi_3)$, then $f'(x_0) = f'(y_0) = f'(z_0) = \lambda_0$ would imply that $x_0 + y_0 + z_0 > 3\xi_3 > \pi$, which is a contradiction. Furthermore, if $\lambda_0 = f'(\xi_1)$, then $x_0 = y_0 = z_0 = \xi_1$ implies that $3\xi_1 = \pi$, another contradiction. Hence, we must have $\frac{23}{128} \le \lambda_0 < f'(\xi_1)$, in which case $f'(x) = \lambda_0$ has exactly two solutions in $[0, \xi_3]$. Therefore, in order for $f'(x_0) = f'(y_0) = f'(z_0) = \lambda_0$ to hold, at least two of x_0, y_0, and z_0 must be the same, and the problem now reduces to finding the maximum value on $[0, \xi_3]$ of the single variable function $\tilde{J}(x) = 2f(x) + f(\pi - 2x)$.

V. Maximum value of $\tilde{J}(x) = 2f(x) + f(\pi - 2x)$ on $[0, \xi_3]$.

Since $\tilde{J}'(x) = 2f'(x) - 2f'(\pi - 2x)$, we see that $\tilde{J}'(x) = 0$ if and only if $f'(x) = f'(\pi - 2x)$. Using (1), this is equivalent to:

$$\begin{aligned}&(f(x))^2 \left[12\tan^4\left(\frac{\pi - x}{4}\right) + 12\tan^2\left(\frac{\pi - x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\right]\\ &\quad = \left[f(\pi - 2x)\right]^2 \left[12\tan^4\left(\frac{x}{2}\right) + 12\tan^2\left(\frac{x}{2}\right) - \cot^2 x - 1\right]\end{aligned}$$

or

$$\begin{aligned}&\left[\cot x + 8\tan^3\left(\frac{x}{2}\right)\right]^2 \left[12\tan^4\left(\frac{\pi - x}{4}\right) + 12\tan^2\left(\frac{\pi - x}{4}\right) - \tan^2\left(\frac{x}{2}\right) - 1\right]\\ &\quad = \left[\tan\left(\frac{x}{2}\right) + 8\tan^3\left(\frac{\pi - x}{4}\right)\right]^2 \cdot\\ &\qquad \left[12\tan^4\left(\frac{x}{2}\right) + 12\tan^2\left(\frac{x}{2}\right) - \cot^2 x - 1\right]. \qquad (5)\end{aligned}$$

Again using $r = \tan\left(\frac{x}{2}\right)$ and $s = \tan\left(\frac{\pi - x}{4}\right)$, we have

$$\cot x = \frac{1}{\tan 2\left(\frac{x}{2}\right)} = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{2\tan\left(\frac{x}{2}\right)} = \frac{1 - r^2}{2r} = \frac{-s^4 + 6s^2 - 1}{4s(1 - s^2)}.$$

Substituting into (5) and simplifying, we obtain the following equation in s, where $s \in (0, 1)$,

$$\begin{aligned}&(4 - 17s^2 + 30s^4 - 17s^6 + 4s^8)^2(48s^6 + 47s^4 - 2s^2 - 1)\\ &= s^2(1 - s^2 + 16s^4)^2 \cdot\\ &\quad \left[12(1 - s^2)^6 + 48s^2(1 - s^2)^4 - s^2(-s^4 + 6s^2 - 1)^2 - 16s^4(1 - s^2)^2\right],\end{aligned}$$

or, on further simplifying and factoring,

$$\begin{aligned}&4(s^2 + 1)(1 - 3s^2)(192s^{18} - 212s^{16} - 1617s^{14} + 4406s^{12}\\ &\qquad - 5404s^{10} + 3258s^8 - 1136s^6 + 154s^4 + 15s^2 - 4) = 0.\end{aligned}$$

On the interval $(0, 1)$, the last equation has exactly three solutions, namely, $\sigma_0 = \frac{1}{\sqrt{3}}$, $\sigma_1 \approx 0.521949$ and $\sigma_2 \approx 0.477039$ yielding the three critical values of $\tilde{J}(x)$ in $(0, \xi_3)$:

$$x_0 = \pi - 4\tan^{-1}(\sigma_0) = \frac{\pi}{3},$$

$$x_1 = \pi - 4\tan^{-1}(\sigma_1) \approx 1.21738,$$

and $$x_2 = \pi - 4\tan^{-1}(\sigma_2) \approx 1.36115.$$

By direct computations, we find that

$$\tilde{J}(x_0) = \frac{9\sqrt{3}}{11}, \quad \tilde{J}(x_1) \approx 1.41514, \quad \text{and} \quad \tilde{J}(x_2) \approx 1.41615.$$

Since $\tilde{J}(0) = 2f(0)+f(\pi) = 0.25$ and $\tilde{J}(\xi_3) = 2f(\xi_3)+f(\pi-\xi_3) \approx 1.4116$, we finally conclude that the maximum value of $\tilde{J}(x)$ on $[0, \xi_3]$, and consequently of the function $J(x, y, z)$ over T, is $\frac{9\sqrt{3}}{11}$, which is attained if and only if $x = y = z = \frac{\pi}{3}$.

There was one incorrect solution.

2758. [2002 : 331] *José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

If $\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \neq 2$, determine all real numbers x, y, z such that

$$\begin{aligned} 0 = {} & (1+2a^2)x^2 + (1+2b^2)y^2 + (1+2c^2)z^2 \\ & + 2xy(ab-a-b) + 2yz(bc-b-c) + 2zx(ca-c-a). \end{aligned}$$

Solution by Michel Bataille, Rouen, France.

The given equation can be written as

$$(ax+by-z)^2 + (ax-y+cz)^2 + (-x+by+cz)^2 = 0.$$

It follows that its solutions are those of the homogeneous linear system

$$\begin{aligned} ax + by - z &= 0, \\ ax - y + cz &= 0, \\ -x + by + cz &= 0. \end{aligned}$$

The determinant of this system is equal to $1 - ab - bc - ca - 2abc$. The condition

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \neq 2$$

is equivalent to $1 - ab - bc - ca - 2abc \neq 0$. Therefore, $x = y = z = 0$ is the only solution of the homogeneous system and the only solution of the given equation.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; JOE HOWARD, Portales, NM, USA; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposers.

2759. [2002 : 331] *Proposed by Michel Bataille, Rouen, France.*

On the line segment AB, let C, D be such that $\frac{AC}{CB} = \frac{BD}{DA} = \frac{1}{3}$. Distinct points M_1, M_2, M_3 lie on a circle passing through B and C and are such that $\angle M_1BC = 2\angle M_1CB$, $\angle M_2BC = 2\angle M_2CB$, and $\angle M_3AD = 2\angle M_3DA$. Show that $\triangle M_1M_2M_3$ is equilateral.

Solution by Toshio Seimiya, Kawasaki, Japan.

Lemma 1. Let $\triangle ABC$ be a triangle with $\angle B = 2\angle C$; let D be the foot of the perpendicular from A to BC; and let M be the mid-point of BC. Then $AB = 2DM$.

Proof. We set $\alpha = \angle C$. Then $\angle B = 2\alpha$. Let N be the mid-point of AC. Then NM is parallel to AB, $NM = \frac{1}{2}AB$, and $ND = NA = NC$. Thus, $\angle NDC = \angle NCD = \alpha$. Since $NM \| AB$, $\angle NMC = \angle ABC = 2\alpha$. Therefore, $\angle MND = \angle NMC - \angle NDC = 2\alpha - \alpha = \alpha = \angle MDN$, so that $DM = NM = \frac{1}{2}AB$; that is, $AB = 2DM$.

Lemma 2. Let $\triangle ABC$ be a triangle with $AB < AC$; let D be the foot of the perpendicular from A to BC; and let M be the mid-point of BC. If $AC = 2DM$, then $\angle ABC = 90^\circ + \frac{1}{2}\angle ACB$.

Proof. Let N be the mid-point of AC. Then NM is parallel to AB, and $ND = NA = NC$. Since $DM = \frac{1}{2}AC = DN$, we deduce that $\angle NMC = 90^\circ + \frac{1}{2}\angle NDM$. Since $ND = NC$, we obtain $\angle NDM = \angle NCD = \angle ACB$, so that $\angle NMC = 90^\circ + \frac{1}{2}\angle ACB$. Since $NM \| AB$, we have $\angle NMC = \angle ABC$. Thus, $\angle ABC = 90^\circ + \frac{1}{2}\angle ACB$.

Now we turn to the original problem. We assume that M_1 and M_2 lie on the major and minor arcs BC, respectively. Then M_3 lies on the major arc BC. We set $\alpha = \angle M_1CB$ and $\beta = \angle M_2CB$. Then $\angle M_1BC = 2\alpha$ and $\angle M_2BC = 2\beta$. Since $\angle M_1CM_2 + \angle M_1BM_2 = 180^\circ$, we have $(\alpha + \beta) + (2\alpha + 2\beta) = 180^\circ$. Thus $\alpha + \beta = 60^\circ$. We set $AB = 4a$; then $AC = a$, $CD = 2a$, and $DB = a$. Let T be the foot of the perpendicular from M_3 to AC. We set $x = TC$. Let M be the mid-point of AD, then $CM = \frac{1}{2}(CD - AC) = \frac{1}{2}(2a - a) = \frac{a}{2}$. Since $\angle M_3AD = 2\angle M_3DA$, we have (by virtue of lemma 1) $M_3A = 2TM = 2\left(x + \frac{1}{2}a\right) = 2x + a$. Since $M_3T \perp AB$, we get $M_3B^2 - M_3A^2 = TB^2 - TA^2$. Thus, we have

$$\begin{aligned} M_3B^2 &= M_3A^2 + TB^2 - TA^2 \\ &= (2x + a)^2 + (x + 3a)^2 - (a - x)^2 \\ &= 4x^2 + 12ax + 9a^2 \;=\; (2x + 3a)^2 . \end{aligned}$$

Hence, $M_3B = 2x + 3a$.

Let N be the mid-point of CB; then $TN = TC + CN = x + \frac{3a}{2}$, or $2TN = 2x + 3a = M_3B$. Set $\gamma = \angle M_3BC$. By lemma 2, we get $\angle M_3CB = 90^\circ + \frac{1}{2}\gamma$. Since $\angle M_3CM_2 + \angle M_3BM_2 = 180^\circ$, we have

$$\left(90^\circ + \tfrac{1}{2}\gamma + \beta\right) + (\gamma + 2\beta) \;=\; 180^\circ\,;$$

that is, $180^\circ + 3(\gamma + 2\beta) = 360^\circ$, from which we get $\gamma + 2\beta = 60^\circ$. Therefore, $\angle M_3M_1M_2 = \angle M_3BM_2 = \gamma + 2\beta = 60^\circ$, which implies that $\angle M_1M_3M_2 = \angle M_1CM_2 = \alpha + \beta = 60^\circ$. Hence, $\triangle M_1M_2M_3$ is equilateral.

Also solved by PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2760. [2002 : 332] (corrected in [2002 : 396]) *Proposed by Michel Bataille, Rouen, France.*

Suppose that A, B, C are the angles of a triangle. Prove that

$$\begin{aligned}8(\cos A + \cos B + \cos C) &\le 9 + \cos(A-B) + \cos(B-C) + \cos(C-A)\\ &\le \csc^2(A/2) + \csc^2(B/2) + \csc^2(C/2)\,.\end{aligned}$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.

Since $f(x) = \csc^2 x$ is convex on $(0,\pi)$ [*Ed:* Direct computations show that $f''(x) = 2\csc^4 x + 4\csc^2 x\cot^2 x > 0$], we have

$$\begin{aligned}\csc^2\left(\frac{A}{2}\right) + \csc^2\left(\frac{B}{2}\right) + \csc^2\left(\frac{C}{2}\right) &\ge 3\csc^2\left(\frac{A+B+C}{6}\right) \;=\; 12\\ &\ge 9 + \cos(A-B) + \cos(B-C) + \cos(C-A)\,.\end{aligned}$$

For the left inequality, note first that

$$\begin{aligned}&\cos A + \cos B + \cos C\\ &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\left(\frac{C}{2}\right)\\ &= 1 + 2\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]\cos\left(\frac{A+B}{2}\right)\\ &= 1 + 2\left[2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right]\sin\left(\frac{C}{2}\right)\\ &= 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \qquad (1)\end{aligned}$$

Also,

$$\begin{aligned}
&\cos(A-B)+\cos(B-C)+\cos(C-A)\\
&= 2\cos\left(\frac{A-C}{2}\right)\cos\left(\frac{A+C-2B}{2}\right)+2\cos^2\left(\frac{C-A}{2}\right)-1\\
&= 2\cos\left(\frac{C-A}{2}\right)\left[\cos\left(\frac{A+C-2B}{2}\right)+\cos\left(\frac{C-A}{2}\right)\right]-1\\
&= 4\cos\left(\frac{C-A}{2}\right)\cos\left(\frac{C-B}{2}\right)\cos\left(\frac{A-B}{2}\right)-1 \qquad (2)
\end{aligned}$$

From (1) and (2), we see that the left inequality is equivalent to

$$8\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \;\le\; \cos\left(\frac{A-B}{2}\right)\cos\left(\frac{B-C}{2}\right)\cos\left(\frac{C-A}{2}\right),$$

which has been shown at least three times previously in Crux: 585 [1981 : 303], 2472 [2000 : 440–441], and 2717 [2003 : 119–120].

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MIHÁLY BENCZE, Brasov, Romania; PIERRE BORNSZTEIN, Pontoise, France; SCOTT H. BROWN, Auburn University, Montgomery, AL, USA; JOE HOWARD, Portales, NM, USA (2 solutions); WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer. There was one incorrect solution.

Most solvers used various known results from the classic book "Geometric Inequalities" by O. Bottema et al. Of course, identity (1) in the solution featured above can also be found in this book (§2.16, p. 22). In particular, Klamkin, Lau, and Loeffler all cited Gerretsen's Inequality, $s^2 \ge 16rR - 5r^2$*, and the famous Euler Inequality,* $2r \le R$*, where* r *and* R *denote the inradius and circumradius of triangle* ABC*, respectively, and* s *denotes its semiperimeter.*

Bencze gave various refinements which were results that he obtained in 1995–96; for example, he showed that the left inequality can be refined to:

$$8F(a,b,c)\cdot\sum_{\text{cyclic}}\cos A \;\le\; 1+8F(a,b,c)+\sum_{\text{cyclic}}\cos(A-B)\,, \qquad (*)$$

where

$$F(a,b,c) \;=\; F \;=\; \left(1+\frac{(\sqrt{a}-\sqrt{b})^2}{8\sqrt{ab}}\right)\left(1+\frac{(\sqrt{b}-\sqrt{c})^2}{8\sqrt{bc}}\right)\left(1+\frac{(\sqrt{c}-\sqrt{a})^2}{8\sqrt{ca}}\right).$$

Clearly, $F(a,b,c) \ge 1$*. Since it is well known that* $\sum_{\text{cyclic}}\cos A > 1$*, we conclude that* $8(F-1)\cdot\sum_{\text{cyclic}}\cos A \ge 8(F-1)$ *which implies by* $(*)$ *that*

$$8\sum_{\text{cyclic}}\cos A \;\le\; 8-8F+8F\cdot\sum_{\text{cyclic}}\cos A \;\le\; 9+\sum_{\text{cyclic}}\cos(A-B)\,.$$

2761★. [2002 : 332] *Proposed by Edgar G. Goodaire, Memorial University, St. John's, NF.*

Give a proof by vectors that the medians of a triangle have a common point of intersection: a proof, however, ***which does not presuppose the answer***.

The vector proofs of this result with which I am familiar answer the question posed this way:

> Prove that the medians of $\triangle ABC$ intersect at $\frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right)$,
> where O is the origin.

The proof, of course, then amounts simply to showing that this point is on each median.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Let A be the origin, and let $\overrightarrow{b}$ and $\overrightarrow{c}$ be representatives of the vectors AB and AC. The midpoints L, M, and N of sides BC, CA, and AB have position vectors $\frac{1}{2}(\overrightarrow{b}+\overrightarrow{c})$, $\frac{1}{2}\overrightarrow{c}$ and $\frac{1}{2}\overrightarrow{b}$, respectively. The vector equations of AL, BM, and CN are as follows:

$$\begin{aligned} AL: \quad & \overrightarrow{r} = \tfrac{1}{2}s\left(\overrightarrow{b}+\overrightarrow{c}\right) \\ BM: \quad & \overrightarrow{r} = \overrightarrow{b}+t\left(\tfrac{1}{2}\overrightarrow{c}-\overrightarrow{b}\right) \\ CN: \quad & \overrightarrow{r} = \overrightarrow{c}+u\left(\tfrac{1}{2}\overrightarrow{b}-\overrightarrow{c}\right). \end{aligned}$$

These three lines pass through a common point if and only if there exist values of s, t, and u such that

$$\tfrac{1}{2}s(\overrightarrow{b}+\overrightarrow{c}) = (1-t)\overrightarrow{b}+\tfrac{1}{2}t\overrightarrow{c} = (1-u)\overrightarrow{c}+\tfrac{1}{2}u\overrightarrow{b}.$$

Since the vectors $\overrightarrow{b}$ and $\overrightarrow{c}$ are linearly independent, the above system of vector equations is equivalent to

$$\begin{aligned} \tfrac{1}{2}s &= (1-t) = \tfrac{1}{2}u, \\ \tfrac{1}{2}s &= \tfrac{1}{2}t = (1-u). \end{aligned}$$

The only solution is $s=u=t=\frac{2}{3}$.

Also solved by MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; M. Benito, O. Ciaurri and E. Fernández, Logroño, Spain; ELIAS BUISSANT DES AMORIE, CJ Castricum, the Netherlands; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines and GIOVANNI MAZZARELLO, Ferrovie dello Stato, Florence, Italy; DOUGLASS L. GRANT, University College of Cape Breton, Sydney, NS; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; GEOFFREY A. KANDALL, Hamden, CT, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB;

DAVID E. MANES, SUNY at Oneonta, Oneonta, NY, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. Most of the submitted solutions are similar to the above.

Bellot has located two solutions published earlier: E. Donath, Die merkwürdigen Punkte und Linien des ebenen Dreiecks, *VEB, Berlin, 1969 (in German), and E.M. Patterson, Vector Algebra, Oliver & Boyd, London, 1968.*

2762. [2002 : 332] *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Quadrilateral $ABCD$ is inscribed in circle Γ. The tangents at A, B, C, D to Γ are t_A, t_B, t_C, t_D, respectively. Given that BD, t_A, and t_C are concurrent, prove that AC, t_B, and t_D are concurrent.

Initial comment. Since t_B and t_D can be parallel, the problem's conclusion should be that these two tangents are concurrent with AC or are parallel to it.

I. *Nearly identical solutions by Michel Bataille, Rouen, France; David Loeffler, student, Trinity College, Cambridge, UK; and Peter Y. Woo, Biola University, La Mirada, CA, USA.*

Let BD, t_A, and t_C be concurrent at E. As the point of intersection of t_A and t_C, E is the pole of AC with respect to Γ. Since the pole of AC lies on BD, the pole of BD, call it F, lies on AC (because a polarity is an incidence-preserving involution). But F is the point where t_B and t_D meet. Thus, AC, t_B, and t_D are concurrent at F. Note that for this projective argument, Γ can be any conic, not just a circle.

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Suppose that BD, t_A, and t_C intersect at E. [Without any loss of generality, we may assume that B and E are on the same side of AC.] Let O be the centre of Γ, and suppose that OE intersects AC at M. Then $OE \perp AC$. If BD is a diameter of Γ, then AC, t_B, and t_D are all perpendicular to OE, in which case $AC \| t_B \| t_D$. Otherwise, AC intersects t_B at some point F. Then O, M, B, F are on a circle Ω with OF as a diameter. Hence, $\angle OFB = \angle EMB$. On the other hand, $EB \cdot ED = EC^2 = EO \cdot EM$; thus, $\triangle EBM \sim \triangle EOD$. Consequently, $\angle EDO = \angle EMB = \angle OFB$, which implies that D is on Ω as well. Therefore $OD \perp FD$, which means that FD coincides with t_D.

III. *Solution by Christopher J. Bradley, Clifton College, Bristol, UK.*

Take the circle Γ to have equation $x^2 + y^2 = 1$, and let

$$T = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$$

be a point on the circumference. The equations of the tangent at A [$t = a$], the tangent at C [$t = c$], and the line BD [from $t = b$ to $t = d$], respectively, are

$$\begin{aligned} t_A : &\quad (1-a^2)x + 2ay \;=\; 1+a^2\,; \\ t_C : &\quad (1-c^2)x + 2cy \;=\; 1+c^2\,; \\ BD : &\quad (1-bd)x + (b+d)y \;=\; 1+bd\,. \end{aligned}$$

If these are concurrent, then

$$\det \begin{bmatrix} 1-a^2 & 2a & -(1+a^2) \\ 1-c^2 & 2c & -(1+c^2) \\ 1-bd & b+d & -(1+bd) \end{bmatrix} \;=\; 0\,.$$

That is,

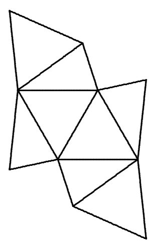
$$-2(a-c)(ab+ad+cb+cd-2ac-2bd) \;=\; 0\,.$$

Since $a \neq c$, we deduce that $(a+c)(b+d) = 2(ac+bd)$. This expression is symmetrical — it does not change when the pair of letters a and c is interchanged with the pair b and d. Consequently, AC, t_B, and t_D are concurrent (or parallel).

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA (a second solution); TITU ZVONARU, Bucharest, Romania; and the proposer.

2763. [2002 : 397] *Proposé par Izidor Hafner, Faculty of Electrical Engineering, Ljubljana, Slovénie.*

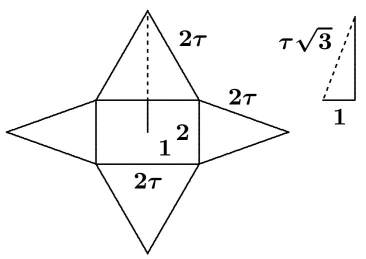
Le développement montré à la droite consiste en 4 triangles équilatéraux dont les côtés mesurent 2τ (deux fois le nombre d'or, $\tau = \frac{\sqrt{5}+1}{2}$), et 4 triangles isocèles dont le petit côté mesure 2. Noter qu'en pliant le développement, on peut obtenir deux polyèdres convexes. Ont-ils le même volume ?



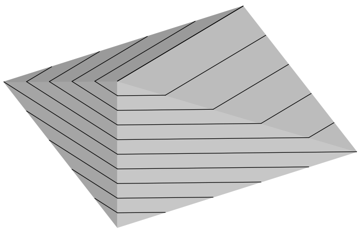
Solution de Robert Bilinski, Outremont, QC.

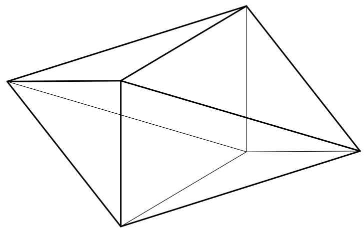
La première configuration est un octaèdre formé de 2 pyramides collées à la base. Le patron de chaque pyramide est formé d'un rectangle ayant pour côtés 2 et 2τ, d'une paire de triangles équilatéraux et d'une paire de triangles isocèles.

La hauteur des triangles équilatéraux mesure $\tau\sqrt{3}$. En utilisant la symétrie des pyramides, on peut calculer leur hauteur en formant un triangle rectangle composé d'une hauteur de triangle équilatéral et une demi-largeur allant de la base de la hauteur du côté au centre du rectangle.



La hauteur de la pyramide est donc $\sqrt{3\tau^2-1}$ par pythagore.





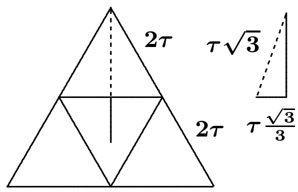
On a donc

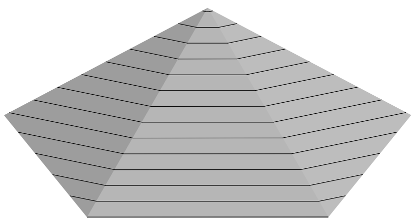
$$\text{Volume Pyramide} = \frac{\text{Base}\cdot\text{hauteur}}{3} = \frac{4\tau\sqrt{3\tau^2-1}}{3}$$

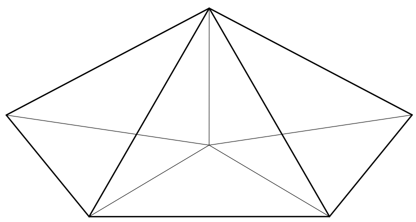
$$\text{et Volume Octaèdre} = \frac{8\tau\sqrt{3\tau^2-1}}{3} \approx 11{,}2962 \text{ unité}^2\,.$$

La deuxième configuration est formée de trois tétraèdres. Il y en a un qui est régulier d'arête 2τ que nous appèlerons Volume A et 2 tétraèdres formés chacun de 2 triangles équilatéraux et 2 triangles isocèles que nous appèlerons volume B.

En utilisant la symétrie du tétraèdre A, on peut calculer sa hauteur en formant un triangle rectangle formé avec une hauteur de triangle équilatéral et un tiers de hauteur de triangle équilatéral allant de la base de la hauteur du côté au centre de gravité de la base.

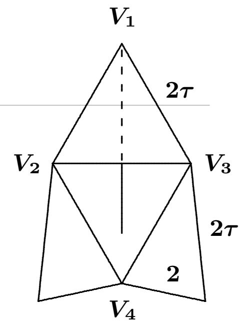






La hauteur de la pyramide est donc $\frac{2\tau\sqrt{6}}{3}$. La base étant un triangle équilatéral a une aire de $\tau^2\sqrt{3}$. Ainsi, le volume $A = \frac{2\tau^3\sqrt{2}}{3}$.

Pour trouver le volume du tétraèdre B, on utilisera le déterminant de Cayley-Menger (voir [1]). Numérotons les sommets du tétraèdre V_1, V_2, V_3 et V_4. Alors les valeurs d_{ij} dans le déterminant représentent les longueurs des côtés du tétraèdre.



$$288V^2 \;=\; \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 \\ 1 & d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 \\ 1 & d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 \\ 1 & d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 \end{vmatrix} \,.$$

Cela devient $288V^2 = 384\tau^4 - 128\tau^2$ qui se simplifie en

$$\text{Volume } B \;=\; \tfrac{2}{3}\tau\sqrt{3\tau^2 - 1}\,.$$

Ainsi la configuration 2 a un volume total de

$$\frac{2\tau^3\sqrt{2}}{3} + \tfrac{4}{3}\tau\sqrt{3\tau^2 - 1} \approx 9,64189 \text{ unité}^2\,.$$

On voit immédiatement que la première configuration est plus volumineuse.

Référence :

[1] http://mathworld.wolfram.com/Tetrahedron.html

Solutioné aussi par MICHEL BATAILLE, Rouen, France ; D. KIPP JOHNSON, Beaverton, OR, USA ; PETER Y. WOO, Biola University, La Mirada, CA, USA ; et le proposeur.

2764. [2002 : 397] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Find an integer-sided scalene triangle in which the lengths of the internal bisectors all have integer values.

Solution by D. Kipp Johnson, Beaverton, OR, USA.

We can find such a triangle by considering Heronian triangles (which have integer sides and integer area). It is known that if p, q, r are positive integers with $r^2 > pq$, then the following formulas produce a Heronian triangle with sides of length a, b, and c:

$$a \;=\; p(q^2 + r^2)\,, \quad b \;=\; q(p^2 + r^2)\,, \quad c \;=\; (p + q)(r^2 - pq)\,.$$

It is also known that the three angle bisectors of the triangle with sides of length a, b, c are

$$w_a \;=\; \sqrt{bc\left(1 - \frac{a^2}{(b + c)^2}\right)}\,, \quad w_b \;=\; \sqrt{ac\left(1 - \frac{b^2}{(a + c)^2}\right)}\,,$$

$$w_c \;=\; \sqrt{ab\left(1-\frac{c^2}{(a+b)^2}\right)}\,.$$

Substituting the above values for the sides of a Heronian triangle and simplifying gives

$$\begin{aligned} w_a &= \frac{2qr(r^2-pq)(p+q)}{pr^2+2qr^2-pq^2}\sqrt{p^2+r^2}\,, \\ w_b &= \frac{2pr(r^2-pq)(p+q)}{2pr^2-p^2q+qr^2}\sqrt{q^2+r^2}\,, \\ w_c &= \frac{2pqr}{r^2+pq}\sqrt{(p^2+r^2)(q^2+r^2)}\,. \end{aligned}$$

(The denominators are positive if $r^2 > pq$.) The three angle bisectors will have rational lengths if p^2+r^2 and q^2+r^2 are perfect squares. If the sides or angle bisectors are then rational but not integral, we can scale up the entire triangle by an appropriate factor to make them all integers. It is not hard to find some acceptable values for p, q, r: $p = 7$, $q = 32$, $r = 24$ (with $r^2 - pq = 352 > 0$ as required), making $p^2 + r^2 = 25^2$ and $q^2 + r^2 = 40^2$. These values yield $a = 11200$, $b = 20000$, $c = 13728$.

We substitute these values of a, b, c into the formulas above for the angle bisectors and get rational numbers with denominators of 527, 779, and 1. Scaling by a factor of $529 \cdot 779$ yields a solution to the problem: $a = 4\,597\,969\,600$, $b = 8\,210\,660\,000$, $c = 4\,635\,797\,024$, which yield angle bisectors of $w_a = 256\,658\,688$, $w_b = 75\,963\,888$, $w_c = 5\,517\,563\,520$.

Also solved by the proposer, whose example had sides of length $315\,409\,500$, $388\,584\,504$, *and* $426\,433\,644$. *For this triangle the internal bisectors are* $312\,405\,600$, $278\,555\,200$, *and* $375\,350\,976$. *As a bonus, the inradius of the triangle is also an integer:* $r = 104\,085\,135$.

2765. [2002 : 397] *Proposed by K.R.S. Sastry, Bangalore, India.*

Derive a set of side length expressions for the family of Heron triangles ABC in which the nine-point centre V lies on side BC. (A Heron triangle has integer sides and integer area.) [See problem 2525(April) [2000 : 177; 2001 : 270].]

Solution by Michel Bataille, Rouen, France.

From problem 2525 we know that V is on the line BC if and only if $\cos(B-C) = 0$; that is, $B = C + \frac{\pi}{2}$ or $C = B + \frac{\pi}{2}$. We will determine the sides of the Heron triangles satisfying $B = C + \frac{\pi}{2}$; those satisfying $C = B + \frac{\pi}{2}$ are obtained by interchanging b and c.

Let K be the area of some suitable $\triangle ABC$. Then the Law of Sines yields $\frac{b}{\cos C} = \frac{c}{\sin C} = \frac{abc}{2K}$, so that $\cos C = \frac{2K}{ac}$ and $\sin C = \frac{2K}{ab}$. Hence,

$4K^2(b^2+c^2) = a^2b^2c^2$, and we see that b^2+c^2 is a perfect square, say $b^2+c^2 = \lambda^2$. Furthermore,

$$\frac{b^2+c^2-a^2}{2bc} = \cos A = \cos\left(\frac{\pi}{2} - 2C\right)$$
$$= \sin 2C = 2\sin C\cos C = \frac{8K^2}{a^2bc};$$

hence, $a^2(b^2+c^2-a^2) = 16K^2$. Thus, $b^2+c^2-a^2 = \mu^2$ for some positive integer μ. Note that $2\lambda K = abc$ and $4K = \mu a$, so that $\lambda\mu = 2bc$. These results easily lead to

$$a^2 = \lambda^2 - \mu^2, \quad (b-c)^2 = \lambda(\lambda-\mu), \quad (b+c)^2 = \lambda(\lambda+\mu).$$

Now, from $\lambda^2 = a^2+\mu^2$, we have either

(1) $a = 2dmn$, $\mu = d(m^2-n^2)$, $\lambda = d(m^2+n^2)$, or

(2) $a = d(m^2-n^2)$, $\mu = 2dmn$, $\lambda = d(m^2+n^2)$,

for some positive integers d, m, n such that m, n are coprime, of opposite parity, and $m > n$.

In case (1), $(b-c)^2 = 2d^2n^2(m^2+n^2)$, which calls for $2(m^2+n^2) = k^2$ for some positive integer k. But this is impossible since m^2+n^2 is odd. Thus, we must be in case (2), which leads to $(b-c)^2 = d^2(m-n)^2(m^2+n^2)$, and $(b+c)^2 = d^2(m+n)^2(m^2+n^2)$.

Hence, $m^2+n^2 = k^2$ for some positive integer k. This gives $b = dkm$ and $c = dkn$ and we may conclude that the sides a, b, c are given by

$$a = d(m^2-n^2), \quad b = dkm, \quad c = dkn$$

where (m,n,k) is a primitive Pythagorean triple (with $m > n$) and d a positive integer.

Conversely, suppose that a, b, and c satisfy these relations. From $\frac{a}{b-c} = \frac{m+n}{k}$, and $\frac{b+c}{a} = \frac{k}{m-n}$, and $(m-n)^2 < k^2 < (m+n)^2$, we easily deduce that $b-c < a < b+c$, and ABC is actually a triangle. Heron's formula gives $4K^2 = d^4m^2n^2(m^2-n^2)^2$; whence, K is an integer (because m or n is even). Hence, $\triangle ABC$ is a Heron triangle. Moreover, the relation $\frac{a^2+c^2-b^2}{2ac} = -\frac{2K}{ab}$ is easily checked and means that $\cos B = -\sin C$; that is, $B = C + \frac{\pi}{2}$ and V is on BC.

Note. V is on the line *segment* BC if the additional condition $\cos(A-B)\cdot\cos(C-A) > 0$ holds (because V, supposed on the line BC, has $(0, b\cos(C-A), a\cos(A-B))$ for *areal* coordinates relative to (A,B,C)). The condition may also be written as $\cos(C-B) + \cos(3A-\pi) > 0$ or

$\cos(3A) < 0$. This yields $C < \frac{\pi}{6}$ or $\cos C > \sqrt{3/2}$. From the results above, it is easy to see that this will be the case if $m^2 > 3n^2$.

Also solved by CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

For explicit examples one can set m equal to the larger of $u^2 - v^2$ or $2uv$ (where u, v are relatively prime positive integers, $v < u < (\sqrt{2}+1)v$), and n equal to the smaller. Again d is an arbitrary positive integer. Then

$$a = d|u^4 - 6u^2v^2 + v^4|\,, \quad b = d(u^4 - v^4)\,, \quad c = 2duv(u^2 + v^2)\,.$$

The area is $uv(u^2 - v^2)|u^4 - 6u^2v^2 + v^4|$.

Sastry, in addition to providing a solution to his problem and an alternative treatment of problem 2525, showed that the following properties are equivalent:

(1) The nine-point centre is on BC.

(2) $|B - C| = \frac{\pi}{2}$.

(3) $\tan B \tan A = -1$.

(4) $OA \| BC$.

(5) AH is tangent to the circumcircle of $\triangle ABC$.

(6) AH is tangent to the nine-point circle of $\triangle ABC$.

(7) BC bisects AH.

(8) $AN = \frac{1}{2}OH$.

(9) AC, BC trisect $\angle OCH$.

2767. [2002 : 398] *Proposed by K.R.S. Sastry, Bangalore, India.*

The points $O(0,0)$, $A(1,0)$, $B(0,1)$ are given. Let $D(a,0)$, $E(1-a,a)$, $F(0,1-a)$ be variable points on the sides of $\triangle OAB$ $(0 < a < 1)$. Let P denote the point of concurrence of the circles ODF, DEA, and BFE. Determine the locus of P.

Combination of solutions by Robert Bilinski, Outremont, QC and Peter Y. Woo, Biola University, La Mirada, CA, USA.

Because P is opposite the right angle at O on the circle $ODPF$ (which implies that FD is the diameter), we see that $\left(\frac{a}{2}, \frac{1-a}{2}\right)$ must be the centre; hence, its equation is

$$x^2 - ax + y^2 - (1-a)y \;=\; 0\,. \tag{1}$$

For circle AED, its centre K is the corner of an isosceles right triangle KNM with right angle at the mid-point $M\left(\dfrac{1+a}{2}, 0\right)$ of DA, while N (where the perpendicular bisector KN of AE meets OA) is $(1-a, 0)$. Hence, K is $\left(\dfrac{1+a}{2}, \dfrac{3a-1}{2}\right)$, and the circle is

$$x^2 - (1+a)x + y^2 - (3a-1)y \;=\; -a\,. \tag{2}$$

For the locus of P we eliminate a by subtracting (2) from (1) to get $a = \dfrac{x-2y}{1-4y}$, then substituting back into (1). Routine algebra leads to

$$\left(x-\frac{3}{8}\right)^2+\left(y-\frac{3}{8}\right)^2 \;=\; \frac{1}{32}\,; \qquad (3)$$

hence, P lies on the circle with centre $\left(\frac{3}{8},\frac{3}{8}\right)$ and radius $1/(4\sqrt{2})$.

We next investigate what portion of the circle (3) is traced by P as a runs from 0 to 1. Note that the point $U\left(\frac{2}{5},\frac{1}{5}\right)$ on the median through B satisfies equation (2) for all a, so that U is on all the circles AED. By symmetry, one can argue that the point $\left(\frac{1}{5},\frac{2}{5}\right)$ on the median through A lies on BFE for all values of a. It follows that the locus can be described as the set of points (other than E) where a circle in the pencil of circles through A and U intersects the corresponding circle of the pencil through B and V. It is easier, however, to obtain the coordinates of P as the point other than $D(a,0)$ satisfying (1) and (2) simultaneously:

$$P \;=\; \left(\frac{4a^2-5a+2}{16a^2-16a+5},\frac{4a^2-3a+1}{16a^2-16a+5}\right)\,.$$

One simply checks that $P=U$ when $a=0$, P is the mid-point of AB when $a=\frac{1}{2}$, and $P=V$ when $a=1$. Thus, P sweeps out the arc of the circle (3) above the line UV from U to V as a goes from 0 to 1.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; D. KIPP JOHNSON, Beaverton, OR, USA; Mª JESÚS VILLAR RUBIO, Santander, Spain; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.