PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 5N3. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or A4 sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 March 2004**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX.) *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5, and 7, English will precede French, and in issues 2, 4, 6, and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

The editor thanks Jean-Marc Terrier and Hidemitsu Saeki of the University of Montreal for translations of the problems.

2827. Correction. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a non-negative integer. Determine

$$\sum_{k=0}^{n} \frac{\tanh\left(2^k\right)}{2+2\sinh^2\left(2^k\right)}\,.$$

. . . .

Soit n un entier non négatif. Calculer

$$\sum_{k=0}^{n} \frac{\tanh\left(2^k\right)}{2+2\sinh^2\left(2^k\right)}\,.$$

2829. Correction. *Proposed by G. Tsintsifas, Thessaloniki, Greece.*
Given $\triangle ABC$ with sides a, b, c, prove that

$$\frac{3\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

. . . .

Montrer que, dans un triangle ABC de côtés a, b, c,

$$\frac{3\left(a^4+b^4+c^4\right)}{\left(a^2+b^2+c^2\right)^2}+\frac{ab+bc+ca}{a^2+b^2+c^2} \;\geq\; 2\,.$$

2839. Correction. *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

Suppose that x, y, and z are real numbers. Prove that

$$\left(x^3+y^3+z^3\right)^2+3(xyz)^2 \;\geq\; 4\left(y^3z^3+z^3x^3+x^3y^3\right)\,.$$

Determine the cases of equality.

. . . .

Si x, y et z sont des nombres réels, montrer que

$$\left(x^3+y^3+z^3\right)^2+3(xyz)^2 \;\geq\; 4\left(y^3z^3+z^3x^3+x^3y^3\right)\,.$$

Déterminer les cas où il y a égalité.

2851★. *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let m, n, and N be non-negative integers such that $m+n\geq 2N+1$. Let $K=m+n-N-1$. Prove that

$$\sum_{j=0}^{\infty}(-1)^j\frac{N+1}{N+1+j}\binom{N}{j}\left[\binom{K-j}{m}+\binom{K-j}{n}\right] \;=\; \frac{\binom{m+n}{m}}{\binom{2N+1}{N}}\,.$$

. . . .

Soit m, n, et N des nombres entiers non négatifs tels que $m+n\geq 2N+1$, et soit $K=m+n-N-1$. Montrer que

$$\sum_{j=0}^{\infty}(-1)^j\frac{N+1}{N+1+j}\binom{N}{j}\left[\binom{K-j}{m}+\binom{K-j}{n}\right] \;=\; \frac{\binom{m+n}{m}}{\binom{2N+1}{N}}\,.$$

2852. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, we have $AB < AC$. The internal bisector of $\angle BAC$ meets BC at D. Let P be an interior point of the line segment AD, and let E and F be the intersections of BP and CP with AC and AB, respectively.

Prove that $\dfrac{PE}{PF} < \dfrac{AC}{AB}$.

. . . .

Dans un triangle ABC, on a $AB < AC$. La bissectrice intérieure de l'angle BAC coupe BC en D. Soit P un point intérieur du segment AD, et soit E et F les intersections respectives de BP et CP avec AC et AB.

Montrer que $\dfrac{PE}{PF} < \dfrac{AC}{AB}$.

2853. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

In $\triangle ABC$, we have $AC = 2AB$. The tangents at A and C to the circumcircle of $\triangle ABC$ meet at P.

Prove that the line BP bisects the arc BAC (of the circumcircle).

. . . .

Dans un triangle ABC, on a $AC = 2AB$. Les tangentes en A et en C au cercle circonscrit du triangle ABC se coupent en P.

Montrer que la droite BP divise l'arc BAC (du cercle circonscrit) en deux parties égales.

2854. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that M and N are the mid-points of the sides AB and CD of quadrilateral $ABCD$, respectively.

Prove that $AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2$.

. . . .

Supposons que M et N sont les points milieu respectifs des côtes AB et CD d'un quadrilatère $ABCD$.

Montrer que $AN^2 + DM^2 + BC^2 = BN^2 + CM^2 + AD^2$.

2855. *Proposed by Antreas P. Hatzipolakis and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given two points B and C, find the locus of the point A such that the centre of the nine-point circle of $\triangle ABC$ lies on the interior bisector of $\angle CAB$.

. . . .

Etant donné deux points B et C, trouver le lieu du point A tel que le centre du cercle des 9-points du triangle ABC soit situé sur la bissectrice intérieure de l'angle CAB.

2856. *Proposed by Óscar Ciaurri, Universidad de La Rioja, Logroño, Spain.*

Let $a_k = \dfrac{q^k - 1}{q - 1}$, where q is a real number, $q \neq 1$. For integers $n \geq 0$ and $k \geq 1$, define $C_{n,k}$ as follows: $C_{n,1} = 1$, $C_{0,k} = 0$ for $k \geq 2$, and $C_{n,k} = \sum\limits_{j=0}^{n-1} \dfrac{a_{k-1}^j}{a_k^{j+1}} C_{j,k-1}$ for $n \geq 1$ and $k \geq 2$.

Show that

$$C_{n,k} \;=\; -(q-1)^{k-1} \sum_{i=1}^{k} \left(\frac{q^i - 1}{q^k - 1} \right)^n \frac{q^i \left\{ q^{-k}, q \right\}_i}{\left\{ q, q \right\}_{i-1} \left\{ q, q \right\}_k},$$

where $\left\{ a, q \right\}_0 = 1$ and $\left\{ a, q \right\}_i = (1-a)(1-aq)\cdots(1-aq^{i-1})$ for $i \geq 1$.

. . . .

Soit q un nombre réel différent de 1 et $a_k = \dfrac{q^k - 1}{q - 1}$. Pour des entiers $n \geq 0$ et $k \geq 1$, on définit $C_{n,k}$ par $C_{n,1} = 1$, par $C_{0,k} = 0$ si $k \geq 2$, et par $C_{n,k} = \sum\limits_{j=0}^{n-1} \dfrac{a_{k-1}^j}{a_k^{j+1}} C_{j,k-1}$ si $n \geq 1$ et $k \geq 2$.

Montrer que

$$C_{n,k} \;=\; -(q-1)^{k-1} \sum_{i=1}^{k} \left(\frac{q^i - 1}{q^k - 1} \right)^n \frac{q^i \left\{ q^{-k}, q \right\}_i}{\left\{ q, q \right\}_{i-1} \left\{ q, q \right\}_k},$$

où $\left\{ a, q \right\}_0 = 1$ et $\left\{ a, q \right\}_i = (1-a)(1-aq)\cdots(1-aq^{i-1})$ si $i \geq 1$.

2857. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Let O be an interior point of $\triangle ABC$, and let D, E, F, be the intersections of AO, BO, CO with BC, CA, AB, respectively.

Suppose that P and Q are points on the line segments BE and CF, respectively, such that $\dfrac{BP}{PE} = \dfrac{CQ}{QF} = \dfrac{DO}{OA}$.

Prove that $PF \parallel QE$.

. . . .

Soit O un point intérieur du triangle ABC, et soit D, E et F, les intersections de AO, BO et CO avec BC, CA et AB, respectivement.

On suppose que P et Q sont des points sur les segments BE et CF, respectivement, tels que $\dfrac{BP}{PE} = \dfrac{CQ}{QF} = \dfrac{DO}{OA}$.

Montrer que $PF \parallel QE$.

2858. *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that P is an interior point of $\triangle ABC$, and that D, E, F are the intersections of AP, BP, CP with BC, CA, AB, respectively. Suppose that

$$\frac{AE+AF}{BC} = \frac{BF+BD}{CA} = \frac{CD+CE}{AB}.$$

Characterize the point P.

. . . .

Soit P un point intérieur d'un triangle ABC et soit D, E et F les intersections respectives de AP, BP et CP avec BC, CA et AB. Supposons que

$$\frac{AE+AF}{BC} = \frac{BF+BD}{CA} = \frac{CD+CE}{AB}.$$

Caractériser le point P.

2859★. *Proposed by Mohammed Aassila, Strasbourg, France.*

Prove that $\sum_{\text{cyclic}} \frac{ab}{c(c+a)} \geq \sum_{\text{cyclic}} \frac{a}{c+a}$, where a, b, c represent the three sides of a triangle.

. . . .

Montrer que $\sum_{\text{cyclic}} \frac{ab}{c(c+a)} \geq \sum_{\text{cyclic}} \frac{a}{c+a}$, où a, b, c représentent les trois côtés d'un triangle.

2860. *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

In $\triangle ABC$ and $\triangle A'B'C'$, the lengths of the sides satisfy $a \geq b \geq c$ and $a' \geq b' \geq c'$. Let h_a and $h_{a'}$ denote the lengths of the altitudes to the opposite sides from A and A', respectively. Prove that

(a) $bb' + cc' \geq ah_{a'} + a'h_a$;

(b) $bc' + b'c \geq ah_{a'} + a'h_a$.

. . . .

Dans les triangles ABC et $A'B'C'$, les longueurs des côtés satisfont $a \geq b \geq c$ et $a' \geq b' \geq c'$. Soit h_a et $h_{a'}$ la longueur des hauteurs issues des sommets A et A'. Montrer que

(a) $bb' + cc' \geq ah_{a'} + a'h_a$;

(b) $bc' + b'c \geq ah_{a'} + a'h_a$.

2861. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

The circle $\Gamma(P, r)$ intersects the side AB of $\triangle ABC$ at A_3 and B_3, the side BC at B_1 and C_1, and the side CA at C_2 and A_2.

Given that $|A_3B_3| : |B_1C_1| : |C_2A_2| = |AB| : |BC| : |CA|$, determine the locus of P.

. . . .

Le cercle $\Gamma(P, r)$ coupe le côté AB du triangle ABC en A_3 et B_3, le côté BC en B_1 et C_1, et le côté CA en C_2 et A_2.

En supposant que $|A_3B_3| : |B_1C_1| : |C_2A_2| = |AB| : |BC| : |CA|$, déterminer le lieu de P.

2862. *Proposed by Mihály Bencze, Brasov, Romania.*

The sequence $\{x_n\}$ is defined by $\left(1+\frac{1}{n}\right)^{n+x_n} = e$.

(a) Prove that $\{x_n\}$ is convergent, and determine its limit.

(b)★ Determine the asymptotic expansion of the sequence.

. . . .

La suite $\{x_n\}$ est définie par $\left(1+\frac{1}{n}\right)^{n+x_n} = e$.

(a) Montrer que $\{x_n\}$ converge et trouver sa limite.

(b)★ Trouver le développement asymptotique de la suite.

2863. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that a, b, c are complex numbers such that $|a| = |b| = |c|$. Prove that

$$\left|\frac{ab}{a^2-b^2}\right| + \left|\frac{bc}{b^2-c^2}\right| + \left|\frac{ca}{c^2-a^2}\right| \geq \sqrt{3}\,.$$

. . . .

On suppose que a, b et c sont des nombres complexes tels que $|a| = |b| = |c|$. Montrer que

$$\left|\frac{ab}{a^2-b^2}\right| + \left|\frac{bc}{b^2-c^2}\right| + \left|\frac{ca}{c^2-a^2}\right| \geq \sqrt{3}\,.$$