THE OLYMPIAD CORNER

No. 231

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Last number we gave ten problems from the shortlist for the 2000 International Mathematical Olympiad. We start this number with the remaining 11 problems of the shortlist. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Seoul, for collecting them.

2000 INTERNATIONAL MATHEMATICAL OLYMPIAD
Shortlisted Problems

11. (*Argentina*) Let $ABCD$ be a convex quadrilateral with AB not parallel to CD, and let Y be the point of intersection of the perpendicular bisectors of AB and CD. If X is a point inside $ABCD$ such that $\angle ADX = \angle BCX < 90^\circ$ and $\angle DAX = \angle CBX < 90^\circ$, prove that $\angle AYB = 2\angle ADX$.

12. (*Belarus*) Find all pairs of functions f and g from the set of real numbers to itself such that $f\big(x+g(y)\big) = xf(y) - yf(x) + g(x)$ for all real numbers x and y.

13. (*India*) Let O be the circumcentre and H the orthocentre of an acute triangle ABC. Prove that there exist points D, E, and F on sides BC, CA, and AB, respectively, such that $OD+DH = OE+EH = OF+FH$ and the lines AD, BE, and CF are concurrent.

14. (*Iran*) Ten gangsters are standing on a flat surface. The distances between them are all distinct. Simultaneously each of them shoots at the one among the other nine who is the nearest. At least how many gangsters will be shot at?

15. (*Ireland*) A non-empty set A of real numbers is called a B_3–set if the conditions a_1, a_2, a_3, a_4, a_5, $a_6 \in A$ and $a_1+a_2+a_3 = a_4+a_5+a_6$ imply that the sequences (a_1, a_2, a_3) and (a_4, a_5, a_6) are identical up to a permutation. For a set X of real numbers, let $D(X)$ denote the difference set $\{|x-y| : x, y \in X\}$. Prove that if $A = \{0 = a_0 < a_1 < a_2 < \cdots\}$ and $B = \{0 = b_0 < b_1 < b_2 < \cdots\}$ are infinite sequences of real numbers with $D(A) = D(B)$, and if A is a B_3–set, then $A = B$.

16. (*The Netherlands*) In the plane we are given two circles intersecting at X and Y. Prove that there exist four points such that for every circle touching the two given circles at A and B, and meeting the line XY at C and D, each of the lines AC, AD, BC, and BD passes through one of those four points.

17. (*Russia*) For a polynomial P with distinct real coefficients, let $M(P)$ be the set of all polynomials that can be obtained from P by permuting its coefficients. Find all integers n for which there exists a polynomial P of degree 2000 with distinct real coefficients such that $P(n) = 0$ and we can get from any $Q \in M(P)$ a polynomial Q' such that $Q'(n) = 0$ by interchanging at most one pair of coefficients of Q.

18. (*Russia*) Let $A_1A_2 \ldots A_n$ be a convex polygon, $n \geq 4$. Prove that $A_1A_2 \ldots A_n$ is cyclic if and only if each vertex A_i can be assigned a pair (b_i, c_i) of real numbers so that $A_iA_j = b_jc_i - b_ic_j$ for all i and j with $1 \leq i < j \leq n$.

19. (*United Kingdom*) Let a, b, and c be positive integers such that $c > 2b > 4a$. Prove that there exists a real number λ such that the three numbers λa, λb, and λc all have their fractional parts in the interval $\left(\frac{1}{3}, \frac{2}{3}\right]$.

20. (*United Kingdom*) A function F is defined from the set of non-negative integers to itself such that, for every non-negative integer n, $F(4n) = F(2n) + F(n)$, $F(4n + 2) = F(4n) + 1$, and $F(2n + 1) = F(2n) + 1$. Prove that, for each positive integer m, the number of integers n with $0 \leq n < 2^m$ and $F(4n) = F(3n)$ is $F(2^{m+1})$.

21. (*United Kingdom*) The tangents at B and A to the circumcircle of an acute triangle ABC meet the tangent at C at T and U, respectively. The lines AT and BC meet at P, and Q is the mid-point of AP; the lines BU and CA meet at R, and S is the mid-point of BR.

(a) Prove that $\angle ABQ = \angle BAS$.

(b) Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

Next we turn to solutions by our readers to problems of the Vietnamese Mathematical Competition 1997 [2001 : 167].

1. In a plane, let there be given a circle with centre O, with radius R and a point P inside the circle, $OP = d < R$. Among all convex quadrilaterals $ABCD$, inscribed in the circle such that their diagonals AC and BD cut each other orthogonally at P, determine the ones which have the greatest perimeter and the ones which have the smallest perimeter. Calculate these perimeters in terms of R and d.

Solved by Mohammed Aassila, Strasbourg, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Aassila's solution, adapted by the editors.

Let $ABCD$ be a quadrilateral satisfying the given conditions, and let p denote its perimeter. Then

$$\begin{aligned} p^2 &= (AB+BC+CD+DA)^2 \\ &= AB^2+CD^2+BC^2+DA^2+2(AB\cdot CD+AD\cdot BC) \\ &\qquad +2(AB\cdot AD+CB\cdot CD)+2(BA\cdot BC+DA\cdot DC)\,. \end{aligned}$$

Now

$$AB^2+CD^2 \;=\; BC^2+DA^2 \;=\; 4R^2\,. \qquad (1)$$

Ptolemy's Theorem gives us

$$AB\cdot CD+AD\cdot BC \;=\; AC\cdot BD\,.$$

Also, with some work one obtains $AC^2+BD^2=8R^2-4d^2$. Hence,

$$2AC\cdot BD \;=\; (AC+BD)^2-8R^2+4d^2\,. \qquad (2)$$

Thus,

$$2(AB\cdot CD+AD\cdot BC) \;=\; (AC+BD)^2-8R^2+4d^2\,. \qquad (3)$$

Furthermore,

$$2(AB\cdot AD+CB\cdot CD) \;=\; 4R\cdot AC\,, \qquad (4)$$

and

$$2(BA\cdot BC+DA\cdot DC) \;=\; 4R\cdot BD\,. \qquad (5)$$

Using (1), (3), (4), and (5) in our expression for p^2, we get

$$p^2 \;=\; (AC+BD)^2+4R(AC+BD)+4d^2\,.$$

Consequently, the maximum (respectively, minimum) of p corresponds to the maximum (respectively, minimum) of $AC+BD$, which, in view of (2), corresponds to the maximum (respectively, minimum) of $AC\cdot BD$. Noting that

$$2AC\cdot BD \;=\; 8R^2-4d^2-(AC-BD)^2\,,$$

we conclude that the maximum (respectively, minimum) of p corresponds to the minimum (respectively, maximum) of $|AC-BD|$. It follows that p is maximized when $AC=BD$, and p is minimized when $AC=2R$ and $BD=2\sqrt{R^2-d^2}$ (the maximum and minimum possible lengths for a chord through P). Hence,

$$p_{\max}^2 \;=\; 16R^2-4d^2+8R\sqrt{4R^2-2d^2}\,,$$

and

$$p_{\min}^2 \;=\; 16R^2+16R\sqrt{R^2-d^2}\,.$$

Editor's note: We can then obtain the following expressions for $p_{\max}$ and $p_{\min}$:

$$\begin{aligned} p_{\max} &= 2\left(\sqrt{2}R + \sqrt{2R^2 - d^2}\right) = \left(\sqrt{\sqrt{2}R - d} + \sqrt{\sqrt{2}R + d}\right)^2, \\ p_{\min} &= 2\sqrt{2R}\left(\sqrt{R+d} + \sqrt{R-d}\right). \end{aligned}$$

2. Let there be given a whole number $n > 1$, not divisible by 1997. Consider two sequences of numbers $\{a_i\}$ and $\{b_j\}$ defined by:

$$a_i = i + \frac{ni}{1997} \quad (i = 1, 2, 3, \ldots, 1996),$$

$$b_j = j + \frac{1997j}{n} \quad (j = 1, 2, 3, \ldots, n-1).$$

By arranging the numbers of these two sequences in increasing order, we get the sequence $c_1 \le c_2 \le \cdots \le c_{1995+n}$.

Prove that $c_{k+1} - c_k < 2$ for every $k = 1, 2, \ldots, 1994 + n$.

Solved by Mohammed Aassila, Strasbourg, France; and Pierre Bornsztein, Pontoise, France. We give the solution of Bornsztein.

First note that $\{a_i\}$ and $\{b_j\}$ are two increasing arithmetical sequences, with difference $\alpha = 1 + \frac{n}{1997}$ and $\beta = 1 + \frac{1997}{n}$, respectively.

Let $i \in \{1, \ldots, 1996\}$ and $j \in \{1, \ldots, n-1\}$, and suppose that $a_i = b_j$. Then $ni = 1997j$. Since 1997 is prime and $n \not\equiv 0 \pmod{1997}$, we deduce that $\gcd(n, 1997) = 1$. From Gauss's Theorem, we then have $i \equiv 0 \pmod{1997}$, which is impossible. It follows that $a_i \neq b_j$.

First Case. $n < 1997$.

We easily see that

$$\alpha < 2 < \beta, \tag{1}$$

$$\text{and} \qquad a_1 < b_1, \tag{2}$$

which implies that $c_1 = a_1$. Moreover, $\frac{a_{1996}}{b_{n-1}} = \frac{1996n}{1997(n-1)} > 1$. Then,

$$b_{n-1} < a_{1996}, \tag{3}$$

which implies that $c_{1995+n} = a_{1996}$.

Lemma. For every $j \in \{1, \ldots, n-2\}$, there exists $i \in \{1, \ldots, 1996\}$ such that $b_j < a_i < b_{j+1}$.

Proof. Suppose, for the purpose of contradiction, that there exists $j \in \{1, \ldots, n-2\}$ such that the interval $[b_j, b_{j+1}]$ does not contain any of the a_i's. Let p be the greatest index such that $a_p < b_j$ (such a p does exist, since $a_1 < b_1 \le b_j$). Then $p < 1996$ (since $b_j < b_{n-1} < a_{1996}$), and $a_p < b_j < b_{j+1} < a_{p+1}$. It follows that $\alpha = a_{p+1} - a_p > b_{j+1} - b_j = \beta$, which contradicts (1). Thus, the lemma is proved.

It follows from the lemma that, for every $k \in \{1, \ldots, 1994 + n\}$, we are in one of the three following cases:

(a) $c_k = a_i$ and $c_{k+1} = a_{i+1}$ for some i. Then $c_{k+1} - c_k = \alpha < 2$.

(b) $c_k = a_i$ and $c_{k+1} = b_j$ for some $i < 1996$ (from (3)) and some $j \le n-1$. Then $b_j < a_{i+1}$ and $c_{k+1} - c_k < a_{i+1} - a_i = \alpha < 2$.

(c) $c_k = b_j$ and $c_{k+1} = a_i$ for some $i > 1$ (from (2)) and some $j \le n - 1$. Then $a_{i-1} < b_j$ and $c_{k+1} - c_k < a_i - a_{i-1} = \alpha < 2$.

In each case, we have $c_{k+1} - c_k < 2$, as desired.

Second Case. $n > 1997$.

This case is essentially the same as the first case: simply interchange n with 1997 and a with b (and also α with β).

3. How many functions $f : \mathbb{N}^* \to \mathbb{N}^*$ are there that simultaneously satisfy the two following conditions:

(i) $f(1) = 1$,

(ii) $f(n) \cdot f(n+2) = \left(f(n+1)\right)^2 + 1997$ for all $n \in \mathbb{N}^*$?

($\mathbb{N}^*$ denotes the set of all positive integers.)

Solved by Mohammed Aassila, Strasbourg, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Athanasias Kalakos, Athens, Greece. We give the solution and comments of Kalakos.

This is, essentially, one of the problems of the 3rd Balkan Mathematical Olympiad (1986). It was proposed by Bulgaria. Here we give it as a lemma with the same proof (slightly modified by the editors) that was given after the competition by the teams that had participated.

Lemma. A sequence is defined by $a_1 = a$, $a_2 = b$, and

$$a_{n+2} = \frac{a_{n+1}^2 + c}{a_n}, \quad n = 1, 2, \ldots,$$

where a, b, c are real numbers and $c > 0$. Then all a_n $(n \ge 1)$ are integers if and only if a, b, and $\dfrac{a^2 + b^2 + c}{ab}$ are integers.

Proof: If $a = 0$, then a_3 is not defined. Thus, $a \ne 0$. Similarly, if $b = 0$, then a_4 is not defined. Thus, $b \ne 0$. It follows inductively that $a_n \ne 0$, for all $n \ge 1$. More precisely, every term exists and is non-zero.

By the recurrence we find, for all $n \ge 2$,

$$a_{n+2}a_n - a_{n+1}^2 = c = a_{n+1}a_{n-1} - a_n^2,$$

and hence,

$$\frac{a_{n+2}+a_n}{a_{n+1}} = \frac{a_{n+1}+a_{n-1}}{a_n}.$$

Therefore, for all $n \geq 1$, we have

$$\frac{a_{n+2}+a_n}{a_{n+1}} = \frac{a_3+a_1}{a_2} = \frac{\frac{b^2+c}{a}+a}{b} = \frac{a^2+b^2+c}{ab},$$

and hence,

$$a_{n+2} = \frac{a^2+b^2+c}{ab}\cdot a_{n+1} - a_n .$$

If a, b, $\frac{a^2+b^2+c}{ab}$ are integers, then an easy induction shows that a_n is an integer for all $n \geq 1$.

Conversely, assume that $a_n \in \mathbb{Z}$, for all $n \geq 1$. Then a_1, $a_2 \in \mathbb{Z}$ implies that a, $b \in \mathbb{Z}$. Moreover, we have $c \in \mathbb{Z}$, since $c = aa_3 - b^2$. Therefore, $\frac{a^2+b^2+c}{ab}$ is rational. Write $\frac{a^2+b^2+c}{ab} = \frac{p}{q}$, where $p \in \mathbb{Z}$, $q \in \mathbb{N}^*$ and $\gcd(p,q) = 1$. For $s \in \mathbb{N}^*$, we will prove inductively on s the following proposition $P(s)$: $q^s \mid a_n$, for all $n \geq s+1$.

For all $k \geq 1$, the recurrence $a_{k+2} = \frac{p}{q}\cdot a_{k+1} - a_k$ gives $\frac{pa_{k+1}}{q} = a_{k+2} + a_k$, and hence, $q \mid (pa_{k+1})$. Since $\gcd(p,q) = 1$, it follows that $q \mid a_{k+1}$. Thus, $q \mid a_n$ for all $n \geq 2$, and $P(1)$ is true.

Suppose that $P(s)$ holds for some $s \geq 1$. Then $q^s \mid a_n$ for all $n \geq s+1$. Consider any $k \geq s+1$. Since $a_{k+2} = \frac{p}{q}a_{k+1} - a_k$, we have

$$\frac{a_{k+2}+a_k}{q^s} = \frac{pa_{k+1}}{q^{s+1}}.$$

By the induction hypothesis, $q^s \mid a_k$ and $q^s \mid a_{k+2}$; hence, $q^s \mid (a_{k+2}+a_k)$. Therefore, $q^{s+1} \mid (pa_{k+1})$. Since $\gcd(p, q^{s+1}) = 1$, we obtain $q^{s+1} \mid a_{k+1}$. Thus, $q^{s+1} \mid a_n$, for all $n \geq s+2$. This proves $P(s+1)$ and completes the induction.

Now let $s \geq 1$ be arbitrary. We have $c = a_{n+2}a_n - a_{n+1}^2$. For $n = s+1$, this yields $c = a_{s+3}a_{s+1} - a_{s+2}^2$. Using $P(s)$,

$$\begin{aligned} q^s \mid a_{s+1}, \quad q^s \mid a_{s+3}, \quad q^s \mid a_{s+2} &\implies q^{2s} \mid (a_{s+3}a_{s+1} - a_{s+2}^2) \\ &\implies q^{2s} \mid c . \end{aligned}$$

Therefore,

$$q^{2s} \leq c \quad \text{for all } s \geq 1 . \tag{1}$$

If we suppose $q > 1$, then $\lim_{s \to +\infty} q^{2s} = +\infty$, which contradicts (1). Thus, $q = 1$ and $\frac{a^2+b^2+c}{ab}$ is an integer. The lemma is proved.

We turn now to the initial problem.

Suppose $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ is any function such that $f(1) = 1$ and

$$f(n+2)f(n) \;=\; (f(n+1))^2 + 1997\,, \qquad \text{for all } n \in \mathbb{N}^*\,.$$

Let $b = f(2)$. Since $f(n)$ is an integer for all n, we have (from the lemma) that $b \in \mathbb{Z}$ and $\frac{1^2+b^2+1997}{1 \cdot b} \in \mathbb{Z}$. Thus, $\frac{b^2+1998}{b} \in \mathbb{Z}$. Then $\frac{1998}{b} \in \mathbb{Z}$, and therefore, $b \mid 1998$. Thus, $f(2) = b$ is a positive divisor of 1998.

Conversely, let b be a positive divisor of 1998. Define $f : \mathbb{N}^* \longrightarrow \mathbb{R}$ by $f(1) = 1$, $f(2) = b$, and $f(n+2) \cdot f(n) = (f(n+1))^2 + 1997$. Since $f(1) \neq 0$ and $f(2) \neq 0$, each $f(n)$ exists and is non-zero (as in the proof of the lemma). Now $b \in \mathbb{Z}$, and

$$\frac{1^2+b^2+1997}{1 \cdot b} \;=\; \frac{b^2+1998}{b} \;=\; b + \frac{1998}{b}$$

is an integer, since $b \mid 1998$. By the lemma, $f(n)$ is an integer for all $n \in \mathbb{N}^*$. Thus, $f : \mathbb{N}^* \longrightarrow \mathbb{Z}^*$. An easy induction shows that $f(n) > 0$ for every n, since $f(1) > 0$ and $f(2) > 0$. Thus, $f : \mathbb{N}^* \longrightarrow \mathbb{N}^*$, and we obtain an admissible sequence.

The above discussion reveals that the number of functions that satisfy both conditions (i) and (ii) in the problem is the same as the number of positive divisors of 1998. Since $1998 = 2 \cdot 3^3 \cdot 37$, the number of positive divisors of 1998 is $(1+1) \cdot (3+1) \cdot (1+1) = 16$. (It is known that the number of divisors of $p_1^{a_1} \cdots p_r^{a_r}$ is $(a_1+1) \cdots (a_r+1)$.)

4. (a) Find all polynomials of least degree, with rational coefficients, such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) \;=\; 3 + \sqrt[3]{3}\,.$$

(b) Does there exist a polynomial with integer coefficients such that

$$f(\sqrt[3]{3} + \sqrt[3]{9}) \;=\; 3 + \sqrt[3]{3}\,?$$

Solved by Pierre Bornsztein, Pontoise, France.

(a) Let $a = 3^{1/3}$.

Lemma. If α, β, $\gamma \in \mathbb{Q}$ such that $\alpha a^2 + \beta a + \gamma = 0$, then $\alpha = \beta = \gamma = 0$.

Proof. Let α, β, $\gamma \in \mathbb{Q}$ such that

$$\alpha a^2 + \beta a + \gamma \;=\; 0\,. \tag{1}$$

Suppose that $\alpha \neq 0$. Since a is a root of the quadratic $\alpha x^2 + \beta x + \gamma = 0$, we must have $a = \dfrac{-\beta \pm \sqrt{\Delta}}{2\alpha}$, where $\Delta = \beta^2 - 4\alpha\gamma \geq 0$. Note that $\sqrt{\Delta} \notin \mathbb{Q}$ (since $a \notin \mathbb{Q}$). Then

$$-24\alpha^3 \;=\; \beta^3 + 3\beta\Delta \pm \sqrt{\Delta}(3\beta^2 + \Delta)\,.$$

It follows that $3\beta^2 + \Delta = 0$, which leads to $\beta = \Delta = 0$. Then we get $\alpha = 0$, a contradiction. Therefore, $\alpha = 0$, and equation (1) becomes $\beta a + \gamma = 0$. Since $a \notin \mathbb{Q}$, we deduce that $\beta = \gamma = 0$, and the lemma is proved.

Let $f \in \mathbb{Q}[x]$ such that $f(a + a^2) = a + 3$. If f has degree 1, then $f(x) = \alpha x + \beta$ for some α, $\beta \in \mathbb{Q}$. Then, $\alpha(a + a^2) + \beta = a + 3$. Then the lemma implies that $\alpha = 0$ and $\alpha = 1$, which is clearly impossible. Therefore, f cannot have degree 1. If f has degree 2, then $f(x) = \alpha x^2 + \beta x + \gamma$ for some α, β, $\gamma \in \mathbb{Q}$. Then, from the lemma, $f(a + a^2) = a + 3$ is equivalent to

$$\begin{aligned} \alpha + \beta \quad\;\; &= 0 \\ 3\alpha + \beta \quad\;\; &= 1 \\ 6\alpha \qquad + \gamma &= 3\,. \end{aligned}$$

The unique solution of this system is $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, and $\gamma = 0$. It follows that there is a unique polynomial f of least degree having rational coefficients such that $f(\sqrt[3]{3} + \sqrt[3]{9}) = 3 + \sqrt[3]{3}$, namely $f(x) = \frac{1}{2}x^2 - \frac{1}{2}x$.

(b) The answer is no.

Suppose, for the purpose of contradiction, that there exists $P \in \mathbb{Z}[x]$ such that $P(a + a^2) = a + 3$. Let $P(x) = \sum_{i=0}^{n} \alpha_i x^i$, where $\alpha_i \in \mathbb{Z}$ for each i. We must have $n \geq 3$, in view of our solution to (a). Note that $(a + a^2)^3 = 12 + 9(a + a^2)$. Then

$$P(a + a^2) \;=\; \sum_{i=0}^{2} \alpha_i (a + a^2)^i + \big(12 + 9(a + a^2)\big) \sum_{i=3}^{n} \alpha_i (a + a^2)^{i-3}\,.$$

It follows that the polynomial

$$Q(x) \;=\; \sum_{i=0}^{2} \alpha_i x^i + (12 + 9x) \sum_{i=3}^{n} \alpha_i x^{i-3}$$

satisfies $Q(a + a^2) = P(a + a^2) = a + 3$, where $Q \in \mathbb{Z}[x]$ with $\deg Q(x) = \deg P(x) - 2$. Now we can apply the same reasoning to Q in place of P. An easy induction leads to a polynomial R of degree at most 2, with integer coefficients, which satisfies $R(a + a^2) = a + 3$. From (a) we must have $R(x) = \frac{1}{2}x^2 - \frac{1}{2}x$, which does not have integer coefficients. This is a contradiction. The conclusion follows.

5. Prove that, for every positive integer n, there exists a positive integer k such that $19^k - 97$ is divisible by 2^n.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Maragoudakis.

We will define by induction a sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ such that $2^n \mid (19^{k_n} - 97)$, for all $n \in \mathbb{N}^*$.

Since $19^8 - 97 \equiv 0 \pmod{64}$, we let $k_1 = k_2 = \cdots = k_6 = 8$. For $n \geq 6$, suppose that $k_n \in \mathbb{N}^*$ satisfies $2^n \mid (19^{k_n} - 97)$. Define

$$k_{n+1} = \begin{cases} k_n & \text{if } 2^{n+1} \mid (19^{k_n} - 97), \\ k_n(2^{n-5} + 1) & \text{if } 2^{n+1} \nmid (19^{k_n} - 97). \end{cases}$$

We will prove that $2^{n+1} \mid (19^{k_{n+1}} - 97)$. This is obvious if k_{n+1} is defined by the first case in the formula above. In the second case, since $2^n \mid (19^{k_n} - 97)$ and $2^{n+1} \nmid (19^{k_n} - 97)$, we must have $19^{k_n} - 97 = 2^n(2m + 1)$ for some $m \in \mathbb{N}$. Then

$$\begin{aligned} 19^{k_{n+1}} - 97 &= 19^{k_{n+1}} - 19^{k_n} + 19^{k_n} - 97 \\ &= \left(19^{k_n \cdot 2^{n-5}} - 1\right) 19^{k_n} + 2^n(2m + 1)\,. \end{aligned}$$

Now, in order to prove that $2^{n+1} \mid (19^{k_{n+1}} - 97)$, it is enough to prove that $19^{k_n \cdot 2^{n-5}} - 1 = 2^n(2x + 1)$, for some $x \in \mathbb{N}$. We start by factoring:

$$\begin{aligned} 19^{k_n \cdot 2^{n-5}} - 1 &= (19^{k_n} - 1) \cdot (19^{k_n} + 1) \cdot (19^{2k_n} + 1) \cdot \\ &\quad \cdot \left(19^{2^2 k_n} + 1\right) \cdots \left(19^{2^{n-6} k_n} + 1\right)\,. \end{aligned}$$

Since $2^n \mid (19^{k_n} - 97)$, where $n \geq 6$, we get that $32 \mid (19^{k_n} - 1)$ and $64 \nmid (19^{k_n} - 1)$. Also, for $v = 1, 2, \ldots$, we have $19^{vk_n} + 1 \equiv 0 \pmod{2}$ and (since k_n is even) $19^{vk_n} + 1 \equiv 2 \pmod{4}$. Therefore,

$$19^{k_n \cdot 2^{n-5}} - 1 = 2^5 \cdot \underbrace{2 \cdot 2 \cdots 2}_{n-5 \text{ factors}} \cdot (2x + 1) = 2^n(2x + 1)\,,$$

for some $x \in \mathbb{N}$.

We have proved that $2^{n+1} \mid (19^{k_{n+1}} - 97)$. The induction is complete.

We next turn to solutions by readers to problems of the Turkey Team Selection Examination for the 38th IMO 1997 [2001 : 168–169].

1. In a triangle ABC which has a right angle at A, let H denote the foot of the altitude belonging to the hypotenuse. Show that the sum of the radii of the incircles of the triangles ABC, ABH, and AHC is equal to $|AH|$.

Solved by Jean-Claude Andrieux, Beaune, France; Mohammed Aassila, Strasbourg, France; Houda Anoun, Bordeaux, France; Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; Pavlos Maragoudakis, Pireas, Greece. We give the solution of Maragoudakis.

Triangles HBA, HAC, ABC are similar. Let r_1, r_2, r be the radii of the incircles of these triangles, respectively. Then

$$\frac{r_1}{AB} = \frac{r_2}{AC} = \frac{r}{BC} = \frac{r_1+r_2+r}{AB+AC+BC}.$$

Thus,

$$r_1+r_2+r = \frac{r(AB+AC+BC)}{BC} = \frac{2[ABC]}{BC} = \frac{BC\cdot AH}{BC} = AH.$$

2. The sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ are defined through $a_1 = \alpha$, $b_1 = \beta$, and $a_{n+1} = \alpha a_n - \beta b_n$, $b_{n+1} = \beta a_n + \alpha b_n$ for all $n \geq 1$. How many pairs (α, β) of real numbers are there such that

$$a_{1997} = b_1 \quad\text{and}\quad b_{1997} = a_1\,?$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Athanasias Kalakos, Athens, Greece. We give the solution by Bataille.

There are 1999 such pairs, namely $(0,0)$ and the pairs $(\cos\theta_k, \sin\theta_k)$ where

$$\theta_k = \frac{\pi}{3996} + \frac{2k\pi}{1998} \quad (k = 0, 1, \ldots, 1997).$$

To prove this result, we remark that, for all $n \geq 1$, the complex number $a_{n+1} + ib_{n+1}$ is given by

$$a_{n+1} + ib_{n+1} = (\alpha + i\beta)(a_n + ib_n),$$

so that $a_n + ib_n = (\alpha + i\beta)^n$ (since $a_1 + ib_1 = \alpha + i\beta$). We will have $a_{1997} = b_1$ and $b_{1997} = a_1$ if and only if $(\alpha + i\beta)^{1997} = i(\alpha - i\beta)$. Letting $z = \alpha + i\beta$, we have the equation $z^{1997} = i\bar{z}$, to be solved for $z \in \mathbb{C}$. An obvious solution is $z = 0$. Any non-zero solution z is necessarily of modulus 1, in which case $\bar{z} = 1/z$ and we have $z^{1998} = i$. Since the solutions of $z^{1998} = i$ are the 1998 complex numbers $\exp\left(i\left(\frac{\pi}{3996} + \frac{2k\pi}{1998}\right)\right)$ with $k = 0, 1, \ldots, 1997$, we have the announced result.

4. The edge AE of a convex pentagon $ABCDE$ whose vertices lie on the unit circle passes through the centre of this circle. If $|AB| = a$, $|BC| = b$, $|CD| = c$, $|DE| = d$ and $ab = cd = \frac{1}{4}$, compute $|AC| + |CE|$ in terms of a, b, c, d.

Solved by Athanasias Kalakos, Athens, Greece; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's write-up.

Let $|AC| = x$, $|CE| = y$, $|AD| = p$, $|BE| = q$. The angles ABE, ACE, ADE are each $90°$, so that $a^2 + q^2 = x^2 + y^2 = p^2 + d^2 = 4$. By Ptolemy's Theorem, $dx + 2c = py$; whence, $d^2x^2 + x + 4c^2 = p^2y^2$. Therefore,

$$\begin{aligned} x &= (4 - d^2)y^2 - 4c^2 - d^2x^2 = 4y^2 - 4c^2 - d^2(x^2 + y^2) \\ &= 4y^2 - 4c^2 - 4d^2 . \end{aligned}$$

Analogously, the relation $ay + 2b = qx$ leads to $y = 4x^2 - 4a^2 - 4b^2$.

Consequently, $x + y = 16 - 4(a^2 + b^2 + c^2 + d^2)$.

5. Prove that, for each prime number $p \geq 7$, there exists a positive integer n and integers $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ which are not divisible by p, such that

$$\begin{aligned} x_1^2 + y_1^2 &\equiv x_2^2 \pmod{p} , \\ x_2^2 + y_2^2 &\equiv x_3^2 \pmod{p} , \\ &\vdots \\ x_{n-1}^2 + y_{n-1}^2 &\equiv x_n^2 \pmod{p} , \\ x_n^2 + y_n^2 &\equiv x_1^2 \pmod{p} . \end{aligned}$$

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bradley's solution (modified by the editors).

We consider two cases:

$$p \equiv 1 \pmod{4} \qquad \text{or} \qquad p \equiv 3 \pmod{4} .$$

(a) $p \equiv 3 \pmod{4}$. Then $p = 4k + 3$ for some integer k (where $k > 0$ since $p \geq 7$), and we observe that

$$1^2 + k^2 \equiv (k + 2)^2 \pmod{p} . \qquad (1)$$

Setting $x_1 = 1$, $y_1 = k$, and $x_2 = k + 2$, we have $x_1^2 + y_1^2 \equiv x_2^2 \pmod{p}$. Suppose now that we are given $x_i^2 + y_i^2 \equiv x_{i+1}^2 \pmod{p}$ for some $i \geq 1$. We will construct integers y_{i+1} and x_{i+2} such that $x_{i+1}^2 + y_{i+1}^2 \equiv x_{i+2}^2 \pmod{p}$. We first multiply (1) by x_{i+1}^2 to yield

$$x_{i+1}^2 + k^2x_{i+1}^2 \equiv (k + 2)x_{i+1}^2 \pmod{p} .$$

Then we choose $y_{i+1} \equiv kx_{i+1} \pmod{p}$ and $x_{i+2} \equiv (k + 2)x_{i+1} \pmod{p}$.

Since, for any prime p, there are a finite number of quadratic residues, eventually we will have $x_j \equiv x_i \pmod{p}$ for some $j > i$. We can then re-label x_i as x_1 and y_i as y_1, and begin the process there.

For example, if $p = 13$, then $p = 4k + 1$ for $k = 3$. We start with

$$1^2 + 9^2 \equiv 11^2 \pmod{13} ,$$

and proceed to get the following circuit:

$$\begin{aligned} 11^2 + 8^2 &\equiv 4^2 \pmod{13} , \\ 4^2 + 10^2 &\equiv 5^2 \pmod{13} , \\ 5^2 + 6^2 &\equiv 3^2 \pmod{13} , \\ 3^2 + 1^2 &\equiv 7^2 \pmod{13} , \\ 7^2 + 11^2 &\equiv 12^2 \equiv 1^2 \pmod{13} . \end{aligned}$$

(b) $p \equiv 1 \pmod 4$. Then $p = 4k + 1$ for some integer k (where $k > 1$ since $p \geq 7$), and we observe that

$$1^2 + (3k)^2 \equiv (3k + 2)^2 \pmod{p} . \qquad (2)$$

Our process is similar to part (a), only this time we multiply (2) by x_{i+1}^2 and choose $y_{i+1} \equiv 3kx_{i+1} \pmod p$ and $x_{i+2} \equiv (3k + 2)x_{i+1} \pmod p$.

6. Given an integer $n \geq 2$, find the minimal value of

$$\frac{x_1^5}{x_2 + x_3 + \cdots + x_n} + \frac{x_2^5}{x_1 + x_3 + \cdots + x_n} + \cdots + \frac{x_n^5}{x_1 + x_2 + \cdots + x_{n-1}}$$

subject to $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, where $x_1, x_2, \ldots, x_n$ are positive real numbers.

Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Bornsztein.

More generally, we will find the minimal value of $\sum_{i=1}^{n} \left(\frac{x_i^p}{S - x_i^r} \right)$ subject to $x_1^s + x_2^s + \cdots + x_n^s = 1$, where $S = x_1^r + x_2^r + \cdots + x_n^r$ and p, r, s are positive real numbers such that $r \leq s \leq \frac{p}{2}$. The given problem is the special case $p = 5$, $r = 1$, $s = 2$.

Applying the Cauchy-Schwarz Inequality, we have

$$\left(\sum_{i=1}^{n} (S - x_i^r) \right) \left(\sum_{i=1}^{n} \frac{x_i^p}{S - x_i^r} \right) \geq \left(\sum_{i=1}^{n} x_i^{p/2} \right)^2 ,$$

with equality if and only if there exists a positive real number λ such that $S - x_i^r = \lambda x_i^{p/2}$.

By the Power-Mean Inequality (since $p/2 \geq s$),

$$\left(\sum_{i=1}^{n} x_i^{p/2}\right)^2 = n^2\left(\sum_{i=1}^{n} \frac{x_i^{p/2}}{n}\right)^2 \geq n^2\left(\sum_{i=1}^{n} \frac{x_i^s}{n}\right)^{p/s} = \frac{n^2}{n^{p/s}},$$

with equality if and only if $p = 2s$ or $x_1 = x_2 = \cdots = x_n = \frac{1}{n^{1/s}}$. Also,

$$\begin{aligned}\sum_{i=1}^{n}(S - x_i^r) &= (n-1)\sum_{i=1}^{n} x_i^r = n(n-1)\sum_{i=1}^{n}\frac{x_i^r}{n} \\ &\leq n(n-1)\left(\sum_{i=1}^{n}\frac{x_i^s}{n}\right)^{r/s} = \frac{n(n-1)}{n^{r/s}},\end{aligned}$$

by the Power-Mean Inequality (since $s \geq r$). Here, equality occurs if and only if $r = s$ or $x_1 = x_2 = \cdots = x_n = \frac{1}{n^{1/s}}$. Since $\sum_{i=1}^{n}(S - x_i^r) > 0$, we have

$$\sum_{i=1}^{n}\frac{x_i^p}{S - x_i^r} \geq \frac{\left(\sum_{i=1}^{n} x_i^{p/2}\right)^2}{\sum_{i=1}^{n}(S - x_i^r)} \geq \frac{\dfrac{n^2}{n^{p/s}}}{\dfrac{n(n-1)}{n^{r/s}}} = \frac{n^{(r+s-p)/s}}{n-1}.$$

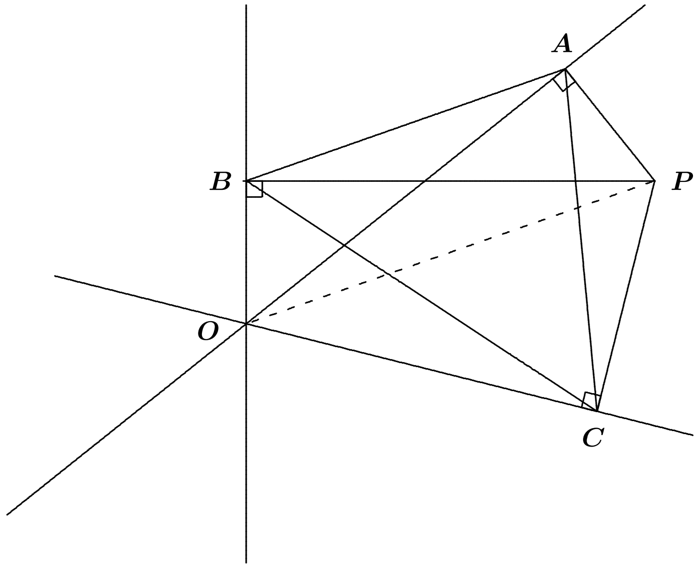
Equality occurs when $x_1 = x_2 = \cdots = x_n = \frac{1}{n^{1/s}}$. Therefore, the expression on the right side above is the minimal value of the sum on the left side, subject to $x_1^s + x_2^s + \cdots + x_n^s = 1$.

Setting $p = 5$, $r = 1$, and $s = 2$, we find that the minimal value in the given problem is $\frac{1}{n(n-1)}$.

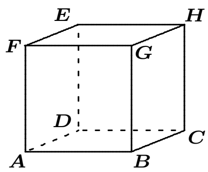
We now turn to readers' solutions of problems of the Chilean Mathematical Olympiads 1994–95 [2001 : 169–170].

1. Given three straight lines in a plane, that concur at point O, consider the three consecutive angles between them (which, naturally, add up to 180°). Let P be a point in the plane not on any of these lines, and let A, B, C be the feet of the perpendiculars drawn from P to the three lines. Show that the internal angles of $\triangle ABC$ are equal to those between the given lines.

Solved by Christopher J. Bradley, Clifton College, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA. We give Kandall's write-up.



The points O, C, P, A, B lie on the circle having $\overline{OP}$ as diameter. Therefore, $\angle ABC = \angle AOC$ and $\angle ACB = \angle AOB$. As a consequence, $\angle BAC$ is equal to the third consecutive angle at O.

2. 

$ABCDEFGH$ is a cube of edge 2. Let M be the mid-point of $\overline{BC}$ and N the mid-point of $\overline{EF}$. Compute the area of the quadrilateral $AMHN$.

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA. We give the solution of Bilinski.

Since clearly $\overrightarrow{NH} = \overrightarrow{AM}$, the four points A, M, H, and N are coplanar. Since $\triangle ENH$, $\triangle FNA$, $\triangle BMA$, and $\triangle CMH$ are right triangles whose sides have lengths 1 and 2, we see that these triangles are congruent and $AM = MH = HN = NA$. Quadrilateral $NHMA$ is thus a rhombus.

Let us find the length of its diagonals AH and NM.

By applying the Pythagorean Theorem in right triangle $\triangle EHC$, we have $EC = \sqrt{EH^2 + HC^2} = 2\sqrt{2}$. Noticing that $NM = EC$, we get $NM = 2\sqrt{2}$. By the same reasoning, we see that $AC = 2\sqrt{2}$. Applying the Pythagorean Theorem in $\triangle ACH$, we get $AH = \sqrt{AC^2 + HC^2} = 2\sqrt{3}$.

Now, the area of a rhombus is half the product of the lengths of its diagonals. Hence, we obtain $2\sqrt{6}$ as the area.

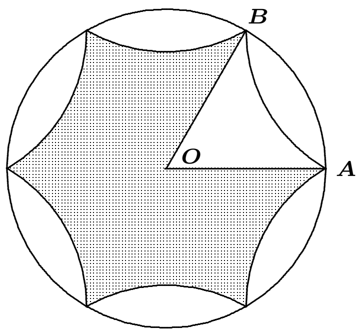
3. Given a trapezoid $ABCD$, where $\overline{AB}$ and $\overline{DC}$ are parallel, and $\overline{AD} = \overline{DC} = \overline{AB}/2$, determine $\angle ACB$.

Solved by Robert Bilinski, Outremont, QC; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; Geoffrey A. Kandall, Hamden, CT, USA; Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Bilinski.

Let M be the mid-point of AB, and let E be the point of intersection of AD and BC. Since $DC = AB/2$ and $DC \| AB$, we see by Thales that D is the mid-point of AE and C is the mid-point of BE. Thus, M, D, and C are the mid-points of the sides of $\triangle ABE$. By the Mid-point Theorem, $AMCD$ is a parallelogram and $MC = AD = AB/2$.

In $\triangle ABC$, MC is a median which is half the length of the side AB. Hence, $\triangle ABC$ has a right angle at C. That is, $\angle ACB = 90^\circ$.

4. In a circle of radius 1 are drawn six equal arcs of circles, radius 1, cutting the original circle as in the figure. Calculate the shaded area.



Solved by Robert Bilinski, Outremont, QC; Christopher J. Bradley, Clifton College, Bristol, UK; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Bilinski's solution.

Since both arcs passing through A and B are of radius 1, they are symmetric about the line AB. Hence, the area enclosed by these two arcs is cut in half by the line segment AB. If line segments are drawn through all vertices of the hexagonal star, we get a regular hexagon inscribed in the circle. From this we can easily calculate the area of the total white border, for it will be twice the area between the circle and the hexagon.

The hexagon's area is six times the area of an equilateral triangle of side 1, namely $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. The circle has area $\pi r^2 = \pi$. Hence, the hexagonal star has area $\pi - 2\left(\pi - \frac{3\sqrt{3}}{2}\right) = 3\sqrt{3} - \pi$.

Thus, since the shaded area has only $5/6$ the area of the hexagonal star, its area is $\frac{5}{6}(3\sqrt{3} - \pi)$.

5. In right triangle ABC the altitude $h_c = \overline{CD}$ is drawn to the hypotenuse $\overline{AB}$. Let P, P_1, P_2 be the radii of the circles inscribed in the triangles ABC, ADC, BCD, respectively. Show that $P + P_1 + P_2 = h_c$.

Solution and observation from Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Geoffrey A. Kandall, Hamden, CT, USA.

This is the same as question 1 of the Turkey Team Selection Test given in the same number of the *Corner*. See the solution given above (p. 287).

6. Consider the product of all the positive multiples of 6 that are less than 1000. Find the number of zeroes with which this product ends.

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Kandall.

The multiples under consideration are the numbers $6n$ $(1 \le n \le 166)$. Their product is $P = 2^{166} \cdot 3^{166} \cdot (166!)$. The prime factorization of P contains exactly 40 5's, since

$$\sum_{n=1}^{\infty} \left\lfloor \frac{166}{5^n} \right\rfloor = \left\lfloor \frac{166}{5} \right\rfloor + \left\lfloor \frac{166}{25} \right\rfloor + \left\lfloor \frac{166}{125} \right\rfloor = 33 + 6 + 1 = 40 .$$

Therefore, P is divisible by 10^{40}, but not by 10^{41}; that is, P ends with 40 zeroes.

7. Let x be an integer of the form

$$x = \underbrace{1\,1\,1\,\ldots\,1}_{n} .$$

Show that, if x is a prime, then n is a prime.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Athanasias Kalakos, Athens, Greece. We use Bradley's write-up.

Note that

$$x = \underbrace{1\,1\,1 \cdots 1}_{n \text{ times}} = 1 + 10 + \cdots + 10^{n-1} = \frac{10^n - 1}{10 - 1} = \frac{1}{9}(10^n - 1) .$$

Now, if n is composite, say $n = n_1 n_2$ (where n_1, $n_2 > 1$), then

$$x = \frac{10^n - 1}{9} = \left(\frac{10^{n_1} - 1}{9}\right)(1 + 10^{n_1} + 10^{2n_1} + \cdots + 10^{(n_2-1)n_1}) .$$

Since $\dfrac{10^{n_1} - 1}{9}$ is an integer greater than 1, then x is composite.

Hence if x is prime, n must be prime.

8. Let x be a number such that

$$x + \frac{1}{x} = -1 .$$

Compute

$$x^{1994} + \frac{-1}{x^{1994}} .$$

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bornsztein's solution.

Let x be a number such that $x + \frac{1}{x} = -1$. Then $x = e^{2i\pi/3} = j$ or $x = \bar{j}$. Since $1/j = \bar{j}$ and $j^2 = \bar{j}$ and $j^3 = 1$, we deduce that

- If $x = j$, then $x^{1994} - \frac{1}{x^{1994}} = j^2 - \frac{1}{j^2} = \bar{j} - j = -i\sqrt{3}$.
- If $x = \bar{j}$, then $x^{1994} - \frac{1}{x^{1994}} = \overline{-i\sqrt{3}} = i\sqrt{3}$.

Then

$$x^{1994} - \frac{1}{x^{1994}} = \pm i\sqrt{3} .$$

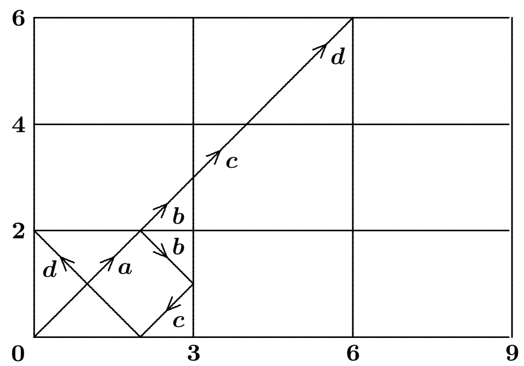
9. Let $ABCD$ be an $m \times n$ rectangle, with $m, n \in \mathbb{N}$. Consider a ray of light that starts from A, is reflected at an angle of 45° on another side of the rectangle, and goes on reflecting in this way.

(a) Show that the ray will finally hit a vertex.

(b) Suppose m and n have no common factor greater than 1. Determine the number of reflections undergone by the ray before it hits a vertex.

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We use Bradley's explanation.

(a) Extend the rectangle in all directions to form a Cartesian grid. (The example in the figure below has $m = 3$ and $n = 2$.)



The actual path is mirrored by a straight line with equation $y = x$. The ray of light will eventually hit the vertex corresponding to the vertex (mn, mn) in the extension.

(b) The ray crosses $m - 1$ lines in one direction and $n - 1$ in the other (in the extension). Thus, the number of reflections is $m + n - 2$.

10. Let a be a natural number. Show that the equation

$$x^2 - y^2 = a^3$$

always has integer solutions for x and y.

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Geoffrey A. Kandall, Hamden, CT, USA; Athanasias Kalakos, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution of Kalakos.

Let $x = \frac{a(a+1)}{2}$, $y = \frac{a(1-a)}{2}$. Clearly, x, $y \in \mathbb{Z}$. Moreover,

$$\begin{aligned} x^2 - y^2 &= (x-y)(x+y) = \frac{a^2 + a - a + a^2}{2} \cdot \frac{a^2 + a + a - a^2}{2} \\ &= a^2 \cdot a = a^3 \,. \end{aligned}$$

Next we move to solutions to problems of the May 2001 number of the *Corner* and the 28th Austrian Mathematics Olympiad 1997 [2001 : 231–232].

1. Let a be a fixed whole number.

Determine all solutions x, y, z in whole numbers to the system of equations

$$\begin{array}{rcrcrcl} 5x &+& (a+2)y &+& (a+2)z &=& a\,, \\ (2a+4)x &+& (a^2+3)y &+& (2a+2)z &=& 3a-1\,, \\ (2a+4)x &+& (2a+2)y &+& (a^2+3)z &=& a+1\,. \end{array}$$

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Pavlos Maragoudakis, Pireas, Greece; Vedula N. Murty, Dover, PA, USA. We give Bradley's write-up.

The second and third equations give

$$(a-1)^2(y-z) = 2(a-1)\,.$$

When $a = 1$, the original system reduces to two independent equations $5x + 3y + 3z = 1$ and $6x + 4y + 4z = 2$, from which we obtain a one-parameter set of whole-number solutions $x = -1$, $y = 1 + t$, $z = 1 - t$, where $t \in Z$.

When $a \neq 1$, we have $y - z = \dfrac{2}{a-1}$. If y, z are whole numbers, then so is their difference, which means that $a - 1 \mid 2$, restricting the possibilities to $a = -1$, $a = 0$, $a = 2$, $a = 3$.

Case 1. $a = -1$. The equations become

$$\begin{aligned} 5x + y + z &= -1\,, \\ 2x + 4y &= -4\,, \\ 2x + 4z &= 0\,, \end{aligned}$$

giving a solution $x = 0$, $y = -1$, $z = 0$.

Case 2. $a = 0$. The equations become

$$\begin{aligned} 5x + 2y + 2z &= 0\,, \\ 4x + 3y + 2z &= -1\,, \\ 4x + 2y + 3z &= 1\,, \end{aligned}$$

giving a solution $x = 0$, $y = -1$, $z = 1$.

Case 3. $a = 2$. The equations become

$$\begin{aligned} 5x + 4y + 4z &= 2\,, \\ 8x + 7y + 6z &= 5\,, \\ 8x + 6y + 7z &= 3\,, \end{aligned}$$

giving a solution $x = -6$, $y = 5$, $z = 3$.

Case 4. $a = 3$. The equations include

$$5x + 5y + 5z = 3\,,$$

which evidently has no whole-number solutions, since $5 \nmid 3$.

2. Let K be a positive whole number. The sequence $\{a_n : n \geq 1\}$ is defined by $a_1 = 1$ and a_n is the n^{th} natural number greater than a_{n-1} which is congruent to n modulo K.

(a) Determine an explicit formula for a_n.

(b) What is the result if $K = 2$?

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the write-up of Bornsztein.

(a) Let n be a positive integer.

Since $a_n \equiv n \bmod K$, the first integer which is greater than a_n and congruent to $n+1$ modulo K is $a_n + 1$. Thus, the $(n+1)^{\text{th}}$ natural number greater than a_n which is congruent to n modulo K is $a_{n+1} = a_n + 1 + nK$. Summing these relations, we get that, for every integer $n \geq 1$,

$$a_n \;=\; a_1 + n - 1 + K\sum_{i=1}^{n-1} i \;=\; n + \frac{n(n-1)}{2}K\,.$$

(b) For $K = 2$, we immediately have $a_n = n^2$ for $n \geq 1$.

4. Determine all quadruples (a,b,c,d) of real numbers satisfying the equation

$$\begin{aligned}256a^3b^3c^3d^3 \;&=\; (a^6+b^2+c^2+d^2)(a^2+b^6+c^2+d^2)\\ &\qquad\times(a^2+b^2+c^6+d^2)(a^2+b^2+c^2+d^6)\,.\end{aligned}$$

Solved by Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bornsztein's write-up.

Let (a,b,c,d) be a quadruple of real numbers satisfying the equation. If one of the numbers is zero, then all are 0. From now on, we suppose that none of the four numbers is 0. Since the right-hand side is positive there must be an even number of negative reals amongst a, b, c, d. Then (a,b,c,d) is a solution if and only if $(|a|,|b|,|c|,|d|)$ is a solution. Thus, we may suppose that a, b, c, d are positive.

From the AM–GM Inequality,

$$a^6+b^2+c^2+d^2 \;\geq\; 4(a^6b^2c^2d^2)^{1/4}\,,$$

and similarly for the other three factors on the right-hand side of the equation. Thus,

$$\begin{aligned}&(a^6+b^2+c^2+d^2)(a^2+b^6+c^2+d^2)\\ &\quad\times(a^2+b^2+c^6+d^2)(a^2+b^2+c^2+d^6)\\ &\geq\; 256(a^6b^2c^2d^2)^{1/4}(a^2b^6c^2d^2)^{1/4}\times(a^2b^2c^6d^2)^{1/4}(a^2b^2c^2d^6)^{1/4}\\ &=\; 256a^3b^3c^3d^3\,,\end{aligned}$$

which indicates that the given equation is the equality case of the AM/GM Inequality. Therefore,

$$a^6 \;=\; b^2 \;=\; c^2 \;=\; d^2 \;=\; a^2 \;=\; b^6 \;=\; c^6 \;=\; d^6\,;$$

that is, $a = b = c = d = 1$.

Then the solutions are $(0,0,0,0)$ and $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ where $\varepsilon_i = \pm 1$ for $i = 1, 2, 3, 4$ and $\prod_{i=1}^{4}\varepsilon_i = 1$.

5. We define the following operation which will be applied to a row of bars being situated side-by-side on positions $1, \ldots, N$:

Each bar situated at an odd-numbered position is left as is, while each bar at an even-numbered position is replaced by two bars. After that, all bars will be put side-by-side in such a way that all bars form a new row and are situated (side-by-side) on positions $1, \ldots, M$.

From an initial number $a_0 > 0$ of bars there originates (by successive application of the above-defined operation) a sequence, $\{a_n : n \geq 0\}$ of natural numbers, where a_n is the number of bars after having applied the operation n times.

(a) Prove that for all $n > 0$ we have $a_n \neq 1997$.

(b) Determine the natural numbers that can only occur as a_0 or a_1.

Solved by Pierre Bornsztein, Pontoise, France; and Christopher J. Bradley, Clifton College, Bristol, UK. We give Bornsztein's solution.

(a) Let n be a non-negative integer. When the operation is applied to a row of a_n bars, the total number of bars increases by $\frac{a_n}{2}$ if a_n is even, and by $\frac{a_n - 1}{2}$ if a_n is odd. Thus, for every $n \geq 0$,

$$a_{n+1} = \begin{cases} \dfrac{3a_n}{2} & \text{if } a_n \text{ is even,} \\ \dfrac{3a_n - 1}{2} & \text{if } a_n \text{ is odd.} \end{cases}$$

Let p be a natural number with $p \equiv 2 \bmod 3$. Let $n \geq 0$ be an integer. Suppose $a_{n+1} = p$. If a_n is even, then $3a_n = 2a_{n+1} = 2p \equiv 1 \pmod 3$, a contradiction, while if a_n is odd, then $3a_n = 2a_{n+1} + 1 = 2p + 1 \equiv 2 \pmod 3$, a contradiction. Thus, $a_{n+1} \neq p$. Since $1997 \equiv 2 \pmod 3$, part (a) is proved.

(b) We have seen that if $p \equiv 2 \pmod 3$, then p can only occur in the sequence as a_0.

Case 1. $p = 9k$, with $k \in \mathbb{N}^*$.
For $a_0 = 4k$, we have $a_1 = 6k$ and $a_2 = 9k$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 2. $p = 9k + 1$, with $k \in \mathbb{N}$.
For $a_0 = 4k + 1$, we have $a_1 = 6k + 1$ and $a_2 = 9k + 1$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 3. $p = 9k + 3$, with $k \in \mathbb{N}$.
Suppose that there exists an integer $n \geq 1$ such that $a_{n+1} = p$. If a_n is even, then $3a_n = 2p = 2(9k + 3)$, and hence, $a_n = 6k + 2 \equiv 2 \pmod 3$ with $n > 0$, a contradiction. If a_n is odd, then $3a_n = 2p + 1 = 2(9k + 3) + 1 \equiv 1 \pmod 3$, a contradiction. Therefore, p cannot occur in the sequence as a_n with $n \geq 2$.

Case 4. $p = 9k + 4$, with $k \in \mathbb{N}$.

For $a_0 = 4k + 2$, we have $a_1 = 6k + 3$ and $a_2 = 9k + 4$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 5. $p = 9k + 6$, with $k \in \mathbb{N}$.

For $a_0 = 4k + 3$, we have $a_1 = 6k + 4$ and $a_2 = 9k + 6$. Thus, p can occur in the sequence as a_n with $n \geq 2$.

Case 6. $p = 9k + 7$, with $k \in \mathbb{N}$.

Suppose that there exists an integer $n \geq 1$ such that $a_{n+1} = p$. If a_n is even, then $3a_n = 2p = 2(9k + 7) \equiv 2 \pmod 3$, a contradiction. If a_n is odd, then $3a_n = 2p + 1 = 2(9k + 7) + 1$, and hence, $a_n = 6k + 5 \equiv 2 \pmod 3$ with $n > 0$, a contradiction. Therefore, p cannot occur in the sequence as a_n with $n \geq 2$.

It follows that the natural numbers that can only occur as a_0 or a_1 are those congruent to 2, 3, 5, 7, or 8 $\pmod 9$.

6. Let n be a fixed natural number. Determine all polynomials $x^2 + ax + b$, where $a^2 \geq 4b$, such that $x^2 + ax + b$ divides $x^{2n} + ax^n + b$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Athanasias Kalakos, Athens, Greece; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bataille's write-up.

If $n = 1$, all polynomials $x^2 + ax + b$ are solutions. We will suppose $n > 1$ from now on. Since $a^2 \geq 4b$, there exist real numbers x_1, x_2 (not necessarily distinct) such that $x^2 + ax + b = (x - x_1)(x - x_2)$. It follows that $x^{2n} + ax^n + b = (x^n - x_1)(x^n - x_2)$.

Now, if $x^2 + ax + b$ divides $x^{2n} + ax^n + b$, then x_1, x_2 are roots of $x^{2n} + ax^n + b$, so that $x_1^n = x_1$ or $x_1^n = x_2$, and $x_2^n = x_2$ or $x_2^n = x_1$. Therefore, x_1 and x_2 must belong to $\{-1, 0, 1\}$.

Now we check the possible cases:

- If $x_1 = x_2 = 0$, then $a = b = 0$ and $x^2 + ax + b = x^2$ divides $x^{2n} + ax^n + b = x^{2n}$.

- If $x_1 = 0$, $x_2 = -1$, then $a = 1$, $b = 0$ and $x^2 + ax + b = x(x + 1)$ divides $x^n(x^n + 1)$ only if n is odd.

- If $x_1 = 0$, $x_2 = 1$, then $x^2 + ax + b = x(x - 1)$ divides $x^n(x^n - 1)$.

- If $x_1 = -1$, $x_2 = -1$, then $x^2 + ax + b = (x + 1)^2$ divides $(x^n + 1)^2$ only if n is odd.

- If $x_1 = 1$, $x_2 = 1$, then $x^2 + ax + b = (x - 1)^2$ divides $(x^n - 1)^2$.

- If $x_1 = -1$, $x_2 = 1$, then $x^2 + ax + b = x^2 - 1$ divides $x^{2n} - 1$.

In conclusion, for n odd ($n > 1$), the solutions are x^2, $x(x+1)$, $x(x-1)$, $(x+1)^2$, $(x-1)^2$, x^2-1; and for n even, the solutions are x^2, $x(x-1)$, $(x-1)^2$, x^2-1.

Editor's comment: Klamkin points out that if the condition $a^2 \geq 4b$ is eliminated, the zeros x_1, x_2 can be complex cube roots of unity, allowing another possibility, x^2+x+1, provided n is not a multiple of 3.

Next we move to readers' solutions for problems of the Íslenzka Staerőfræðikeppni Framhaldsskólanema 1995–1996 [2001 : 232–233].

1. Calculate the area of the region in the plane determined by the inequality

$$|x| + |y| + |x+y| \;\leq\; 2\,.$$

Solved by Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the solution of Diaz-Barrero.

Let $\mathcal{A} = \{(x,y) \in \mathbb{R}^2 : |x| + |y| + |x+y| \leq 2\}$ be the given region. We claim that $\mathcal{A}$ has area 3. To prove this, we start by recalling that for all $a \in \mathbb{R}$, $|a| = |-a|$. It follows that $\mathcal{A}$ is symmetric with respect to reflection through the origin. Hence, it suffices to investigate the given inequality only when $y \geq 0$.

In the first quadrant, where $x \geq 0$ and $y \geq 0$, we have $x+y \geq 0$, and the inequality becomes $x+y+x+y \leq 2$. Therefore, the part of $\mathcal{A}$ in this quadrant is the triangle

$$AOB \;=\; \{(x,y) \in \mathbb{R}^2 : x \geq 0\,, y \geq 0\,, x+y \leq 1\}\,.$$

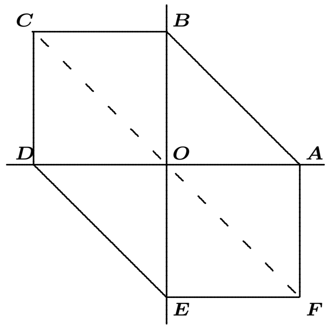
In the second quadrant, where $x \leq 0$ and $y \geq 0$, we have two possibilities: (a) $x+y \geq 0$ or (b) $x+y \leq 0$. In case (a), the inequality becomes $-x+y+x+y \leq 2$, and we have the triangle

$$BOC \;=\; \{(x,y) \in \mathbb{R}^2 : x \leq 0\,, 0 \leq y \leq 1\}\,,$$

In case (b), the inequality becomes $-x+y-(x+y) \leq 2$, determining the triangle

$$COD \;=\; \{(x,y) \in \mathbb{R}^2 : -1 \leq x \leq 0\,, y \geq 0\}\,.$$

These triangles can be seen in the figure on the next page. Reflecting them through the origin, we see that $\mathcal{A}$ is the hexagon $ABCDEF$, with area 3.



2. Suppose that a, b, and c are the three roots of the polynomial $p(x) = x^3 - 19x^2 + 26x - 2$. Calculate

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\,.$$

Solved by Rahul Banotra, student, Sir Winston Churchill High School and Samapti Samapti, Western Canada High School, Calgary, AB; Marcus Emmanuel Barnes, student, York University; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain; Elizabeth Park, Western Canada High School, Calgary, AB. We give the solution of Banotra and Samapti.

If a, b, c are the roots of the polynomial, then

$$(x-a)(x-b)(x-c) \;=\; x^3 - 19x^2 + 26x - 2\,. \qquad (1)$$

Expanding yields:

$$\begin{aligned} &(x-a)(x-b)(x-c) \\ &= (x^2 - (a+b)x + ab)(x-c) \\ &= x^3 - (a+b)x^2 + abx - cx^2 + (a+b)cx - abc \\ &= x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc\,. \qquad (2) \end{aligned}$$

From (1) and (2) we get

$$\begin{aligned} a+b+c &= 19\,, \\ ab+bc+ac &= 26\,, \\ abc &= 2\,. \end{aligned}$$

Thus, we have $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{bc+ac+ab}{abc} = \frac{26}{2} = 13$.

3. A collection of 52 integers is given. Show that amongst these numbers it is possible to find two such that 100 divides either their sum or their difference.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Hongyi Li, student, Sir Winston Churchill High School, Calgary, AB; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Pavlos Maragoudakis, Pireas, Greece. We give the solution of Klamkin.

This problem appeared as a Russian Olympiad problem quite some time ago. Unfortunately, I do not have a reference.

Each of the numbers can be expressed in the form $100a + b$, where $-49 \leq b \leq 50$. Since there are only 51 possible values for $|b|$, at least two of them must be the same. If the corresponding numbers have opposite signs associated with b, then the sum of the numbers is divisible by 100; if they have the same signs, then the difference is divisible by 100.

The result is true more generally if we replace 100 by n and 52 by $\lfloor n/2 \rfloor + 2$.

4. (i) Show that the sum of the digits of every integer multiple of 99, from $1 \cdot 99$ up to and including $100 \cdot 99$, is 18.

(ii) Show that the sum of the digits of every integer multiple of the number $10^n - 1$, from $1 \cdot (10^n - 1)$ up to and including $10^n \cdot (10^n - 1)$, is $n \cdot 9$.

Solved by Michel Bataille, Rouen, France; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Samapti Samapti, Western Canada High School, Calgary, AB. We give the solution of Klamkin.

(i) $99n = 100n - n$. Thus, if n is a single-digit number, the digits resulting from subtracting n from the 3–digit number $n00$ are $n - 1$, 9, and $10 - n$, for a sum of 18. If n is a 2–digit number ab with $b = 0$, the digits of $100n - n$ are $a - 1$, 9, $10 - a$, and 0, which still sum to 18; if $b \neq 0$, then the digits of $ab00 - ab$ are a, $b - 1$, $10 - a - 1$, and $10 - b$, again summing to 18. (In a similar way it follows that the sum of the digits of *every* multiple of 99 is 18).

(ii) Proceeding in a similar way as in (i), it follows that the sum of the digits of the number $99\ldots9m$, where there are n 9's, is obtained by subtracting m from $m00\ldots0$. For example, if m is the 2–digit number ab not ending in 0, then the successive digits are a, $b - 1$, 9, 9, $\ldots$, 9, $10 - a - 1$, and $10 - b$. Hence, the sum of the digits is $9n$.

5. The sequence $\{a_n\}$ is defined by $a_1 = 1$ and, for $n \geq 1$,

$$a_{n+1} = \frac{a_n}{1 + na_n}\,.$$

Find a_{1996}.

Solved by Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; Samapti Samapti, Western Canada High School, Calgary, AB; Heinz-Jürgen Seiffert, Berlin, Germany; D.J. Smeenk, Zaltbommel, the Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution of Bataille.

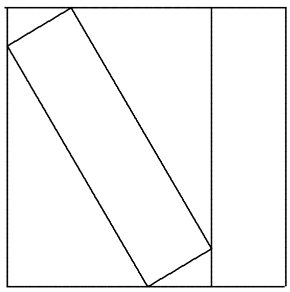
By an immediate induction, $a_n > 0$ for all n.

Now, $\dfrac{1}{a_{n+1}} = \dfrac{1 + na_n}{a_n} = \dfrac{1}{a_n} + n$ for all n. It follows that

$$\frac{1}{a_{1996}} - \frac{1}{a_1} = \sum_{k=1}^{1995}\left(\frac{1}{a_{k+1}} - \frac{1}{a_k}\right) = \sum_{k=1}^{1995} k = \frac{1995 \times 1996}{2}\,.$$

Since $a_1 = 1$, we deduce that $\dfrac{1}{a_{1996}} = 1 + \dfrac{1995 \times 1996}{2} = 1991011$. Therefore, $a_{1996} = \dfrac{1}{1991011}$.

6. In a square bookcase two identical books are placed as shown in the figure. Suppose the height of the bookcase is 1. How thick are the books?



Solved by Rahul Banotra, student, Sir Winston Churchill High School, Calgary, AB; Robert Bilinski, Outremont, QC; Christopher J. Bradley, Clifton College, Bristol, UK; Murray S. Klamkin, University of Alberta, Edmonton, AB; and D.J. Smeenk, Zaltbommel, the Netherlands. We give Bilinski's solution.

The solution is illustrated by a diagram at the end.

Due to the abundance of right angles and complementary angles, we have two pairs of congruent triangles: $\triangle ABI \cong \triangle FGJ$, $\triangle CJB \cong \triangle HIG$. All four of these triangles are similar, since they have corresponding angles equal.

Since $GI = BJ = 1$ (the book height), we have $HI = CJ = \cos\theta$ and $HG = BC = \sin\theta$. Since $AH = CF = 1$, we quickly conclude that $AI = JF = 1 - \cos\theta$.

In $\triangle ABI$, we have $\sin\theta = \frac{1-\cos\theta}{BI}$, which gives $BI = \frac{1-\cos\theta}{\sin\theta}$, the width of the book ($BI = GJ = CD = FE$). In a similar fashion, we find that $AB = \frac{\cos\theta(1-\cos\theta)}{\sin\theta}$.

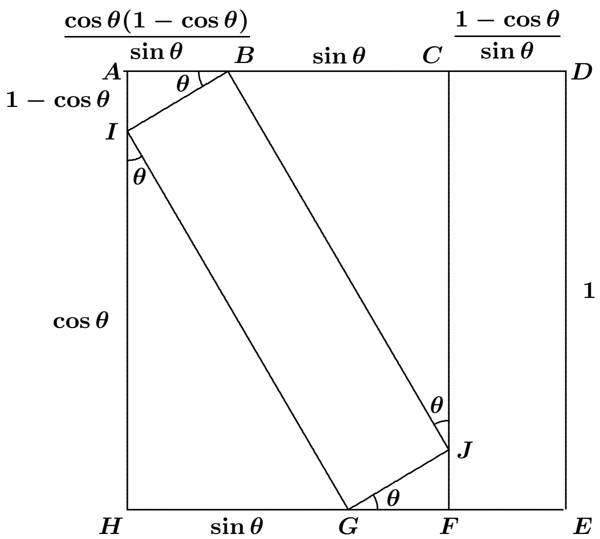
Since

$$1 = AD = AB + BC + CD = \frac{\cos\theta(1-\cos\theta)}{\sin\theta} + \sin\theta + \frac{1-\cos\theta}{\sin\theta},$$

we get

$$\begin{aligned}\frac{\cos\theta - \cos^2\theta + \sin^2\theta + 1 - \cos\theta}{\sin\theta} &= 1,\\ \frac{2\sin^2\theta}{\sin\theta} &= 1,\\ \sin\theta &= \frac{1}{2}.\end{aligned}$$

Hence, $\theta = 30^\circ$, and the book width is $BI = \frac{1-\cos 30^\circ}{\sin 30^\circ} = 2 - \sqrt{3}$.



That concludes this issue of the *Corner*. Please keep sending me Olympiad contests and your nice solutions.