SKOLIAD No. 71

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We are looking for solutions especially from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 March 2004*. A copy of **MATHEMATICAL MAYHEM Vol. 5** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

We will only print solutions to problems marked with an asterisk ($*$) if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

The first item this issue comes from the 2003 first annual Fryer Contest. This and two other contests, the Galois and the Hypatia, were introduced this year for students in grades 9, 10, and 11, respectively, by the Canadian Mathematics Competitions. My thanks go out to Ian VanderBurgh and Peter Crippin of The University of Waterloo for forwarding the material to me. We especially invite students in grade 10 (or equivalent) or earlier to send in solutions.

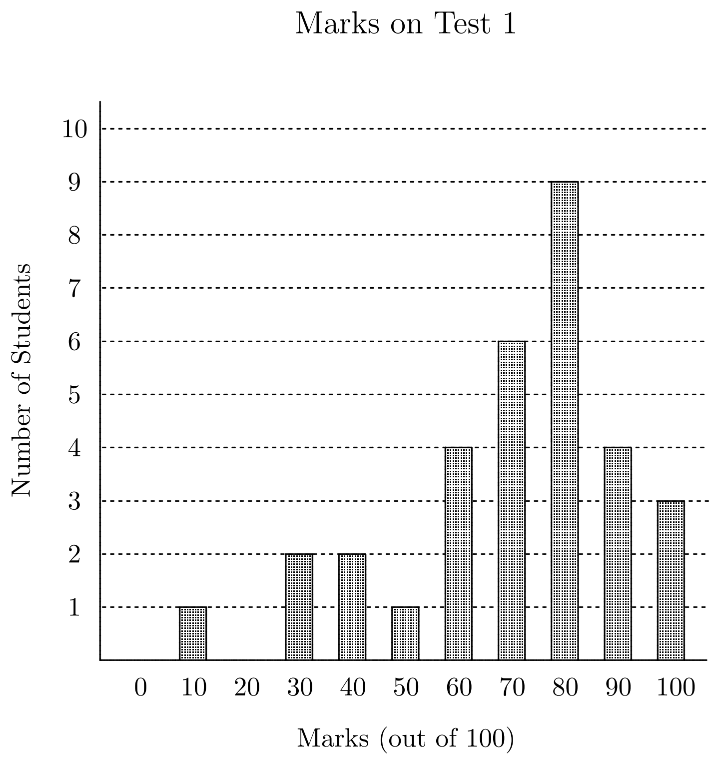
2003 Fryer Contest
April 16, 2003

1. (a) ($*$) The marks of 32 mathematics students on Test 1 are all multiples of 10 and are shown on the bar graph. What was the average (mean) mark of the 32 students in the class?

(b) ($*$) After his first 6 tests, Paul has an average of 86. What will his average be if he scores 100 on his next test?

(c) ($*$) Later in the year, Mary realizes that she needs a mark of 100 on the next test in order to achieve an average of 90 for all her tests. However, if she gets a mark of 70 on the next test, her average will be 87. After she writes the next test, how many tests will she have written?

Extension to #1: ($*$) Mary's teacher records the final marks of the 32 students. The teacher calculates that, for the entire class, the median mark is 80. The teacher also calculates that the difference between the highest and lowest marks is 40 and calculates that the average mark for the entire class is 58. Show that the teacher has made a calculation error.



(a) (∗) Les notes de 32 élèves, lors de leur première épreuve de mathématiques, sont toutes des multiples de 10. Elles sont indiquées dans le diagramme à bâtons. Quelle est la moyenne des notes des 32 élèves de la classe ?

(b) (∗) Après 6 épreuves, Paul a une moyenne de 86. Quelle sera sa moyenne s'il obtient une note de 100 lors de la prochaine épreuve ?

(c) (∗) Plus tard dans l'année, Marie se rend compte qu'elle a besoin d'une note de 100 lors de la prochaine épreuve pour que sa moyenne, dans toutes les épreuves, soit égale à 90. Or si elle obtient une note de 70 dans la prochaine épreuve, sa moyenne sera égale à 87. Lorsqu'elle aura terminé la prochaine épreuve, combien d'épreuves aura-t-elle écrites ?

Prolongement du Problème 1 *:* (∗) L'enseignante de Marie inscrit la note finale des 32 élèves. L'enseignante calcule la médiane de la classe et obtient une note de 80. Elle calcule aussi l'étendue des notes, soit la différence entre la note la plus haute et la note la plus basse, et obtient 40. Elle calcule enfin la moyenne de la classe et obtient 58. Démontrer que l'enseignante a commis une erreur.

2. In a game, Xavier and Yolanda take turns calling out whole numbers. The first number called must be a whole number between and including 1 and 9. Each number called after the first must be a whole number which is 1 to 10 greater than the previous number called.

(a) ($*$) The first time the game is played, the person who calls the number 15 is the winner. Explain why Xavier has a winning strategy if he goes first and calls 4.

(b) ($*$) The second time the game is played, the person who calls the number 50 is the winner. If Xavier goes first, how does he guarantee that he will win?

Extension to #2: ($*$) In the game described in b), the target number was 50. For what different values of the target number is it guaranteed that Yolanda will have a winning strategy if Xavier goes first?

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Xavier et Yvonne participent à un jeu dans lequel chacun, tour à tour, annonce un numéro qu'il ou elle a choisi. Le premier numéro doit être un entier de 1 à 9. Chaque numéro subséquent doit être un entier qui est de 1 à 10 de plus que le numéro précédent.

(a) ($*$) Lors de la première partie, la personne qui annoncera le numéro 15 sera déclarée gagnante. Expliquer que Xavier a une stratégie gagnante s'il joue premier en annonçant le numéro 4.

(b) ($*$) Lors de la deuxième partie, la personne qui annoncera le numéro 50 sera déclarée gagnante. Si Xavier joue premier, comment peut-il s'assurer de gagner ?

Prolongement du Problème 2 : ($*$) Dans la partie b), le nombre-cible était 50. Quelles sont les valeurs du nombre-cible qui peuvent assurer à Yvonne une stratégie gagnante si Xavier joue premier ?

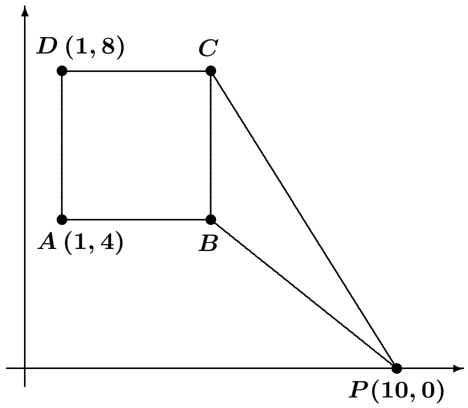
3. In the diagram, $ABCD$ is a square and the coordinates of A and D are as shown.

(a) ($*$) The point P has coordinates $(10, 0)$. Show that the area of triangle PCB is 10.

(b) ($*$) Point $E(a, 0)$ is on the x–axis such that triangle CBE lies entirely outside square $ABCD$. If the area of the triangle is equal to the area of the square, what is the value of a?

(c) ($*$) Show that there is no point F on the x–axis for which the area of triangle ABF is equal to the area of square $ABCD$.

Extension to #3: ($*$) G is a point on the line passing through the points $M(0, 8)$ and $N(3, 10)$ such that $\triangle DCG$ lies entirely outside the square. If the area of $\triangle DCG$ is equal to the area of the square, determine the coordinates of G.



$ABCD$ est un carré et les coordonnées de A et de D sont indiquées.

(a) ($*$) Le point P a pour coordonnées $(10, 0)$. Montrer que le triangle PCB a une aire de 10.

(b) ($*$) Soit un point $E(a, 0)$, sur l'axe des abscisses, de manière que le triangle CBE soit situé complètement à l'extérieur du carré $ABCD$. Si l'aire du triangle est égale à l'aire du carré, quelle est la valeur de a?

(c) ($*$) Démontrer qu'il n'existe aucun point F, sur l'axe des abscisses, pour lequel l'aire du triangle ABF est égale à l'aire du carré $ABCD$.

Prolongement du Problème 3 : ($*$) Soit G un point sur la droite qui passe par les points $M(0, 8)$ et $N(3, 10)$, de manière que le triangle DCG soit situé complètement à l'extérieur du carré. Déterminer les coordonnées de G, sachant que l'aire du triangle est égale à l'aire du carré.

4. For the set of numbers {1, 10, 100} we can obtain 7 distinct numbers as totals of one or more elements of the set. These totals are 1, 10, 100, $1 + 10 = 11$, $1 + 100 = 101$, $10 + 100 = 110$, and $1 + 10 + 100 = 111$. The *power-sum* of this set is the sum of these totals, in this case, 444.

(a) ($*$) How many distinct numbers may be obtained as a sum of one or more different numbers from the set {1, 10, 100, 1000} ? Calculate the power-sum for this set.

(b) ($*$) Determine the power-sum of the set

$$\{1,\ 10,\ 100,\ 1000,\ 10\,000,\ 100\,000,\ 1\,000\,000\}.$$

Extension to #4: ($*$) Consider the set {1, 2, 3, 6, 12, 24, 48, 96}. How many different totals are now possible if a total is defined as the sum of 1 or more elements of a set?

Étant donné l'ensemble $\{1, 10, 100\}$, on peut obtenir 7 totaux distincts en additionnant un nombre ou plus de cet ensemble. Ces totaux sont 1, 10, 100, $1+10 = 11$, $1+100 = 101$, $10+100 = 110$, et $1+10+100 = 111$. La *somme-puissance* de cet ensemble est la somme de ces totaux. Elle est égale à 444.

(a) ($*$) Étant donné l'ensemble $\{1, 10, 100, 1000\}$, combien peut-on obtenir de totaux distincts en additionnant un nombre ou plus de cet ensemble ? Calculer la somme-puissance de cet ensemble.

(b) ($*$) Déterminer la somme-puissance de l'ensemble

$$\{1, 10, 100, 1000, 10\,000, 100\,000, 1\,000\,000\}.$$

Prolongement du Problème 4 : ($*$) Soit l'ensemble $\{1, 2, 3, 6, 12, 24, 48, 96\}$. Combien peut-on obtenir de totaux distincts en additionnant un nombre ou plus de cet ensemble ?

Our second item this month is the 2002 W.J. Blundon Mathematics Contest. My thanks go out to Don Rideout of Memorial University for forwarding the material to me.

The Nineteenth W.J. Blundon Mathematics Contest

Sponsored by
The Canadian Mathematical Society
in cooperation with
The Department of Mathematics and Statistics
Memorial University of Newfoundland

February 20, 2002

1. ($*$) Five years ago Janet was one sixth of her mother's age. In thirteen years she will be half her mother's age. What is Janet's present age?

2. ($*$) If $a + b + c = 0$, prove that $a^3 + b^3 + c^3 = 3abc$.

3. ($*$) A certain rectangle has area 6 and diagonal of length $2\sqrt{5}$. What is its perimeter?

4. Find all positive numbers x such that $x^{x\sqrt{x}} = (x\sqrt{x}\,)^x$.

5. Rationalize the denominator: $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{6}}$.

6. Points A and B are on the parabola $y = 2x^2 + 4x - 2$. The origin is the mid-point of the line segment joining A and B. Find the length of this line segment.

7. If $\log_{125} 2 = a$ and $\log_9 25 = b$, find $\log_8 9$ in terms of a and b.

8. Point P lies in the first quadrant on the line $y = 2x$. Point Q is a point on the line $y = 3x$ such that PQ has length 5 and is perpendicular to the line $y = 2x$. Find the point P.

9. For what conditions on a and b is the line $x + y = a$ tangent to the circle $x^2 + y^2 = b$?

10. In $\triangle ABC$, we have $\angle ACB = 120$ degrees, $AC = 6$ and $BC = 2$. The internal bisector of $\angle ACB$ meets the side AB at the point D. Determine the length of the line segment CD.

Next, we present the solutions to the 2^{nd} Junior Balkan Mathematical Olympiad (1998) that appeared in the December 2002 issue ([2002 : 522]).

1. (Yugoslavia) Prove that the number

$$\underbrace{11\ldots111}_{1997}\underbrace{22\ldots222}_{1998}5$$

is a perfect square.

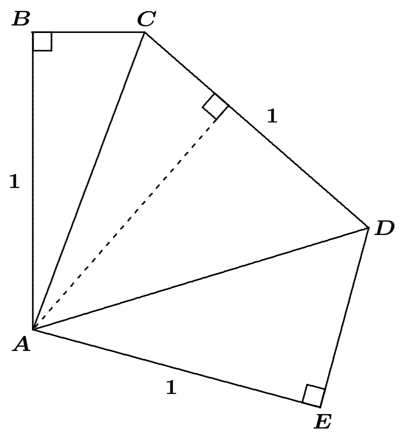
Official Solution.

$$\begin{aligned}
\underbrace{11\ldots111}_{1997}\underbrace{22\ldots222}_{1998}5 &= \underbrace{11\ldots111}_{1997}\times 10^{1999} + \underbrace{22\ldots222}_{1998}\times 10 + 5 \\
&= \frac{10^{1997}-1}{9}\cdot 10^{1999} + 2\cdot\frac{10^{1998}-1}{9}\cdot 10 + 5 \\
&= \frac{1}{9}\left(10^{3996} - 10^{1999} + 2\cdot 10^{1999} - 20 + 45\right) \\
&= \frac{1}{9}\left(10^{3996} + 2\cdot 5\cdot 10^{1998} + 25\right) \\
&= \frac{1}{9}\left(10^{1998} + 5\right)^2 \\
&= (\underbrace{33\ldots33}_{1997}5)^2\,.
\end{aligned}$$

2. (Greece) Let us consider a convex pentagon $ABCDE$, with $AB = AE = CD = 1$, $\angle ABC = \angle DEA = 90^\circ$ and $BC + DE = 1$. Find the area of the pentagon.

Official Solution.

Draw the diagonals AC and AD. Since $\angle B = \angle E = 90^\circ$ and $AB = AE$, we can create a triangle from the triangles ABC and AED having altitude $AB = AE = 1$ and base $BC + DE = 1$. The area of this new triangle is $\frac{1}{2}$. This newly created triangle is then congruent to $\triangle ACD$; hence, the total area of the pentagon is 1.



3. (Albania) Find all pairs of positive integers (x, y) that satisfy the following equation: $x^y = y^{x-y}$.

Official Solution.

If (x, y) is a solution of the given equation and $x \neq 1$, then $x > y$. Otherwise, we would have $y^{x-y} \leq 1$, while $x^y > 1$. Obviously, $(1, 1)$ is a solution of the given equation, and this is the only solution with $y = 1$. Let $x > y \geq 2$. Then we obtain the equation

$$\left(\frac{x}{y}\right)^y = y^{x-2y}\,.$$

Since $\frac{x}{y} > 1$, we get $x - 2y > 0$ and $\frac{x}{y} > 2$. Also, $\frac{x}{y} \in \mathbb{N}$. Our equation above can be written as

$$\frac{x}{y} = y^{\frac{x}{y}-2}\,.$$

Since $y^{\frac{x}{y}-2} \geq 2^{\frac{x}{y}-2}$, we obtain $\frac{x}{y} \leq 4$. We conclude that $2 < \frac{x}{y} \leq 4$.

- If $\frac{x}{y} = 3$, then $y = 3$, $x = 9$.
- If $\frac{x}{y} = 4$, then $y = 2$, $x = 8$.

Finally, the set of solutions of the given equation is:

$$\{(1,1)\,,(8,2)\,,(9,3)\}\ .$$

4. (Bulgaria) Using only three digits can one write 16 three-digit numbers, such that no two of them are giving the same remainder divided by 16?

Official Solution.

Let us suppose that we can write down 16 such numbers. It is clear that 8 of these numbers must be even and the rest must be odd. Therefore, the digits cannot be all even or all odd. Let us examine the case where the given digits are two even and one odd. (The case with two odd digits and one even digit is similar.) Let e_1, e_2, and o denote the two even digits and the odd digit, respectively. We can write exactly 9 odd three-digit numbers with the given digits:

$$e_1e_1o\,,\,e_1e_2o\,,\,e_1oo\,,\,e_2e_1o\,,\,e_2e_2o\,,\,e_2oo\,,\,oe_1o\,,\,oe_2o\,,\,ooo\,.$$

If we rewrite these as a_1o, a_2o, $\dots$, a_9o, where a_1, a_2, $\dots$, a_9 are two-digit numbers formed by the first two digits of the numbers in the first list, then $a_io - a_jo$ is divisible by 16 if and only if $a_i - a_j$ is divisible by 8. But there are only three two-digit odd numbers among a_1, a_2, $\dots$, a_9, whereas four are required to obtain four different odd remainders when divided by 8. Therefore, we cannot write down 16 such three-digit numbers.

Finally, we present the answers to the 2002 British Columbia Colleges Junior High School Mathematics Contest, Preliminary Round that appeared in the December 2002 issue ([2002 : 523–525]).

1. b **2**. b **3**. b **4**. c **5**. a **6**. e **7**. b **8**. c
9. e **10**. d **11**. d **12**. e **13**. a **14**. c **15**. e

That brings us to the end of another issue of Skoliad. Continue sending in contests and solutions.