

THE OLYMPIAD CORNER

No. 228

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We begin this number with problems of the 2000 Belarusian Mathematical Olympiad. My thanks go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

2000 BELARUSIAN MATHEMATICAL OLYMPIAD

1. Pete and Bill play the following game. At the beginning, Pete chooses a number a , then Bill chooses a number b , and then Pete chooses a number c . Can Pete choose his numbers in such a way that the three equations $x^3 + ax^2 + bx + c = 0$, $x^3 + bx^2 + cx + a = 0$ and $x^3 + cx^2 + ax + b = 0$ have a common

(a) real root?

(b) negative root?

2. How many pairs (n, q) satisfy $\{q^2\} = \{\frac{n!}{2000}\}$, where n is a positive integer and q is a non-integer rational number such that $0 < q < 2000$? [Editor's comment: $\{r\}$ means the "fractional part" of r .]

3. Given a fixed integer $N \geq 5$, and any sequence e_1, e_2, \dots, e_N , where $e_i \in \{1, -1\}$ for $i = 1, 2, \dots, N$, a move is made by choosing any five consecutive terms and changing their signs. Two such sequences are said to be similar if one of them can be obtained from the other in a finite number of moves. Find the maximal number of sequences no two of which are similar to each other.

4. Let $ABCD$ be a quadrilateral with AB parallel to DC . A line ℓ intersects AD , AC , BD , and BC , forming three segments of equal length between consecutive points of intersection. Does it follow that ℓ is parallel to AB ?

5. Nine points are given on a plane, no three on a line. Every pair of points is connected by a segment. Is it possible to colour these segments by some colours so that for each colour used, there are exactly three segments of this colour, and these three form a triangle with vertices among the given points?

6. A vertex of a tetrahedron is called *perfect* if one can construct a triangle using edges from this vertex as its sides. What are the possible numbers of perfect vertices a tetrahedron can have?

7. (a) Find all positive integers n such that $(a^a)^n = b^b$ has at least one solution in integers a and b , both exceeding 1.

(b) Find all positive integers a and b such that $(a^a)^5 = b^b$.

8. To any triangle ABC with $AB = c$, $BC = a$, $CA = b$, $\angle A = \alpha$, $\angle B = \beta$, and $\angle C = \gamma$, we assign the sextuple $(a, b, c, \alpha, \beta, \gamma)$, where the angles are measured in radians. Find the minimal value of n for which there is a non-isosceles triangle ABC such that there are exactly n distinct numbers in $(a, b, c, \alpha, \beta, \gamma)$.

As a second problem set we give the 2000 Taiwanese Mathematical Olympiad. Again, thanks to Andy Liu for collecting them for our use while Team Leader for the Canadian Team to the IMO in Korea.

2000 TAIWANESE MATHEMATICAL OLYMPIAD

1. Find all pairs (x, y) of positive integers such that $y^{x^2} = x^{y+2}$.

2. In an acute triangle ABC , $AC > BC$ and M is the mid-point of AB . Let AP be the altitude from A . Let BQ be the altitude from B meeting AP at H . Let the lines AB and PQ meet at R . Prove that the lines RH and CM are perpendicular to each other.

3. Let $S = \{1, 2, 3, \dots, 100\}$, and let \mathcal{P} denote the family of all subsets T of S with $|T| = 49$. For each set T in \mathcal{P} , we label it with a number chosen at random from $\{1, 2, \dots, 100\}$. Prove that there exists a subset M of S with $|M| = 50$ such that for each $x \in M$, $M - \{x\}$ is not labelled with x .

4. Let $\phi(k)$ denote the number of positive integers $n \leq k$ such that $\gcd(n, k) = 1$. Suppose that $\phi(5^m - 1) = 5^n - 1$ for some positive integers m and n . Prove that $\gcd(m, n) > 1$.

5. Let $A = \{1, 2, 3, \dots, n\}$, where n is a positive integer. A subset of A is said to be *connected* if it consists of one element or some consecutive integers. Determine the greatest integer k for which A contains k distinct subsets such that the intersection of any two of them is connected.

6. Let f be a function from the set of positive integers to the set of non-negative integers such that $f(1) = 0$ and

$$f(n) = \max\{f(j) + f(n-j) + j\}$$

for all $n \geq 2$. Determine $f(2000)$.

Now we turn to the problems of the Composition de Mathématiques, Classe Terminale S given [2000 : 453–454]. We received reader solutions by Mohammed Aassila, Strasbourg, France. However, we will only present here the web address, provided by Pierre Bornsztein, Pontoise, France, where the official solutions may be found. Go to

www.ac-poitiers.fr/voir.asp?r=88

Then click successively on “Pour s’y retrouver”, “Plan du site”, “Concours général”, “Concours 97: indications et corrigé”, and finally on “Corrigé des 5 exercices”.

Next, we give solutions to the Ukrainian Mathematical Olympiad, Selected Problems 1997 [2001 : 5–6].

1. (9th Grade) Cells of some rectangular board are coloured as chess-board cells. In each cell an integer is written. It is known that the sum of the numbers in each row is even and the sum of numbers in each column is even. Prove that the sum of all numbers in the black cells is even.

Solved by Bruce Crofoot, University College of the Cariboo, Kamloops, BC; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We present the solution by Crofoot.

First observe that in any particular row or column, the sum of the numbers in the black cells has the same parity (even or odd) as the sum of the numbers in the white cells, because the sum over all cells in the column is even.

Now consider the sum over all the black cells on the board. We imagine this sum to be calculated by columns. Since we are interested only in the parity of the sum, we are free to sum over all the white cells instead of the black cells in any given column. Thus, we sum over the black cells in the odd-numbered columns and over the white cells in the even-numbered columns. Thinking now in terms of rows, our sum is effectively over *all* cells (both black and white) in every second row. This sum is clearly even, since the sum over all the cells in any row is even.

2. (10th Grade) Solve the system in real numbers

$$\begin{cases} x_1 + x_2 + \cdots + x_{1997} = 1997 \\ x_1^4 + x_2^4 + \cdots + x_{1997}^4 = x_1^3 + x_2^3 + \cdots + x_{1997}^3 \end{cases}$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We first give the solution by Bornsztein.

More generally, let n be a positive integer. We will prove that the unique solution in real numbers of the system

$$\begin{cases} x_1 + x_2 + \cdots + x_n = n \\ x_1^4 + x_2^4 + \cdots + x_n^4 = x_1^3 + x_2^3 + \cdots + x_n^3 \end{cases}$$

is $(1, 1, \dots, 1)$. It is easy to see that $(1, 1, \dots, 1)$ is a solution of the system. Conversely, if x_1, x_2, \dots, x_n are real numbers satisfying the system, then

$$\begin{aligned} 0 &= \sum_{i=1}^n x_i^4 - \sum_{i=1}^n x_i^3 - \sum_{i=1}^n x_i + n = \sum_{i=1}^n (x_i^4 - x_i^3 - x_i + 1) \\ &= \sum_{i=1}^n (x_i - 1)^2 (x_i^2 + x_i + 1) = \sum_{i=1}^n (x_i - 1)^2 \left(\left(x_i + \frac{1}{2} \right)^2 + \frac{3}{4} \right). \end{aligned}$$

Since, for each i , we have $(x_i - 1)^2 \left(\left(x_i + \frac{1}{2} \right)^2 + \frac{3}{4} \right) \geq 0$, the equality occurs only if $x_i = 1$ for $i = 1, 2, \dots, n$.

Next we give an alternate approach by Maragoudakis.

By Chebyshev's inequality, if $a_1 \leq \dots \leq a_n$ and $b_1 \leq \dots \leq b_n$, then

$$n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

Equality occurs if and only if at least one sequence is constant [2001 : 514].

Without loss of generality, we assume that $x_1 \leq x_2 \leq \dots \leq x_{1997}$. Then $x_1^3 \leq x_2^3 \leq \dots \leq x_{1997}^3$. Therefore,

$$1997(x_1^4 + x_2^4 + \cdots + x_{1997}^4) \geq (x_1 + x_2 + \cdots + x_{1997})(x_1^3 + x_2^3 + \cdots + x_{1997}^3).$$

Since $x_1 + x_2 + \cdots + x_{1997} = 1997$ and

$$x_1^4 + x_2^4 + \cdots + x_{1997}^4 = x_1^3 + x_2^3 + \cdots + x_{1997}^3,$$

the above inequality becomes equality. Thus,

$$x_1 = x_2 = \cdots = x_{1997} \quad \text{or} \quad x_1^3 = x_2^3 = \cdots = x_{1997}^3.$$

In either case, $x_1 = x_2 = \cdots = x_{1997} = 1$.

3. (10th Grade) Let $d(n)$ denote the greatest odd divisor of the natural number n . We define the function $f : \mathbb{N} \rightarrow \mathbb{N}$ as follows: $f(2n - 1) = 2^n$, $f(2n) = n + \frac{2n}{d(n)}$ for all $n \in \mathbb{N}$.

Find all k such that $f(f(\dots f(1)\dots)) = 1997$, where f is iterated k times.

Solution by Pierre Bornsztein, Pontoise, France.

Let (x_k) be the sequence defined by $x_1 = 1$ and $x_{k+1} = f(x_k)$ for all $k \geq 1$. We want to find the integers k such that $x_{k+1} = 1997$.

The first terms of the sequence (x_k) are 1, 2, 3, 4, 6, 5, 8, 12, 10, 7, 16, 24, We present them in successive rows, R_1, R_2, R_3, \dots , where R_j contains exactly j terms:

$$\begin{aligned} R_1 &: 1 \\ R_2 &: 2, 3 \\ R_3 &: 4, 6, 5 \\ R_4 &: 8, 12, 10, 7 \\ R_5 &: 16, 24, 20, 14, 9 \\ &\vdots \end{aligned}$$

We will prove that, for all positive integers i, j , with $j \leq i$, the j^{th} number in R_i is $(2j - 1)2^{i-j}$.

This is clearly true for $i = 1$. Let $i \geq 1$ be fixed. Suppose that the result is true for R_i . Then the last term of R_i (the one at the right) is $2i - 1$. It follows that the first term of R_{i+1} is $f(2i - 1) = 2^i = (2 \times 1 - 1)2^{i+1-1}$. Thus, the desired formula is true for this first term.

Suppose that the result is true for the j^{th} number in row R_{i+1} , where $1 \leq j < i + 1$. Then the following term in R_{i+1} is:

$$f((2j - 1)2^{i+1-j}) = (2j - 1)2^{i-j} + 2^{i-j+1} = (2(j + 1) - 1)2^{i+1-(j+1)}.$$

Therefore, the formula is true for the value $j + 1$. By induction, it is true for all $j \in \{1, 2, \dots, i + 1\}$.

Thus, the formula is true for all of R_{i+1} . By induction, it is true for all the rows.

Let $a = m2^n$, where m, n are non-negative integers and m is odd. Then $m2^n = (2j - 1)2^{i-j}$ if and only if

$$\begin{cases} m = 2j - 1 \\ n = i - j \end{cases} ; \quad \text{that is,} \quad \begin{cases} k = \frac{m + 1}{2} \\ i = n + \frac{m + 1}{2} \end{cases}.$$

It follows that a appears exactly once in the sequence, in position $\frac{m + 1}{2}$ in $R_{n + \frac{m + 1}{2}}$.

If $a = 1997$, then $m = 1997$ and $n = 0$. Thus, $k = i = 999$. Therefore, $x_k = 1997$ if and only if x_k is the last term of R_{999} , in which case

$$k = 1 + 2 + 3 + \dots + 999 = 999 \cdot 500 = 499500.$$

5. (11th Grade) It is known that the equation $ax^3 + bx^2 + cx + d = 0$ with respect to x has three distinct real roots. How many roots does the equation $4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2$ have?

Comment by Murray S. Klamkin, University of Alberta, Edmonton, AB.

As stated, the problem is trivial. Since the degree of the given equation is 4, it must have 4 roots. Perhaps the original version asked for the number of real roots?

Solution by Bruce Crofoot, University College of the Cariboo, Kamloops, BC.

Let $p(x) = ax^3 + bx^2 + cx + d$, and let

$$f(x) = 4(ax^3 + bx^2 + cx + d)(3ax + b) - (3ax^2 + 2bx + c)^2.$$

We are told that the equation $p(x) = 0$ has three distinct real roots, and we are asked about the roots of the equation $f(x) = 0$. Clearly, the problem is interested in real roots only. Note that $f(x) = 2p(x)p''(x) - [p'(x)]^2$. Hence, $f'(x) = 2p(x)p'''(x) = 12ap(x)$ and $f''(x) = 12ap'(x)$.

Let the roots of $p(x)$ be r_1, r_2 , and r_3 , where $r_1 < r_2 < r_3$. Since these roots are distinct, $p'(r_i) \neq 0$ (for $i = 1, 2, 3$). Since there are three roots, the degree of $p(x)$ cannot be less than three. Therefore, $a \neq 0$. We can assume $a > 0$. (Otherwise we replace $p(x)$ by $-p(x)$, with no effect on $f(x)$.) Then $\lim_{x \rightarrow \infty} p(x) = \infty$ and $\lim_{x \rightarrow -\infty} p(x) = -\infty$. Hence $p'(r_1) > 0, p'(r_2) < 0$ and $p'(r_3) > 0$.

For each i , $f'(r_i) = 12ap(r_i) = 0$, and there are no other points where $f'(x) = 0$. Since $f''(x)$ has the same sign as $p'(x)$, we have $f''(r_1) > 0$, $f''(r_2) < 0$ and $f''(r_3) > 0$. Thus, f has local minima at r_1 and r_3 , a local maximum at r_2 , and no other local maxima or minima. For each i , $f(r_i) = -[p'(r_i)]^2 < 0$. Therefore, $f(x) < 0$ on an interval containing r_1, r_2 and r_3 . Furthermore, $\lim_{x \rightarrow \pm\infty} f(x) = \infty$ (since the highest-degree term in $f(x)$ is $3a^2x^4$). All of this implies that the equation $f(x) = 0$ has exactly two real roots, one of which is less than r_1 (where $f(x)$ changes from positive to negative) and the other greater than r_3 (where $f(x)$ changes from negative to positive).

6. (11th Grade) Let \mathbb{Q}^+ denote the set of all positive rational numbers. Find all functions $f: \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$:

(a) $f(x+1) = f(x) + 1$,

(b) $f(x^2) = (f(x))^2$.

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give the write-up of Bataille.

Let f be any such function. By property (a) and an immediate induction, we get

$$f(x+n) = f(x) + n \quad \text{for all } x \in \mathbb{Q}^+ \text{ and } n \in \mathbb{N}.$$

On the other hand,

$$\begin{aligned} (f(x+n))^2 &= f((x+n)^2) = f(x^2 + 2nx + n^2) \\ &= f(x^2 + 2nx) + n^2. \end{aligned}$$

Comparing, we obtain

$$f(x^2 + 2nx) = f(x^2) + 2nf(x) \quad (1)$$

for all $x \in \mathbb{Q}^+$ and $n \in \mathbb{N}$.

Now, let $r = \frac{p}{q}$ be any element of \mathbb{Q}^+ , where $p \in \mathbb{N}$ and $q \in \mathbb{N}$. From (1), with $x = r$ and $n = q$, we get $f(r^2 + 2p) = f(r^2) + 2qf(r)$. Then

$$f(r^2) + 2p = f(r^2) + 2qf(r),$$

which yields $f(r) = \frac{2p}{2q} = r$. Therefore, f is the identity on \mathbb{Q}^+ .

Conversely, the identity of \mathbb{Q}^+ clearly satisfies conditions (a) and (b), whence it is the unique solution.

7. (11th Grade) Find the smallest n such that among any n integers there are 18 integers whose sum is divisible by 18.

Solved by Pierre Bornsztejn, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Bornsztejn's account.

The smallest n is 35.

Erdős has proved, more generally, that for any given integer $k > 1$, among any $2k - 1$ integers there are k integers whose sum is divisible by k (see [1]). Consider any $2k - 2$ integers such that $k - 1$ of them are equal to 0 modulo k , and the other $k - 1$ are equal to 1 modulo k . It is easy to see that, among these $2k - 2$ integers, we cannot find k integers whose sum is divisible by k . Thus, the value $2k - 1$ is indeed minimal. _____

Reference:

[1] R. Graham, *The Mathematical Intelligencer*, 1979, p. 250.

Next on the list are solutions to problems of the Tenth Irish Mathematical Olympiad 1997 given [2001 : 6–8].

1. Find (with proof) all pairs of integers (x, y) satisfying the equation

$$1 + 1996x + 1998y = xy.$$

Solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztejn, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution and comment by Amengual Covas.

We find all pairs of integers (x, y) satisfying the more general equation

$$1 + (p - 1)x + (p + 1)y = xy,$$

where $p > 1$ is a prime number. This equation is equivalent to

$$px + py = xy + x - y - 1,$$

which can be rewritten as

$$p((x - 1) + (y + 1)) = (x - 1)(y + 1). \quad (1)$$

We observe that $(x, y) = (1, -1)$ is a solution and that no other candidates for (x, y) with $x = 1$ or $y = -1$ can satisfy (1).

Now suppose that $x \neq 1$ and $y \neq -1$, and denote by d the greatest common divisor of $x - 1$ and $y + 1$. We have

$$x - 1 = du, \quad y + 1 = dv, \quad (2)$$

where u and v are relatively prime integers. Substituting these expressions for $x - 1$ and $y + 1$ into (1) and dividing both sides by d gives

$$p(u + v) = duv. \quad (3)$$

Hence, uv divides the product $p(u + v)$ and is relatively prime to $u + v$. By the Fundamental Theorem of Arithmetic, uv is a divisor of p . Thus,

$$uv \in \{1, -1, p, -p\}.$$

Since $p > 0$ and $d > 0$, it follows from (3) that $u + v$ and uv agree in sign. This leads to the following possibilities:

- $u = v = 1$. Then, by (3), $d = 2p$. Substituting these values into (2), we find that

$$x = 2p + 1, \quad y = 2p - 1.$$

- $u = 1, v = p$. This yields $d = p + 1$ and

$$x = p + 2, \quad y = p(p + 1) - 1.$$

- $u = 1, v = -p$. This yields $d = p - 1$ and

$$x = p, \quad y = -p(p - 1) - 1.$$

- $u = p, v = 1$. This yields $d = p + 1$ and

$$x = p(p + 1) + 1, \quad y = p.$$

- $u = -p, v = 1$. This yields $d = p - 1$ and

$$x = 1 - p(p - 1), \quad y = p - 2.$$

We conclude that the set of solutions for (x, y) is

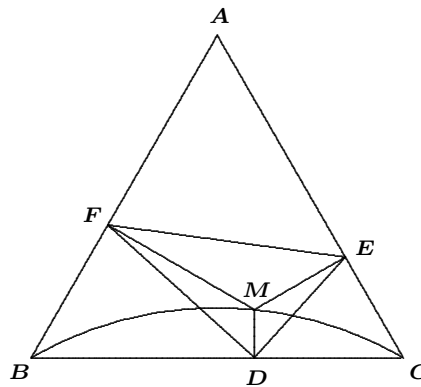
$$\{(1, -1), (2p + 1, 2p - 1), (p + 2, p(p + 1) - 1), (p, -p(p - 1) - 1), \\ (p(p + 1) + 1, p), (1 - p(p - 1), p - 2)\}.$$

The given problem is the special case when $p = 1997$.

Comment. For two different methods of finding the integer solutions of the equation $p(x + y) = xy$, see problem 4 of the 31st Spanish Mathematical Olympiad given in the *Corner* in the May 2000 number [2000 : 206] and Problem E:10322 in the Romanian *Gazeta Matematica* Nr 5/1992, pp. 186–187.

2. Let ABC be an equilateral triangle. For a point M inside ABC , let D, E, F be the feet of the perpendiculars from M onto BC, CA, AB , respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.

Solution by Michel Bataille, Rouen, France.



For all interior points M whose projections onto BC, CA, AB are D, E, F , respectively, points B, D, M, F are concyclic (they lie on the circle with diameter BM). Similarly, M, D, C, E are concyclic. It follows that $\angle FBM = \angle FDM$ and $\angle ECM = \angle EDM$. Therefore,

$$\angle FDE = \angle FBM + \angle ECM.$$

Thus,

$$\begin{aligned} \angle FDE = 90^\circ &\iff \angle FBM + \angle ECM = 90^\circ \\ &\iff \angle MBD + \angle MCD = 30^\circ \\ &\quad (\text{since } \angle B = \angle C = 60^\circ) \\ &\iff \angle BMC = 150^\circ. \end{aligned}$$

We may now conclude that the locus of M is the arc of the circle interior to $\triangle ABC$ subtending 150° on the line segment BC (as shown in the figure).

3. Find all polynomials $p(x)$ satisfying the equation

$$(x - 16)p(2x) = 16(x - 1)p(x)$$

for all x .

Solved by Jean-Claude Andrieux, Beaune, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztein, Pontoise, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use the solution by Andrieux.

Posons

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

avec $a_n \neq 0$. Dans $(x - 16)p(2x) = 16(x - 1)p(x)$, l'égalité des coefficients des termes de plus haut degré donne $2^n a_n = 16a_n$. On en déduit donc que $n = 4$. D'où

$$p(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0.$$

On a

$$\begin{aligned} (x - 16)p(2x) &= 16a_4 x^5 + (8a_3 - 256a_4)x^4 + (4a_2 - 128a_3)x^3 \\ &\quad + (2a_1 - 64a_2)x^2 + (a_0 - 32a_1)x - 16a_0 \end{aligned}$$

et

$$\begin{aligned} 16(x - 1)p(x) &= 16a_4 x^5 + (16a_3 - 16a_2)x^4 + (16a_2 - 16a_3)x^3 \\ &\quad + (16a_1 - 16a_2)x^2 + (16a_0 - 16a_1)x - 16a_0. \end{aligned}$$

On obtient par identification des coefficients

$$a_3 = 30a_4, \quad a_2 = 280a_4, \quad a_1 = -960a_4, \quad a_0 = 1024a_4,$$

d'où

$$\begin{aligned} p(x) &= a_4(x^4 - 30x^3 + 280x^2 - 960x + 1024) \\ &= a_4(x - 2)(x - 4)(x - 8)(x - 16). \end{aligned}$$

4. Let a, b, c be non-negative real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq abc$.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Heinz-Jürgen Seiffert, Berlin, Germany; Panos E. Tsaoussoglou, Athens, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Seiffert's solution.

More generally, let $n \geq 2$ be an integer and $1 \leq p \leq n$. We claim that if a_1, a_2, \dots, a_n are non-negative real numbers such that $\sum a_i \geq \prod a_i$, then

$$\sum a_i^p \geq n^{(p-1)/(n-1)} \prod a_i.$$

(Here and below, all sums and products are extended over $i = 1, 2, \dots, n$.)

Proof. We consider two cases:

Case 1. $\prod a_i \leq n^{n/(n-1)}$.

From the AM-GM Inequality, we have

$$\begin{aligned} \sum a_i^p &\geq n \left(\prod a_i^p \right)^{1/n} = n \left(\prod a_i \right)^{(p-n)/n} \prod a_i \\ &\geq n^{(p-1)/(n-1)} \prod a_i, \end{aligned}$$

because $p \leq n$.

Case 2. $\prod a_i > n^{n/(n-1)}$.

Using the Power-Mean Inequality and the condition $\sum a_i \geq \prod a_i$, we obtain

$$\left(\frac{1}{n} \sum a_i^p \right)^{1/p} \geq \frac{1}{n} \sum a_i \geq \frac{1}{n} \prod a_i,$$

which implies

$$\begin{aligned} \sum a_i^p &\geq n^{1-p} \left(\prod a_i \right)^p = n^{1-p} \left(\prod a_i \right)^{p-1} \prod a_i \\ &\geq n^{(p-1)/(n-1)} \prod a_i. \end{aligned}$$

This completes the proof of the claim.

Taking $n = 3$ and $p = 2$, and renaming a_1, a_2, a_3 by a, b, c , we see that under the conditions given in the proposal, there holds the better inequality $a^2 + b^2 + c^2 \geq \sqrt{3}abc$. This inequality is stronger than the one proposed.

We also give Klamkin's solution and remarks.

We need only consider two cases.

(1) If a, b, c are all ≥ 1 , then clearly $a^2 + b^2 + c^2 \geq a + b + c \geq abc$.

(2) If at least one of $a, b, c \leq 1$ (say $c \leq 1$), then $a^2 + b^2 \geq ab \geq abc$.

Comment. The same problem without the condition that a, b, c be non-negative is given as a problem without solution in D. Fomin, S. Genkin, I. Itenberg, *Mathematical Circles*, Amer. Math. Soc., 1996, p. 185.

We now conclude the proof by allowing negative numbers. If just one of a, b, c is negative or if all three are negative, then $abc \leq 0$, in which case the result is immediate. Thus, we may assume that only two of them, say b and c , are negative. Then, letting $x = -b$ and $y = -c$, we have to show that when $a \geq x + y + axy$, we can conclude that $a^2 + x^2 + y^2 \geq axy$.

The assumed inequality implies that $xy \leq 1$. If $a \geq 1$, then $a^2 \geq axy$, which implies the desired result. If $a \leq 1$, then $x \leq 1$ and $y \leq 1$. In this case $a > x + y \geq xy$, since the latter inequality can be rewritten as $1 \geq (1-x)(1-y)$. Thus, we still have $a^2 \geq axy$, with the same conclusion as before.

5. Let S be the set of all odd integers greater than one. For each $x \in S$, denote by $\delta(x)$ the unique integer satisfying the inequality

$$2^{\delta(x)} < x < 2^{\delta(x)+1}.$$

For $a, b \in S$, define

$$a * b = 2^{\delta(a)-1}(b-3) + a.$$

[For example, to calculate $5 * 7$ note that $2^2 < 5 < 2^3$, so that $\delta(5) = 2$, and hence, $5 * 7 = 2^{2-1}(7-3) + 5 = 13$. Also $2^2 < 7 < 2^3$, so that $\delta(7) = 2$ and $7 * 5 = 2^{2-1}(5-3) + 7 = 11$].

Prove that if $a, b, c \in S$, then

- (a) $a * b \in S$ and
 (b) $(a * b) * c = a * (b * c)$.

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Pontoise, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

(a) Since $b-3$ is even, $a*b$ is clearly odd. Also, since $2^{\delta(a)-1}(b-3) \geq 0$, we have $a * b \geq a > 1$. Hence, $a * b \in S$.

(b) Note first that

$$\begin{aligned} (a * b) * c &= 2^{\delta(a*b)-1}(c-3) + (a * b) \\ &= 2^{\delta(a*b)-1}(c-3) + 2^{\delta(a)-1}(b-3) + a \end{aligned} \quad (1)$$

and

$$\begin{aligned} a * (b * c) &= a * (2^{\delta(b)-1}(c-3) + b) \\ &= 2^{\delta(a)+\delta(b)-2}(c-3) + 2^{\delta(a)-1}(b-3) + a. \end{aligned} \quad (2)$$

By (1) and (2) it clearly suffices to show that

$$\delta(a * b) = \delta(a) + \delta(b) - 1. \quad (3)$$

By definition, $2^{\delta(a)} < a < 2^{\delta(a)+1}$ and $2^{\delta(b)} < b < 2^{\delta(b)+1}$.

Using the inequalities $a < 2^{\delta(a)+1}$ and $b < 2^{\delta(b)+1}$, we have

$$\begin{aligned} 2^{\delta(a)-1}(b-3) + a &< 2^{\delta(a)-1}(b-3) + 2^{\delta(a)+1} \\ &= 2^{\delta(a)-1}(b-3+4) \\ &= 2^{\delta(a)-1}(b+1) \leq 2^{\delta(a)-1} \cdot 2^{\delta(b)+1}; \end{aligned}$$

that is,

$$a * b < 2^{\delta(a)+\delta(b)}. \quad (4)$$

Similarly, using the inequalities $2^{\delta(a)} < a$ and $2^{\delta(b)} < b$, we have

$$\begin{aligned} 2^{\delta(a)-1}(b-3) + a &> 2^{\delta(a)-1}(b-3) + 2^{\delta(a)} \\ &= 2^{\delta(a)-1}(b-3+2) \\ &= 2^{\delta(a)-1}(b-1) \geq 2^{\delta(a)-1} \cdot 2^{\delta(b)}; \end{aligned}$$

that is,

$$a * b > 2^{\delta(a)+\delta(b)-1}. \quad (5)$$

From (4) and (5) we have $2^{\delta(a)+\delta(b)-1} < a * b < 2^{\delta(a)+\delta(b)}$. Therefore,

$$\delta(a * b) = \delta(a) + \delta(b) - 1,$$

which is (3). This completes the proof.

6. Given a positive integer n , denote by $\sigma(n)$ the sum of all the positive integers which divide n . [For example, $\sigma(3) = 1 + 3 = 4$, $\sigma(6) = 1 + 2 + 3 + 6 = 12$, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$.]

We say that n is abundant if $\sigma(n) > 2n$. (Thus, for example, 12 is abundant). Let a, b be positive integers and suppose that a is abundant. Prove that ab is abundant.

Solved by Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Pavlos Maragoudakis, Pireas, Greece; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Maragoudakis' solution.

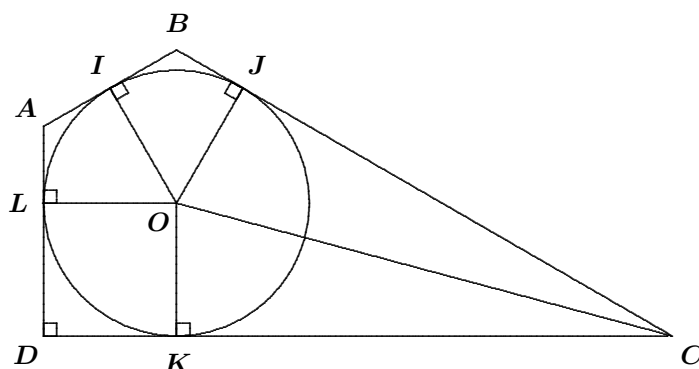
Let d_1, d_2, \dots, d_k be all the positive integers which divide a . Then $d_1 + d_2 + \dots + d_k > 2a$. Also, d_1b, d_2b, \dots, d_kb are all different positive integers that divide ab . Thus,

$$\sigma(ab) \geq d_1b + d_2b + \dots + d_kb > 2ab,$$

whence ab is abundant.

7. $ABCD$ is a quadrilateral which is circumscribed about a circle Γ (that is, each side of the quadrilateral is tangent to Γ). If $\angle A = \angle B = 120^\circ$, $\angle D = 90^\circ$ and BC has length 1, find, with proof, the length of AD .

Solved by Michel Bataille, Rouen, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Pavlos Maragoudakis, Pireas, Greece. We give the solution by Bataille.



Let O be the centre of Γ , and let I, J, K , and L be the points at which AB, BC, CD , and DA touch Γ , respectively. Triangle IBJ is isosceles with $\angle B = 120^\circ$. Therefore, $\angle BIJ = \angle BJI = 30^\circ$; whence, $\angle OIJ = 60^\circ$. It follows that $\triangle IOJ$ is equilateral and, consequently, $IJ = OI = OJ = R$, the radius of Γ . Since $\frac{\sqrt{3}}{2} = \cos 30^\circ = \frac{IJ/2}{BJ}$, we get $BJ = \frac{R}{\sqrt{3}}$.

Now, observing that $OKDL$ is a square and that $\triangle IOJ$ and $\triangle IOL$ are equilateral triangles, we obtain $\angle KOJ = 150^\circ$. Then $\angle OCJ = 15^\circ$; whence, $2 - \sqrt{3} = \tan 15^\circ = \frac{OJ}{CJ}$. This implies that $CJ = R(2 + \sqrt{3})$. The relation $1 = BC = BJ + CJ$ now yields $R = \frac{\sqrt{3}}{4 + 2\sqrt{3}}$. Using $DL = R$ and $AL = BJ = \frac{R}{\sqrt{3}}$, we can compute $AD = AL + DL = \frac{R}{\sqrt{3}} + R$, which readily gives $AD = \frac{\sqrt{3} - 1}{2}$.

8. Let A be a subset of $\{0, 1, 2, 3, \dots, 1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k with k a non-negative integer) or there exist two distinct elements $a, b \in A$ such that $a + b$ is a power of 2.

Solved by Pierre Bornsztein, Pontoise, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's solution.

Let $S = \{0, 1, 2, \dots, 1997\}$. We prove the following stronger result: if A is a subset of S containing more than 997 elements, then A contains a power of 2, or A contains two distinct elements a, b such that $a + b$ is a power of 2. Furthermore, the bound 997 is the best possible.

Suppose $A \subseteq S$ does *not* have the described property. We will show that $|A| \leq 997$.

Note that $2^{10} = 1024 < 1997 < 2048 = 2^{11}$. We arrange the numbers

in $S - \{0, 1, 2, 8, 32, 1024\}$ into an array as follows:

3	13	14	50	51	1997
4	12	15	49	52	1996
5	11	16	48	53	1995
6	10	17	47	54	1994
7	9	18	46	55	1993
		·	·	·	·
		·	·	·	·
		31	33	·	·
				·	·
				·	·
				1023	1025

Let C_i denote the i^{th} column of the array ($i = 1, 2, \dots, 6$). Note that in $C_1 \cup C_2$, the two numbers in the same row add up to 16; in $C_3 \cup C_4$, the two numbers in the same row add up to 64; and in $C_5 \cup C_6$, the two numbers in the same row add up to 2048. Since 16, 64, and 2048 are powers of 2, and since $|C_1| = |C_2| = 5$, $|C_3| = |C_4| = 18$, and $|C_5| = |C_6| = 973$, we have $|A \cap (C_1 \cup C_2)| \leq 5$, $|A \cap (C_3 \cup C_4)| \leq 18$, and $|A \cap (C_5 \cup C_6)| \leq 973$. Allowing $0 \in A$, we find that $|A| \leq 1 + 5 + 18 + 973 = 997$, as claimed.

Furthermore, if we take $A = \{0\} \cup C_2 \cup C_4 \cup C_6$, then clearly $|A| = 997$, and it is easy to check that A does not contain a power of 2 or two elements which add up to a power of 2. Thus, the upper bound of 997 is the best possible.

Remark: A very interesting problem “deserving” a nice solution, which hopefully is from “The Book” (by Erdős’ definition).

9. Let S be the set of all natural numbers n satisfying the following conditions:

- (a) n has 1000 digits,
- (b) all the digits of n are odd, and
- (c) the absolute value of the difference between adjacent digits of n is 2.

Determine the number of distinct elements of S .

Solved by Michel Bataille, Rouen, France; and Pierre Bornshtein, Pontoise, France. We give Bornshtein’s solution.

Let k be a positive integer. Denote by S_k the set of all k -digit natural numbers n satisfying (b) and (c). Let U_k be the set of elements of S_k ending in 1 or 9, V_k the set of elements of S_k ending in 3 or 7, and W_k the set of elements of S_k ending in 5. Let $s_k = |S_k|$, $u_k = |U_k|$, $v_k = |V_k|$, and $w_k = |W_k|$.

Note that each element of S_{k+1} is obtained from an element of S_k by adding an odd number at the right of its decimal expansion, following (c). Any $n \in U_k$ may be used to construct exactly one number of S_{k+1} because

the only possible digit that may be added to n is **3** if the rightmost digit of n is **1**, and **7** if the rightmost digit of n is **9**. Then we obtain u_k elements belonging to V_{k+1} . Similarly, any $n \in V_k$ may be used to construct exactly one number of U_{k+1} and one number of W_{k+1} ; and any $n \in W_k$ may be used to construct exactly two numbers of V_{k+1} .

Since n cannot belong to more than one of the cases above, the construction leads to different numbers in S_{k+1} . Thus,

$$\begin{cases} u_{k+1} = v_k \\ v_{k+1} = u_k + 2w_k \\ w_{k+1} = v_k \end{cases},$$

with $u_1 = v_1 = 2$ and $w_1 = 1$. For $k \geq 2$, we have $u_k = w_k = v_{k-1}$, which leads to $v_{k+1} = 3v_{k-1}$. Therefore, $v_{2k+1} = 3^k v_1 = 2 \times 3^k$ and $v_{2k+2} = 3^k v_2 = 4 \times 3^k$ for all $k \geq 0$. Then $u_{2k+1} = w_{2k+1} = 4 \times 3^{k-1}$ and $u_{2k} = w_{2k} = 2 \times 3^{k-1}$ for all $k \geq 1$.

We deduce that, for all $k \geq 1$,

$$\begin{aligned} s_{2k} &= 2 \times 3^{k-1} + 4 \times 3^{k-1} + 2 \times 3^{k-1} = 8 \times 3^{k-1}, \\ s_{2k+1} &= 4 \times 3^{k-1} + 2 \times 3^k + 4 \times 3^{k-1} = 14 \times 3^{k-1}. \end{aligned}$$

In particular, $s_{1000} = 8 \times 3^{499}$.

10. Let p be a prime number and n a natural number, and let $T = \{1, 2, 3, \dots, n\}$. Then n is called *p-partitionable* if there exist non-empty subsets T_1, T_2, \dots, T_p of T such that

- (i) $T = T_1 \cup T_2 \cup \dots \cup T_p$,
- (ii) T_1, T_2, \dots, T_p are disjoint (that is, $T_i \cap T_j$ is the empty set for all i, j with $i \neq j$), and
- (iii) the sum of the elements in T_i is the same for $i = 1, 2, \dots, p$.

[For example, **5** is **3**-partitionable since, if we take $T_1 = \{1, 4\}$, $T_2 = \{2, 3\}$, $T_3 = \{5\}$, then (i), (ii) and (iii) are satisfied. Also **6** is **3**-partitionable since, if we take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$, $T_3 = \{3, 4\}$, then (i), (ii) and (iii) are satisfied.]

(a) Suppose that n is p -partitionable. Prove that p divides n or $n + 1$.

(b) Suppose that n is divisible by $2p$. Prove that n is p -partitionable.

Solved by Michel Bataille, Rouen, France; Pierre Bornshtein, Pontoise, France; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Bataille's solution.

(a) Denote by $s(A)$ the sum of the elements of the finite subset A of \mathbb{N} . Then

$$\frac{n(n+1)}{2} = s(T) = \sum_{i=1}^p s(T_i) = pm,$$

where $m = s(T_i)$, $i = 1, 2, \dots, p$. Thus, $2pm = n(n+1)$. The prime number p , which divides the product $n(n+1)$, must divide n or $n+1$.

(b) Let k be the integer such that $n = 2pk$, and set

$$\begin{aligned} T_1 &= \{1, 2, \dots, k\} \cup \{(2p-1)k+1, (2p-1)k+2, \dots, 2pk\}, \\ T_2 &= \{k+1, k+2, \dots, 2k\} \cup \\ &\quad \{(2p-2)k+1, (2p-2)k+2, \dots, (2p-1)k\}, \\ &\quad \vdots \\ T_{j+1} &= \{jk+1, \dots, (j+1)k\} \cup \{(2p-j)k+1, \dots, (2p-(j-1))k\}, \\ &\quad \vdots \\ T_p &= \{(p-1)k+1, \dots, pk, pk+1, pk+2, \dots, (p+1)k\}. \end{aligned}$$

Now we have $s(T_i) = k(2pk+1)$ for $i = 1, 2, \dots, p$, and clearly (i) and (ii) are also satisfied. It follows that n is p -partitionable. _____



That completes this number of the *Corner*. This is Olympiad Season—send me Olympiad Contests as well as your nice solutions and generalizations.