

## Pólya's Paragon

Paul Ottaway

In this installment of Pólya's Paragon, we will examine the *Extreme Principle*. Here is the basic premise:

*If possible, assume that the elements of your problem are "in order". Focus on the "largest" and "smallest" elements, as they may be constrained in interesting ways.*

Using this simple idea, we will be able to provide elegant solutions to difficult problems. Let's illustrate this with several examples.

**Problem 1:** On the plane, we colour a *finite* number of points either black or white. We choose the points and their colours so that every line segment which joins two points of the same colour contains a point of the other colour. Prove that all the points must lie on a single line segment.

**Solution:** Suppose that the points do not all lie on a single line segment. Then there must exist at least one set of three points that form a triangle (that is, these three points are not all on the same line). Of all such triangles that can be formed, consider the triangle  $ABC$  of *smallest* area. We will obtain a contradiction by finding a triangle whose area is smaller than the area of  $\triangle ABC$ .

Each of the points  $A$ ,  $B$ , and  $C$  are coloured black or white. So at least two of the points must be coloured the same. Without loss of generality, suppose that  $B$  and  $C$  are both coloured white. Then, there must be a black point  $D$  somewhere between  $B$  and  $C$ . Then  $\triangle ABD$  is a triangle whose area is strictly smaller than the area of  $\triangle ABC$ , which contradicts the fact that  $\triangle ABC$  was the triangle of smallest area.

Since we have a contradiction, we conclude that all the points must lie on a single line segment.

**Problem 2:** Let  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Determine all functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all  $x$  and  $y$  in  $\mathbb{N}$ .

**Solution:** This problem appeared as the last question of the 2002 Canadian Mathematical Olympiad. The proposer's solution was purely number-theoretic and spanned several pages. It was fully expected that very few people would solve this problem, and that the only correct solutions would be similar to the highly technical approach obtained by the proposer.

To the CMO committee's surprise (and delight!), the following solution was submitted by David Han from Woburn Collegiate Institute, who was the winner of the 2002 CMO Contest.

We claim that  $f$  is a constant function. Suppose, for a contradiction, that there exist  $x$  and  $y$  with  $f(x) < f(y)$ . Choose  $x$  and  $y$  such that  $f(y) - f(x) = d > 0$  is minimal. Then,

$$f(x) = \frac{xf(x) + yf(x)}{x + y} < \frac{xf(y) + yf(x)}{x + y} < \frac{xf(y) + yf(y)}{x + y} = f(y).$$

Letting  $z = x^2 + y^2$ , we have shown that  $f(x) < f(z) < f(y)$ . Hence, the integers  $x$  and  $z$  satisfy  $0 < f(z) - f(x) < d$ , which contradicts the minimality of  $d$ . Therefore, no such  $x$  and  $y$  exist, and so we conclude that  $f$  must be a constant function. [*Editor's Note:* The above argument requires  $x \neq 0$  and  $y \neq 0$ . To complete the proof, we need only show that  $f(0) = f(x)$  for some non-zero  $x \in \mathbb{N}$ . Now, select  $y = 0$  and  $x = 1$  in the given functional equation, and we are done.]

We quickly see that for all  $c \in \mathbb{N}$ , the function  $f(x) = c$  satisfies the given functional equation. Therefore, we have solved the problem.

**Problem 3:** Prove that the equation  $x^4 + y^4 = z^4$  has no solutions in positive integers  $x, y, z$ .

**Solution:** This is the most famous application of the Extreme Principle. This is the  $n = 4$  case of Fermat's Last Theorem.

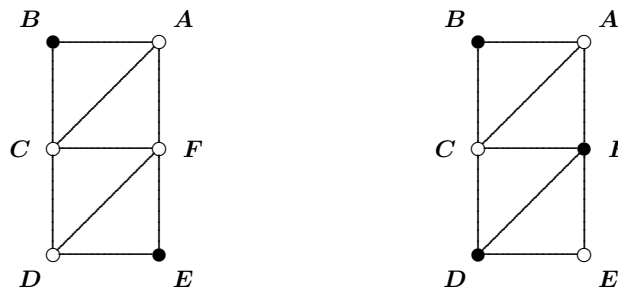
We outline the proof here, and we invite you to fill in the details!

1. Define  $(a, b, c)$  to be a *special triple* if  $a, b, c$  are positive integers for which  $a^4 + b^4 = c^2$ .
2. Suppose that a special triple  $(a, b, c)$  exists. Consider the *smallest* one; that is, one where  $c$  is minimized.
3. We have  $a^4 + b^4 = c^2$ , so  $(a^2, b^2, c)$  is a Pythagorean Triple. Explain why there must exist positive integers  $p$  and  $q$  for which  $a^2 = p^2 - q^2$ ,  $b^2 = 2pq$ , and  $c = p^2 + q^2$ .
4. Show that there exist positive integers  $d$  and  $e$  for which  $a = d^2 - e^2$ ,  $q = 2de$ , and  $p = d^2 + e^2$ .
5. Explain why  $d, e$ , and  $d^2 + e^2$  are all perfect squares. Conclude that there must exist a special triple  $(r, s, t)$  with  $t < c$ , which gives you the desired contradiction.
6. Explain why no special triples  $(r, s, t)$  exist, and thus conclude that the equation  $x^4 + y^4 = z^4$  has no solutions in positive integers  $x, y, z$ .

We conclude this article by providing some more questions where the Extreme Principle may be used.

1. Imagine an infinite chessboard that contains a positive integer in each square. If the value in each square is equal to the average of its four neighbours to the north, south, west, and east, prove that the values in all the squares are equal.

2. Consider a graph with finitely many points, some of which are joined to one another by lines. We shall colour each point either black or white, and call the graph “integrated” if each white point has at least as many black as white neighbours, and vice versa. The example below shows two different colourings of the same graph. The one on the left is not integrated, because point  $A$  has two white neighbours ( $C$  and  $F$ ), and only one black neighbour ( $B$ ). The graph on the right is integrated.



Given any graph, can we colour the points so that the graph is integrated?

3. Prove that the equation  $x^2 + y^2 = 3z^2$  has no solutions in positive integers  $(x, y, z)$ .
4. On a large flat field,  $n$  people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and, at a given signal, fires and hits the person who is closest. When  $n$  is odd, show that there is at least one person left dry. Is this always true when  $n$  is even?  
(1987 CMO, Question 4.)
5. Consider finitely many points in the plane such that, if we choose any three points  $A, B, C$  among them, the area of triangle  $ABC$  is always less than 1. Prove that all of these points lie within the interior or on the boundary of a triangle with area less than 4.  
(1995 Korean Mathematical Olympiad.)