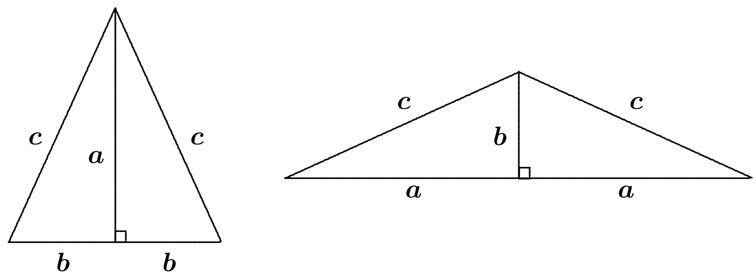
Mayhem Solutions

M26. *Proposed by the Mayhem staff.*

Find two isosceles triangles, with two sides 106 units long and the other side an integer, that have the same area.

Solution by Geneviève Lalonde, Massey, ON.

Two different isosceles triangles can be created that have the same length for their double side and the same area by creating them from two scalene right-angled triangles as in the diagram below, where (a, b, c) is a Pythagorean triple.



For our problem we need a Pythagorean triple $(a, b, 106)$. If we can find two positive integers x and y with $x > y$, such that $x^2 + y^2 = 106$, then we can let $a = x^2 - y^2$ and $b = 2xy$. Since $106 = 9^2 + 5^2$, we have $a = 9^2 - 5^2 = 56$ and $b = 2 \cdot 9 \cdot 5 = 90$. Thus, the triangles with sides 106, 106, 112 and 106, 106, 180 both have area $ab = 5040$.

M27. *Proposed by the Mayhem staff.*

Find $\sqrt{ab+1}$ where $a = \overbrace{111\cdots11}^{2002\text{ 1's}}$ and $b = 1\overbrace{00\cdots00}^{2001\text{ 0's}}5$.

Solution by Mihály Bencze, Brasov, Romania.

In general, if $a = \overbrace{111\cdots11}^{n\text{ 1's}} = \dfrac{10^n - 1}{9}$ and $b = 1\overbrace{00\cdots00}^{n-1\text{ 0's}}5 = 10^n + 5$, then

$$\sqrt{ab+1} = \sqrt{\left(\frac{10^n+2}{3}\right)^2} = \frac{10^n+2}{3}.$$

Thus, for this problem the result is $\dfrac{10^{2002}+2}{3} = \overbrace{333\cdots3}^{2001\text{ 3's}}4$.

Also solved by Gustavo Krimker, Universidad CAECE, Argentina. One incomplete solution was received.

M28. *Proposed by the Mayhem staff.*

Shawn tosses 2001 fair coins and Bruce tosses 2002 fair coins. What is the probability that Bruce gets more heads than Shawn?

Solution by Geneviève Lalonde, Massey, ON.

To start out, let Shawn toss his 2001 coins and Bruce toss 2001 of his. By symmetry P(Bruce more heads) $=$ P(Shawn more heads) $= \frac{1-p}{2}$ where p is the probability that they have the same number of heads. If Bruce has more heads, he has already won and the last toss is immaterial. If they have the same number of heads, then Bruce can get more by flipping heads, but if Shawn already has more heads, Bruce cannot get more with his last toss. Thus, the probability that Bruce will have more heads after his 2002^{nd} toss is

$$\frac{1-p}{2} + \frac{1}{2}p \;=\; \frac{1}{2}.$$

Also solved by José L. Díaz–Barrero and Juan J. Egozcue, UPC, Barcelona, Spain.

M29. *Proposed by the Mayhem staff.*

Define the "silly product" of two numbers as the sum of the product of all the corresponding digits. So $235 \times_s 718 = 2 \times 7 + 3 \times 1 + 5 \times 8 = 57$. Find two numbers A and B so that $A \times_s B = 2002$ and $A+B$ is a minimum.

Solution by Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK.

A and B should have the same number of digits. Otherwise, some digits would be multiplied by zero which has no contribution to the "silly product". It contradicts the condition that $A + B$ is minimum.

Let $A = a_n a_{n-1} \cdots a_1$, and $B = b_n b_{n-1} \cdots b_1$ where the a_i and b_i are the digits of A and B. Then

$$A \times_s B \;=\; a_n b_n + a_{n-1} b_{n-1} + \cdots + a_1 b_1 \;\le\; 9 \times 9 + 9 \times 9 + \cdots + 9 \times 9\,,$$

whence $2002 \le 81n$. Thus, $n \ge \frac{2002}{81}$, but since n is an integer we must have $n \ge 25$.

In order to keep $A + B$ minimum, we want the least number of digits. Therefore, $a_i b_i$ must be as great as possible. But $a_i b_i$ is at most 81 when $a_i = b_i = 9$, and $25 \times 81 = 2025 > 2002$. Hence, only the least significant 24 digits can be 9. Hence, there remains $2002 - 24 \times 81 = 58 = 2 \times 29$. Since 29 is prime, we cannot make 58 with the silly product of two one-digit numbers. Thus, we need two more digits for our number, say a_n, a_{n+1} for A and b_n, b_{n+1} for B. Hence, $a_{n+1}b_{n+1} + a_n b_n = 58$ and a_{n+1} and b_{n+1} should be kept as small as possible. The smallest occurs for $1\times 2+7\times 8 = 58$. Thus, the two possibilities for A and B that keep the sum $A+B$ a minimum are $1899\ldots9$, $2799\ldots9$ and $1799\ldots9$, $2899\ldots9$ (there are twenty-four 9's in each number).

Also solved by Robert Bilinski, Outremont, PQ; Jack Gu, grade 11, Rachel Li, grade 12, Alvin Miao, grade 10, Molly Yan, grade 11, and Corey Zhou, grade 12, Dalian Maple Leaf International School, Dalian, China. Three incorrect solutions were received.

M30. *Proposed by Haralampy Steryion, Chalkis, Greece.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property

$$f(x+y) = f(x)e^{f(y)-1} \qquad \text{for every } x,\, y \in \mathbb{R}.$$

Solution by Shien Jin Ong, MIT, USA.

Claim: $f(x) = 1$ or $f(x) = 0$ for all $x \in \mathbb{R}$.

Note that the function $f(x) = 0$ for all $x \in \mathbb{R}$ is a solution. Now assume that $f(x) \neq 0$ for some $x \in \mathbb{R}$, say for $x = x_1$. Substitute $y = 0$ and $x = x_1$ into the equation. We conclude that $f(0) = 1$. Next, substitute $x = 0$ into the same equation. We get $f(y) = e^{f(y)-1}$ for all $y \in \mathbb{R}$. Let $g(t) = t - e^{t-1}$. Note that $g'(t) < 0$ if $t > 1$, $g'(t) > 0$ if $t < 1$, and $g(1) = 0$. Hence, $g(t)$ has only one root, at $t = 1$. This means that the only solution to $f(y) = e^{f(y)-1}$ is $f(y) = 1$ for all $y \in \mathbb{R}$. A simple check verifies that the solution indeed fits into the given equation.

Also solved by Jack Gu, grade 11, Rachel Li, grade 12, and Corey Zhou, grade 12, Dalian Maple Leaf International School, Dalian, China. One incorrect solution was received.