SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We apologize for omitting the name of NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina from the list of solvers of 2634 and 2683, and the name of DAVID LOEFFLER, student, Trinity College, Cambridge, UK from the list of solvers of 2690.

2701★. [2002 : 52] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Do there exists infinitely many triplets $(n, n+1, n+2)$ of adjacent natural numbers such that all of them are sums of two positive perfect squares?

(Examples are $(232, 233, 234)$, $(520, 521, 522)$ and $(808, 809, 810)$.)

Compare the 2000 Putnam problem A2 [2001 : 3]

Amalgamated solution by Paul Jefferys, student, Berkhamsted Collegiate School, UK; Gottfried Perz, Pestalozzigymnasium, Graz, Austria; and the proposer.

For all integers $a \geq 3$ consider the triples

$$\begin{aligned} (a^2-a)^2+(a^2-a)^2 &= 2a^4-4a^3+2a^2 \\ (a^2-2a)^2+(a^2-1)^2 &= 2a^4-4a^3+2a^2+1 \\ (a^2-a-1)^2+(a^2-a+1)^2 &= 2a^4-4a^3+2a^2+2 \end{aligned}$$

Since these expressions are increasing with respect to a, there are infinitely many triples.

Several solutions began with the observation that $8n^2 = (2n)^2+(2n)^2$ and $8n^2+2 = (2n-1)^2+(2n+1)^2$. Thus, the problem reduces to expressing $8n^2+1$ as a sum of two squares.

$$(2n-a)^2+(2n+a-1)^2 \;=\; 8n^2+1-(4n-(2a^2-2a))\,.$$

Hence, we require n and a such that $4n-(2a^2-2a) = 0$, or $2n \;=\; a^2-a$.

It is possible to require that all members of the triple are the sum of two *different* positive perfect squares. Note that $4n^4+4n^2 = (2n^2)^2+(2n)^2$ and $4n^4+4n^2+2 = (2n^2+1)^2+1$. Finally, if $2n^2+1 = r^2+s^2$, then

$$4n^4+4n^2+1 \;=\; (2n^2+1)^2 \;=\; (r^2-s^2)^2+(2rs)^2\,.$$

By above $2n^2+1 = r^2+s^2$ admits infinitely many solutions.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; PETER HURTHIG, Columbia College, Vancouver, BC; DAVID LOEFFLER, student, Trinity College, *Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; and LI ZHOU, Polk Community College, Winter Haven, FL, USA. There was one incorrect solution.*

Two solutions constructed triples from solutions to Pell's equation: $u^2 - 2v^2 = 1$. Li Zhou asks the question: Are there infinitely many different similarity classes of Pythagorean triples the hypotenuses of which appear in the set $\{2n^2+1\}_{n\in\mathbb{N}}$?

2702. [2002 : 53] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let λ be an arbitrary real number. Show that

$$\left(\frac{s}{r}\right)^{2\lambda} s^2 \;\geq\; 3^{3\lambda+1}\left(s^2 - 8Rr - 2r^2\right)\,,$$

where R, r and s are the circumradius, the inradius and the semi-perimeter of a triangle, respectively.

Determine the cases of equality.

[*Editor's Remark*. The condition $\lambda > 0$ was added in a footnote in [2002 : 248].]

I. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

The statement is incorrect, even with the correction $\lambda > 0$. The given statement is equivalent to

$$\left(\frac{s^2}{27r^2}\right)^{\lambda} \;\geq\; \frac{3}{s^2}\left(s^2 - 8Rr - 2r^2\right)\,. \tag{1}$$

Now, recall Guerretsen's Inequalities ([1996 : 130]):

$$16Rr - 5r^2 \;\leq\; s^2 \;\leq\; 4R^2 + 4Rr + 3r^2\,.$$

We also have

$$\begin{aligned} R \;\geq\; 2r \quad &\Longleftrightarrow\quad 4Rr \;\geq\; 8r^2 \\ &\Longleftrightarrow\quad 16Rr - 5r^2 \;\geq\; 12Rr + 3r^2\,, \end{aligned}$$

which implies $s^2 \geq 12Rr + 3r^2$. Hence,

$$3s^2 - 24Rr - 6r^2 \;\geq\; s^2 \qquad\Longleftrightarrow\qquad \frac{3}{s^2}\left(s^2 - 8Rr - 2r^2\right) \;\geq\; 1\,.$$

We note that, crucially, this is strictly greater than 1 if the triangle is not equilateral.

Thus, in the adjusted inequality (1), let $\lambda \to 0$ from above. The left-hand side tends to 1, while for any non-equilateral triangle the right-hand side is strictly greater than 1. Therefore, for some positive λ the inequality does not hold.

II. *Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

We show that the inequality is valid for $\lambda \geq 2$. The case $0 < \lambda < 2$ is left open.

Dividing the given inequality by r^2 and setting $x = (s/r)^2$, the inequality becomes

$$x^{\lambda+1} + 3^{3\lambda+1}\left(\frac{8R}{r} + 2\right) \;\geq\; 3^{3\lambda+1}x\,.$$

Since $R/r \geq 2$, it suffices to prove the stronger inequality

$$t(t^\lambda - 1) \;\geq\; 2(t-1)\,,$$

where now $t = x/27$ and it is known that $t \geq 1$. Since $t^\lambda - 1 \geq t^2 - 1$ and since $t(t+1) \geq 2$, we have

$$t(t^\lambda - 1) \;\geq\; t(t^2 - 1) \;\geq\; 2(t-1)\,.$$

There is equality only if the triangle is equilateral.

Also shown incorrect by PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; and LI ZHOU, Polk Community College, Winter Haven, FL, USA.

In an addendum to his solution, Loeffler makes the following comment:

We have

$$R \;\geq\; 2r \qquad \Longleftrightarrow \qquad 16Rr - 5r^2 \;\geq\; 27r^2\,.$$

Thus, from Gerretsen, we see that the left-hand side of (1) is also strictly greater than 1 for non-equilateral triangles. It follows that for each non-equilateral triangle ABC, there is some λ_0 such that the inequality holds if and only if $\lambda \geq \lambda_0$, with equality only for $\lambda = \lambda_0$. (Clearly, equality always holds for equilateral trinagles.)

The limit of $\lambda_0(a,b,c)$ as $a,b,c \longrightarrow 1,1,1$ is $\frac{2}{3}$. Over all triangles the largest value of λ_0, and hence the minimum value of λ for which the inequality holds for all triangles, is (to 30 digits):

$$0.702543072697378209700856413\,.$$

This is achieved for a triangle with sides 1, 1, and α where α is a constant equal to about

$$0.77373714143340760389115253968466 71\,.$$

(Loeffler has run these numbers through the Inverse Symbolic Calculator website, and states that they do not appear to be obviously related to any known constants.)

The constant α may be identified as the value of z maximizing the expression

$$\frac{\log\left[\dfrac{3(3z^2 - 4z + 4)}{(z+2)^2}\right]}{\log\left[\dfrac{(z+2)^3}{27z^2(2-z)}\right]}$$

and the maximum value of this expression is the sought-after maximum of λ_0.

2703. [2002 : 53]
Proposed by Mihály Bencze, Brasov, Romania.

Suppose that a, b, c, d, u, $v \in \mathbb{R}$ and $a + c \neq 0$. Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(ax+b) + f(cx+d) = ux + v$.

Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA (*modified slightly by the editor*).

Suppose f satisfies the required condition. Let

$$g(x) \;=\; f(x) - \frac{u}{a+c}x + \frac{1}{2}\left(\frac{b+d}{a+c}u - v\right) .$$

Then g is continuous and

$$\begin{aligned} g(ax+b) + g(cx+d) \;&=\; ux + v - \frac{u}{a+c}\Big((a+c)x + (b+d)\Big) \\ &\qquad + \frac{b+d}{a+c}u - v \\ &=\; 0\,. \end{aligned}$$

Depending on the values of a, b, c, and d, we consider three cases separately:

Case (i). If $a = c \neq 0$ and $b = d$, then $2g(ax+b) = 0$ for all $x \in \mathbb{R}$. Hence, $g \equiv 0$ and $f(x) = \dfrac{u}{2a}x + \dfrac{1}{2}\left(\dfrac{b}{a}u - v\right)$.

Case (ii). If $a = c \neq 0$ and $b \neq d$, then $g(ax+b) + g(ax+d) = 0$. Letting $y = ax + d$, we then obtain $g(y + b - d) = -g(y)$. Hence,

$$f(x) \;=\; g(x) + \frac{u}{a+c}x - \frac{1}{2}\left(\frac{b+d}{2a}u - v\right) ,$$

where g is a continuous function satisfying $g(x + b - d) = -g(x)$. (Thus, g is periodic with period $2(b-d)$.) There are clearly infinitely many such functions; for example, $g(x) = \sin\left(\dfrac{\pi x}{b-d}\right)$.

Case (iii). If $a \neq c$, then $|a| \neq |c|$ since $a \neq -c$ by assumption. Without loss of generality, we may assume that $|a| > |c|$. Let $p = \dfrac{d-b}{a-c}$ and $q = \dfrac{ad-bc}{a-c}$. Then $ap + b = cp + d = q$. Now, for all $x \in \mathbb{R}$, we have

$$g(ax+q) \;=\; g\Big(a(x+p)+b\Big) \;=\; -g\Big(c(x+p)+d\Big) \;=\; -g(cx+q) \quad (1)$$

If $c = 0$, then $q = d$ and $g(ax+q) = -g(q)$ for all $x \in \mathbb{R}$, implying that $g \equiv 0$. If $c \neq 0$, then letting $x = \dfrac{ay}{c}$, we get from (1) that

$$g\left(a\Big(\frac{a}{c}\Big)y + q\right) \;=\; -g(ay+q) = g(cy+q)\,.$$

Iterating this substitution, we obtain inductively that

$$g\left(a\left(\frac{a}{c}\right)^n z+q\right) \;=\; (-1)^{n+1}g(cz+q)$$

for all $z \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Let $z_n = z_n(x) = \dfrac{x-q}{a(a/c)^n}$. Then

$$g(x) \;=\; g\left(\left(\frac{a}{c}\right)^n z_n+q\right) \;=\; (-1)^{n+1}g(cz_n+q)\,. \qquad (2)$$

Since $\left|\frac{a}{c}\right| > 1$ we have $z_n(x) \to 0$ as $n \to \infty$. Let $n \to \infty$ through the odd and even integers, respectively. Then from (2) and the continuity of g, we conclude that $g(x) = g(q)$ and $g(x) = -g(q)$ for all $x \in \mathbb{R}$. Therefore, $g(x) = g(q) = 0$. Thus, $g \equiv 0$ again, from which it follows that

$$f(x) \;=\; \frac{u}{a+c}x - \frac{1}{2}\left(\frac{b+d}{a+c}u - v\right)\,.$$

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JOEL SCHLOSBERG, student, New York University, NY, USA; and the proposer.

2704. [2002 : 53] (Corrected [2002 : 174]) *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$R-2r \;\geq\; \frac{1}{12}\left(\sum_{\text{cyclic}}\sqrt{2(b^2+c^2)-a^2} - \frac{s^2+r^2+4Rr}{R}\right) \;\geq\; 0\,,$$

where a, b and c are the sides of a triangle, and R, r and s are the circumradius, the inradius and the semi-perimeter of the triangle, respectively.

Solution by G. Tsintsifas, Thessaloniki, Greece.

We prove the stronger inequality:

$$R-2r \;\geq\; \frac{1}{8}\left(\sum_{\text{cyclic}}\sqrt{2(b^2+c^2)-a^2} - \frac{s^2+r^2+4Rr}{R}\right) \;\geq\; 0\,,$$

From elementary geometry we use the following formulas:

$$b^2+c^2 \;=\; 2m_a^2+\frac{a^2}{2} \qquad \text{and} \qquad s^2+r^2+4Rr = ab+bc+ca\,,$$

where m_a is the length of the median through A. We set

$$A \;=\; \sum_{\text{cyclic}}\sqrt{2(b^2+c^2)-a^2} - \frac{s^2+r^2+4Rr}{R}\,.$$

This simplifies to:

$$\begin{aligned} A &= 2\sum_{\text{cyclic}} m_a - \frac{ab+bc+ca}{R} = 2\sum_{\text{cyclic}} m_a - \frac{2R\sum_{\text{cyclic}} h_a}{R} \\ &= 2\left[\sum_{\text{cyclic}} m_a - \sum_{\text{cyclic}} h_a\right]. \end{aligned}$$

From [1] 8.2 and 7.12, we have

$$\sum_{\text{cyclic}} m_a \le 4R + r \qquad \text{and} \qquad 9r \le \sum_{\text{cyclic}} h_a ,$$

which yields

$$A \le 2[4R + r - 9r] = 8(R - 2r).$$

References.

[1] D.S. Mitrinovic et. al., *Recent Advances in Geometric Inequalities*. Kluwer, Dordrecht 1989.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; JOE HOWARD, Portales, NM, USA; KEE-WAI LAU, Hong Kong, China; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VEDULA N. MURTY, Dover, PA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. Most solutions were not easily amended to show the stronger inequality above.

2705. [2002 : 53] *Proposed by Angel Dorito, Geld, ON.*

The interior of a rectangular container is 1 metre wide and 2 metres long, and is filled with water to a depth of $\frac{1}{2}$ metre. A cube of gold is placed flat in the tub, and the water rises to exactly the top of the cube without overflowing.

Find the length of the side of the cube.

Solution by Gavin Johnstone, student, Dame Alice Owen's School, Potters Bar, UK.

The volume of water in the tub is $(1)(2)(\frac{1}{2}) = 1$ m^3, and this is invariant. After the cube is placed in the tub,

$$\text{vol. water} = \text{vol. water and cube} - \text{vol. cube}.$$

Letting the side length of the cube be x, we have $1 = 2x - x^3$, and thus

$$\begin{aligned} x^3 - 2x + 1 &= 0 \qquad (1) \\ (x-1)(x^2+x-1) &= 0. \end{aligned}$$

Hence, $x = 1$ or $x = (-1 \pm \sqrt{5})/2$. The solution $x = (-1 - \sqrt{5})/2$ is negative and therefore inadmissible. The solution $x = (-1 + \sqrt{5})/2$, the Golden Ratio, is an acceptable side length for the cube. Whether $x = 1$ is acceptable depends on whether an exact fit is allowed for the cube in the tub. [*Editor's note:* The equation (1) is the same as was found for the solution of 2670 [2002 : 464].]

Also solved in essentially the same manner by AUSTRIAN IMO TEAM 2002; CHARLES ASHBACHER, Hiawatha, IA, USA; MICHEL BATAILLE, Rouen, France; BRIAN D. BEASLEY, Presbyterian College, Clinton, SC, USA; CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; PAOLO CUSTODI, Fara Novarese, Italy; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT MULLËR, Landesshule Pforta, Schulpforte, Germany; VICTOR PAMBUCCIAN, ASU West, Phoenix, AZ, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; HARRY SEDINGER, St. Bonaventure University, St. Bonaventure, NY, USA; D.J. SMEENK, Zaltbommel, the Netherlands; Mª JESÚS VILLAR RUBIO, Santander, Spain; OLOV WILANDER, student, Christ's College, Cambridge, UK; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposer. There was one incorrect solution.

Strictly speaking, the depth of the container should be considered in deciding which solutions are possible lengths for a side of the cube. The depth is not given, but we are told that the water does not overflow when the cube is placed in the tub. Consequently, there are two cases: (1) depth ≥ 1, in which case both positive solutions are admissible; (2) $(\sqrt{5}-1)/2 \leq$ depth < 1, in which case only the solution $(\sqrt{5}-1)/2$ is admissible. If $1/2 <$ depth $< (\sqrt{5}-1)/2$, then no solution is admissible, but this possibility seems to be ruled out by the wording of the problem. Only the Austrian IMO-Team and Natalio Guersenzvaig noted the different cases. All the other solutions assumed implicitly that the tub was deep enough for both cubes.

David Loeffler observes that the proposer's name, Angel Dorito, "is clearly a pseudonym, being an anagram of his stated address of Geld, Ontario".

2706. [2002 : 54] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Suppose that Γ_1 and Γ_2 are two circles having at least one point S in common. Take an arbitrary line ℓ through S. This line intersects Γ_k again at P_k (if ℓ is tangent to Γ_k, then $P_k = S$).

Let λ be a (fixed) real number, and let $R_\lambda = \lambda P_1 + (1-\lambda)P_2$.

Determine the locus of R_λ as ℓ varies over all possible lines through S.

I. *Solution by G. Tsintsifas, Thessaloniki, Greece.*

Let A and B be the opposite ends of the diameters through S in circles Γ_1 and Γ_2 respectively; denote the second intersection point of the circles by T (with $T = S$ when the circles are tangent). Since AT and BT both make right angles with ST, T must lie on AB. Let Q be the point of AB for which $Q = \lambda A + (1-\lambda)B$. By definition, S is on P_1P_2, so P_1P_2 makes right angles with AP_1 and BP_2. Thus $AP_1 || BP_2$ and so these lines are parallel to QR_λ (since QR_λ cuts transversals P_1P_2 and AB proportionally). Therefore, also

$\angle QR_\lambda S$ is a right angle for all positions of R_λ, whose locus is consequently the circle with diameter SQ. Note that because O_1 is the mid-point of SA while O_2 is the mid-point of SB, the centre of the locus (which is the mid-point of SQ) must be $\lambda O_1 + (1-\lambda)O_2$.

II. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

The condition on R is equivalent to stating that the cross-ratio of the four points P_1, P_2, R, ∞ is constant and equal to $-\lambda$, where ∞ represents the point at infinity in the inversive plane. Let T be the second intersection of Γ_1 and Γ_2, which may coincide with S. Let us invert the diagram in some circle centred at S. Then the circles Γ_1 and Γ_2 become lines through the point T'. Since l passes through S it inverts into itself. This inversion has mapped ∞ into S, so since inversions preserve cross-ratios, $[P_1', P_2', R', S] = -\lambda$. This implies that the locus of R' is a fixed line through T', since P_1', P_2' and S are collinear. (Note that T' may be ∞). So the locus of R is a circle through S and T.

Also solved by MICHEL BATAILLE, Rouen, France (2 solutions); CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; GERRY LEVERSHA, St. Paul's School, London, UK; TOSHIO SEIMIYA, Kawasaki, Japan; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2707. [2002 : 54] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

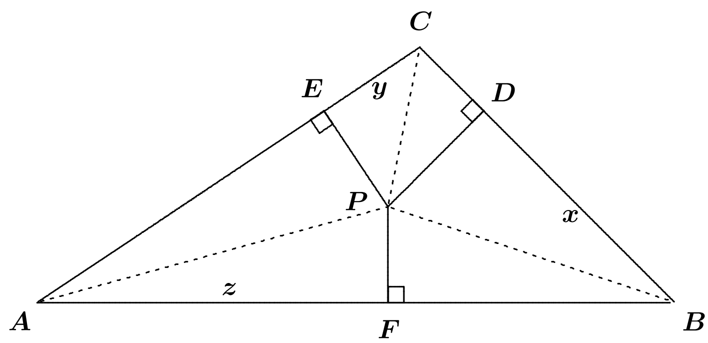
Let ABC be a triangle and P a point in its plane. The feet of the perpendiculars from P to the lines BC, CA and AB are D, E and F, respectively.

Prove that

$$\frac{AB^2 + BC^2 + CA^2}{4} \leq AF^2 + BD^2 + CE^2 ,$$

and determine the cases of equality.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.



Let $AF = z$, $BD = x$, and $CE = y$ (signed lengths). [Also, $AB = c$, $BC = a$ and $CA = b$.] Then $FB = c - z$, $DC = a - x$, and $EA = b - y$. A well-known theorem (proved by applying the theorem of Pythagoras to the six triangles such as $\triangle PBD$) is that

$$x^2 + y^2 + z^2 \;=\; (a-x)^2 + (b-y)^2 + (c-z)^2\,,$$

so that

$$ax + by + cz \;=\; \tfrac{1}{2}(a^2 + b^2 + c^2)\,.$$

Hence,

$$\tfrac{1}{4}(a^2+b^2+c^2)^2 \;=\; (ax+by+cz)^2 \;\le\; (a^2+b^2+c^2)(x^2+y^2+z^2)\,,$$

where the last step follows by the Cauchy-Schwarz Inequality. Thus,

$$\tfrac{1}{4}(a^2+b^2+c^2) \;\le\; (x^2+y^2+z^2)\,.$$

Equality holds when $a : b : c = x : y : z$, which is when D, E and F are the mid-points of the sides, making P the circumcentre of $\triangle ABC$.

References.

[1] ***CRUX with MAYHEM*** 23, No. 2 (March, 1997), p. 122.

[2] D. S. Mitronović, J. E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, 1989, p. 336.

Also solved by AUSTRIAN IMO TEAM 2002; MICHEL BATAILLE, Rouen, France; JOHN G. HEUVER, Grande Prairie, AB; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GAVIN JOHNSTONE, student, Dame Alice Owen's School, Potters Bay, UK; MURRAY S. KLAMKIN, University of Alberta, Edmonton, AB; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; TOSHIO SEIMIYA, Kawasaki, Japan; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIȚĂ, Bucharest, Romania. There was one incorrect solution. Two other solutions failed to treat the case of equality, and were considered to be incomplete.

The well-known theorem mentioned in the featured solution is discussed in [1]. *Seimiya obtains the equation* $ax + by + cz = \frac{1}{2}(a^2+b^2+c^2)$ *by a different approach. He applies the Law of Cosines to each of the triangles* BAP, CBP *and* ACP, *eliminates the cosines using* $AP\cos(\angle BAP) = AF$, $BP\cos(\angle CBP) = BD$ *and* $CP\cos(\angle ACP) = CE$, *and adds the resulting three equations.*

Heuver notes that this problem may be found in [2].

2708. [2002 : 54] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that

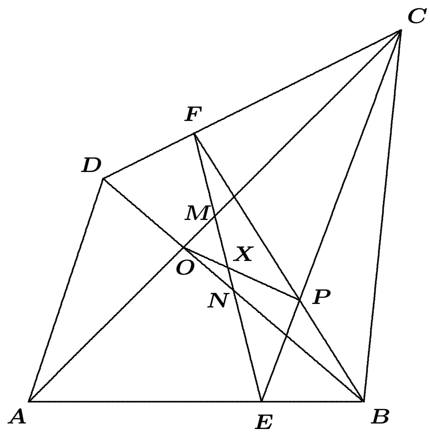
1. O is the intersection of diagonals AC and BD of quadrilateral $ABCD$,
2. $OA < OC$ and $OD < OB$,
3. M and N are the mid-points of AC and BD, respectively,

4. MN meets AB and CD at E and F, respectively, and

5. P is the intersection of BF and CE.

Prove that OP bisects the line segment EF.

Solution by Gerry Leversha, St. Paul's School, London, UK.



We use vectors. Taking point O as the origin, define the position vectors of points M and N as $\overrightarrow{m}$ and $\overrightarrow{n}$, respectively. Then the position vectors of points C and B can be taken as $c\,\overrightarrow{m}$ and $b\,\overrightarrow{n}$, where b, $c > 1$, and, since M and N are mid-points, the points A and D have position vectors $(2-c)\,\overrightarrow{m}$ and $(2-b)\,\overrightarrow{m}$, correspondingly.

The point E lies on both AB and MN, so that there exist λ and μ such that

$$\lambda(2-c)\,\overrightarrow{m} + (1-\lambda)b\,\overrightarrow{n} \;=\; \mu\,\overrightarrow{m} + (1-\mu)\,\overrightarrow{n}\,.$$

Hence, by linear independence, we have $\lambda(2-c)+(1-\lambda)b = 1$, so that

$$\lambda \;=\; \frac{b-1}{b+c-2} \qquad \text{and} \qquad 1-\lambda \;=\; \frac{c-1}{b+c-2}\,.$$

Therefore, the position vector of E is given by

$$(b+c-2)\,\overrightarrow{e} \;=\; (2-c)(b-1)\,\overrightarrow{m} + b(c-1)\,\overrightarrow{n}\,.$$

Similarly, the position vector of F is given by

$$(b+c-2)\,\overrightarrow{f} \;=\; (2-b)(c-1)\,\overrightarrow{n} + c(b-1)\,\overrightarrow{m}\,.$$

The mid-point X of EF has position vector

$$\overrightarrow{x} \;=\; \frac{(b-1)\,\overrightarrow{m} + (c-1)\,\overrightarrow{n}}{b+c-2}\,.$$

It remains to find the position vector of P. Since P lies on both BF and CE, there must exist α and β such that

$$\begin{aligned} & \alpha(b+c-2)b\,\overrightarrow{\mathrm{n}} + (1-\alpha)(2-b)(c-1)\,\overrightarrow{\mathrm{n}} + (1-\alpha)c(b-1)\,\overrightarrow{\mathrm{m}} \\ = \; & \beta(b+c-2)c\,\overrightarrow{\mathrm{m}} + (1-\beta)(2-c)(b-1)\,\overrightarrow{\mathrm{m}} + (1-\beta)b(c-1)\,\overrightarrow{\mathrm{n}}\,, \end{aligned}$$

and therefore,

$$\alpha(b+c-2)b + (1-\alpha)(2-b)(c-1) \;=\; (1-\beta)b(c-1)$$

and

$$(1-\alpha)c(b-1) \;=\; \beta(b+c-2)c + (1-\beta)(2-c)(b-1)\,.$$

These in turn reduce to

$$(b^2+2bc-3b-2c+2)\alpha + b(c-1)\beta \;=\; 2(b-1)(c-1)$$

and

$$(c^2+2bc-3c-2b+2)\beta + c(b-1)\alpha \;=\; 2(b-1)(c-1)\,.$$

Eliminating β, we obtain

$$\alpha \;=\; \frac{c-1}{b+c-1}\,,$$

so that

$$1-\alpha \;=\; \frac{b}{b+c-1}\,.$$

Thus, the position vector of the point P is given by

$$\overrightarrow{\mathrm{p}} \;=\; \frac{bc[(b-1)\,\overrightarrow{\mathrm{m}} + (c-1)\,\overrightarrow{\mathrm{n}}]}{b+c-1}\,.$$

Comparing the expressions for $\overrightarrow{\mathrm{x}}$ and $\overrightarrow{\mathrm{p}}$, we see that the points O, X and P are collinear, which completes the proof.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Editorial comments. Janous has proved the more general result: If condition 3 is replaced by "M and N are the points dividing both segments AC and BD in ratio $\lambda : (1-\lambda)$" then OP divides EF in ratio $\lambda : (1-\lambda)$ too. (Here, $0 < \lambda < 1$.)

2709. [2002 : 55] *Proposed by Toshio Seimiya, Kawasaki, Japan.*
Suppose that

1. P is an interior point of $\triangle ABC$,

2. AP, BP and CP meet BC, CA and AB at D, E and F, respectively,
3. A' is a point on AD produced beyond D such that $DA' : AD = \kappa : 1$, where κ is a fixed positive number,
4. B' is a point on BE produced beyond E such that $EB' : BE = \kappa : 1$, and
5. C' is a point on CF produced beyond F such that $FC' : CF = \kappa : 1$.

Prove that $[A'B'C'] \leq \frac{(3\kappa+1)^2}{4}[ABC]$, where $[PQR]$ denotes the area of $\triangle PQR$.

Solution by Francisco Bellot Rosado, I.B. Emilio Ferrari, Valladolid, Spain.

Victor Thébault investigated areas in a more general setting in the former Belgian journal *Mathesis* (1940), p. 67. In particular, he does not assume that AD, BE, and CF concur. We shall postpone that assumption until after having developed the relevant area formulas.

Lemma (Thébault). *Given $\triangle ABC$ with points D dividing BC internally in the ratio $(1-m) : m$, E dividing CA internally in the ratio $(1-n) :$ n, and F dividing CA internally in the ratio $(1-p) : p$, we define A', B', C' as in the statement of problem 2709. Then*

$$\frac{[DEF]}{[ABC]} = mn + np + pm - (m+n+p) + 1 \tag{1}$$

and

$$\frac{[A'B'C']}{[ABC]} = [mn + np + pm - (m+n+p) + 3](k+1)^2 - 3k - 2\,. \tag{2}$$

Proof. For the proof we introduce barycentric (areal) coordinates with $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ from which we obtain

$$D(0,m,1-m)\,,\ E(1-n,0,n)\,,\ F(p,1-p,0)\,,$$
$$A'(-k, m(1+k), (1-m)(1+k))\,, \text{ etc.}$$

Therefore,

$$\begin{aligned} \frac{[DEF]}{[ABC]} &= \begin{vmatrix} 0 & m & 1-m \\ 1-n & 0 & n \\ p & 1-p & 0 \end{vmatrix} \\ &= mn + np + pm - (m+n+p) + 1\,, \quad \text{and} \\ \frac{[A'B'C']}{[ABC]} &= \begin{vmatrix} -k & m(1+k) & (1-m)(1+k) \\ (1-n)(1+k) & -k & n(1+k) \\ p(1+k) & (1-p)(1+k) & -k \end{vmatrix} \\ &= [mn + np + pm - (m+n+p) + 3](k+1)^2 - 3k - 2\,, \end{aligned}$$

as claimed.

From (1) and (2) we can eliminate m, n, p to obtain

$$[A'B'C'] = (k+1)^2[DEF] + k(2k+1)[ABC]\,. \qquad (3)$$

We now turn to our problem 2709 and assume that the cevians AD, BE, CF concur; by Ceva's Theorem we have

$$\begin{aligned} \frac{1-m}{m}\cdot\frac{1-n}{n}\cdot\frac{1-p}{p} &= 1\text{ , or equivalently} \\ (1-m)(1-n)(1-p) &= mnp\text{ , or} \qquad (4) \\ mn+np+pm-(m+n+p) &= 2mnp-1 \end{aligned}$$

Thus (from the last line of (4)), when the given cevians are concurrent, equations (1) and (2) become

$$\begin{aligned} \frac{[DEF]}{[ABC]} &= 2mnp \\ \frac{[A'B'C']}{[ABC]} &= 2(1+k)^2(mnp+1)-3k-2\,. \end{aligned}$$

In the special case when our cevians are the medians of $\triangle ABC$, we have $m = n = p = \frac{1}{2}$ so that the last formula becomes

$$[A'B'C'] = \frac{(3k+1)^2}{4}[ABC]\,.$$

(Compare a note by Thébault, *Mathesis* (1939) page 311.)

To finish the problem we must show that the above value of $[A'B'C']$, when D, E, F are mid-points, is its maximum value (as the intersection point of the cevians ranges over the triangle's interior). Note that formula (3) tells us that to maximize $[A'B'C']$ we need only find the maximum of $[DEF]$. [*Editor's comment.* Seimiya stated in his solution that this is known: The mid-point triangle is the "cevian triangle" of largest area. Bellot's argument is so simple, however, we shall continue with it.] We just saw that when the cevians concur, $[DEF] = 2mnp[ABC]$, so the problem reduces to finding the maximum of the product mnp or, more conveniently from (4), of

$$(mnp)^2 = mnp(1-m)(1-n)(1-p)\,.$$

The product is composed of three pairs of factors such as $m(1-m)$, numbers that have a constant sum 1. Such a product achieves its maximum when both factors are equal, which means $m = \frac{1}{2}$. Similarly, $n = \frac{1}{2}$ and $p = \frac{1}{2}$, so that DEF is the mid-point triangle of $\triangle ABC$, and the argument is complete.

Also solved by MICHEL BATAILLE, Rouen, France; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GAVIN JOHNSTONE, student, Dame Alice Owen's School, Potters Bay, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; G. TSINTSIFAS, Thessaloniki, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

2710. [2002 : 56] *Proposed by Jaroslav Švrček, Palacký University, Olomouc, Czech Republic.*

Determine the point P on the semicircle Γ, constructed externally over the side AB of the square $ABCD$, such that $AP^2 + CP^2$ is maximal.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

Without loss of generality, assume the side length of the square is two. Consider a rectangular coordinates system with the mid-point of AB as the origin so that $A = (-1, 0)$, $C = (1, -2)$ and $P = (\cos\theta, \sin\theta)$, $0 \le \theta \le \pi$. Then

$$\begin{aligned} AP^2 + CP^2 &= (1+\cos\theta)^2 + \sin^2\theta + (1-\cos\theta)^2 + (2+\sin\theta)^2 \\ &= 8 + 4\sin\theta\,. \end{aligned}$$

This is maximal when $\theta = \pi/2$ and so P is the mid-point of $\widehat{AB}$.

Also solved by the AUSTRIAN IMO TEAM; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; PIERRE BORNSZTEIN, Pontoise, France; PAOLO CUSTODI, Fara Novarese, Italy; NIKOLAOS DERGIADES, Thessaloniki, Greece; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; GAVIN JOHNSTONE, student, Dame Alice Owen's School, Potters Bay, UK; VÁCLAV KONEČNÝ, Ferris State University, Big Rapids, MI, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; VICTOR PAMBUCCIAN, ASU West, Phoenix, AZ, USA; JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; ROBERT P. SEALY, Mount Allison University, Sackville, NB; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; KIRSTIN STROKORB, Winckelmann-Gymnasium, Stendal, Germany; Mª JESÚS VILLAR RUBIO, Santander, Spain; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU, Bucharest, Romania; and the proposer.

2711★. [2002 : 56] *Proposed by Catherine Shevlin, Wallsend upon Tyne, UK.*

Two circles, centres O_1 and O_2, of radii R_1 and R_2 ($R_1 > R_2$), respectively, are externally tangent at P. A common tangent to the two circles, not through P, meets O_1O_2 produced at Q, the circle with centre O_1 at A_1 and the circle with centre O_2 at A_2.

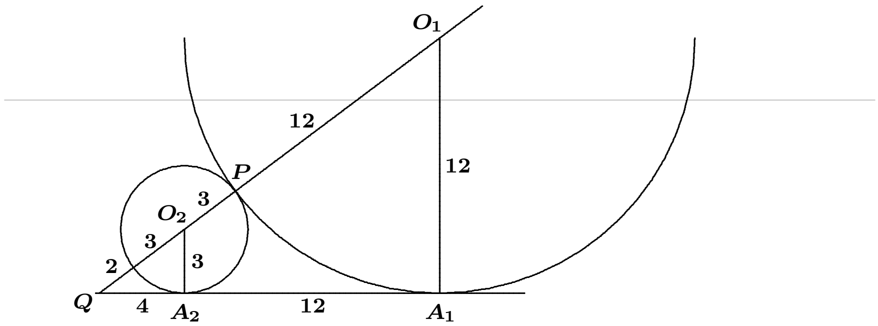
Prove or disprove that there exist simultaneously integer triangles QO_1A_1 and QO_2A_2.

Solution by Christopher J. Bradley, Clifton College, Bristol, UK.

A parameterization that works is

$$\begin{aligned} R_1 &= u^2(u^2-v^2) \qquad & R_2 &= v^2(u^2-v^2) \\ O_2Q &= v^2(u^2+v^2) \qquad & O_1Q &= u^2(u^2+v^2) \end{aligned}$$

where u and v are coprime and of opposite parity.



Another parameterization is

$$\begin{aligned} R_1 &= (u+v)^2\,(2uv) & R_2 &= (u-v)^2\,(2uv) \\ O_2Q &= (u-v)^2\,(u^2v^2) & O_1Q &= (u+v)^2\,(u^2+v^2) \end{aligned}$$

where u and v are coprime and of opposite parity.

Apart from scale factors, the above two parameterizations are complete, as can be seen, since we require

$$O_2Q = \frac{R_2(R_1+R_2)}{(R_1-R_2)},$$

together with the usual Pythagorean triples.

Also solved by MANUEL BENITO and EMILIO FERNÁNDEZ, I.B. Praxedes Mateo Sagasta, Logroño, Spain; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; DANIEL REISZ, Voncelles, France; Mª JESÚS VILLAR RUBIO, Santander, Spain (2 solutions); PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and TITU ZVONARU, Bucharest, Romania. There was one incorrect submission.

2712. [2002 : 56] *Proposed by Antreas P. Hatzipolakis, Athens, Greece; and Paul Yiu, Florida Atlantic University, Boca Raton, FL, USA.*

Given $\triangle ABC$, let Y and Z be the feet of the altitudes from B and C. Suppose that the bisectors of $\angle BYC$ and $\angle BZC$ meet at X. Prove that $\triangle BXC$ is isosceles.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.

The circle with diameter BC passes through Y and Z. The bisector of $\angle BYC$ intersects the circle in the point X such that the arc BX equals the arc CX. The bisector of $\angle BZC$ intersects the circle in the same point. Therefore, $\triangle BXC$ is an isosceles right triangle.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; AUSTRIAN IMO TEAM 2002; MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, *Bristol, UK; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; GEOFFREY A. KANDALL, Hamden, CT, USA; MITKO KUNCHEV, Baba Tonka School of Mathematics, Rousse, Bulgaria; GERRY LEVERSHA, St. Paul's School, London, UK; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; JOEL SCHLOSBERG, student, New York University, NY, USA; TOSHIO SEIMIYA, Kawasaki, Japan; IFTIMIE SIMION, Roslyn H.S., N.Y. City State; D.J. SMEENK, Zaltbommel, the Netherlands; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; TITU ZVONARU and BOGDAN IONIŢĂ, Bucharest, Romania; and the proposers.*

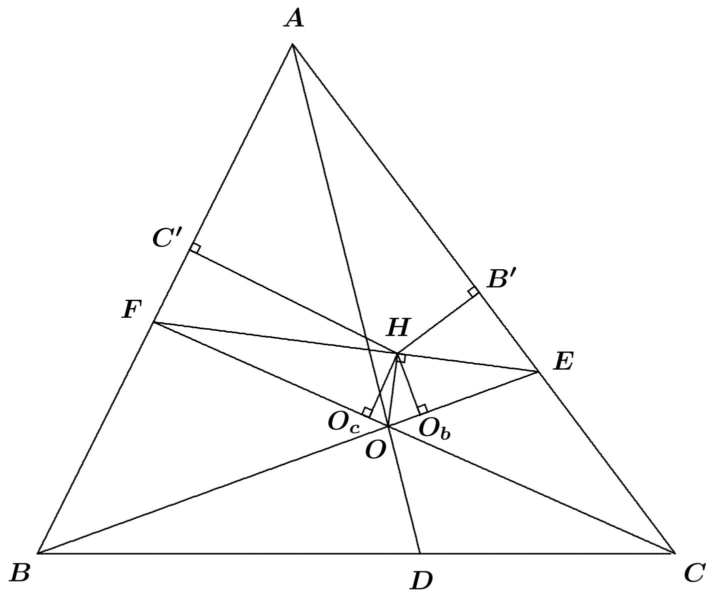
Editorial comments. Several solvers (Jefferys, Kunchev, Seimiya, Specht, Zvonaru and Ioniţă) noted that the claim of the problem is not true if either of the angles B and C of the triangle ABC is greater than 90°. Seimiya noted that the point X does not exist for a right triangle ABC.

2713. [2002 : 110] *Proposed by Toshio Seimiya, Kawasaki, Japan.*

Suppose that O is an interior point of $\triangle ABC$, and that AO, BO and CO meet BC, CA and AB at D, E and F, respectively. Let H be the foot of the perpendicular from D to EF.

Prove that the feet of the perpendiculars from H to AF, FO, OE and EA are concyclic.

Solution by Nikolaos Dergiades, Thessaloniki, Greece.



If the line EF meets the line BC at D', then it is known that the points D and D' are harmonic conjugates of B and C, correspondingly, and since $\angle D'HD = 90^\circ$, the point H lies on the Apolonian circle with diameter DD'. Hence HD is a bisector of the angle BHC and therefore

$$\angle FHB = \angle CHE\,. \qquad (1)$$

If the line EF is parallel to the line BC, then D is the mid-point of BC and HD is a median, altitude and bisector in $\triangle BHC$, so that the equality (1) still holds.

Let C', O_c, O_b and B' be the feet of the perpendiculars from H to AF, FO, OE and EA. Then the points H, O_b, E and B' are concyclic and so are the points H, O_b, B and C'. Hence,

$$\angle B'O_bH = \angle B'EH \quad \text{and} \quad \angle C'O_bH = \angle C'BH\,.$$

Thus,

$$\angle B'O_bC' = \angle B'EH + \angle C'BH\,.$$

Similarly,

$$\angle B'O_cC' = \angle B'CH + \angle C'FH\,.$$

Then

$$\begin{aligned}\angle B'O_bC' - \angle B'O_cC' &= (\angle B'EH + \angle C'BH) - (\angle B'CH + \angle C'FH)\\ &= (\angle B'EH - \angle B'CH) - (\angle C'FH - \angle C'BH)\\ &= \angle CHE - \angle FHB\\ &= 0;\end{aligned}$$

the last equality follows from (1). Therefore,

$$\angle B'O_bC' = \angle B'O_cC'\,,$$

which shows that the points C', O_c, O_b and B' are concyclic.

Also solved by WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; G. TSINTSIFAS, Thessaloniki, Greece; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.