PROBLEMS

Problem proposals and solutions should be sent to Jim Totten, Department of Mathematics and Statistics, University College of the Cariboo, Kamloops, BC, Canada, V2C 5N3. Proposals should be accompanied by a solution, together with references and other insights which are likely to be of help to the editor. When a proposal is submitted without a solution, the proposer must include sufficient information on why a solution is likely. An asterisk (⋆) after a number indicates that a problem was proposed without a solution.

In particular, original problems are solicited. However, other interesting problems may also be acceptable provided that they are not too well known, and references are given as to their provenance. Ordinarily, if the originator of a problem can be located, it should not be submitted without the originator's permission.

To facilitate their consideration, please send your proposals and solutions on signed and separate standard $8\frac{1}{2}''\times 11''$ *or* A4 *sheets of paper. These may be typewritten or neatly hand-written, and should be mailed to the Editor-in-Chief, to arrive no later than* **1 September 2003**. *They may also be sent by email to* **crux-editors@cms.math.ca**. (*It would be appreciated if email proposals and solutions were written in* LaTeX.) *Graphics files should be in epic format, or encapsulated postscript. Solutions received after the above date will also be considered if there is sufficient time before the date of publication. Please note that we do not accept submissions sent by FAX.*

Each problem is given in English and French, the official languages of Canada. In issues 1, 3, 5 and 7, English will precede French, and in issues 2, 4, 6 and 8, French will precede English.

In the solutions section, the problem will be given in the language of the primary featured solution.

2801. *Proposed by Heinz-Jürgen Seiffert, Berlin, Germany.*

Suppose that $\triangle ABC$ is not obtuse. Denote (as usual) the sides by a, b, and c and the circumradius by R. Prove that

$$\left(\frac{2A}{\pi}\right)^{\frac{1}{a}}\left(\frac{2B}{\pi}\right)^{\frac{1}{b}}\left(\frac{2C}{\pi}\right)^{\frac{1}{c}} \;\le\; \left(\frac{2}{3}\right)^{\frac{\sqrt{3}}{R}} .$$

When does equality hold?

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Supposons que le triangle ABC n'aie pas d'angle obtus et soit a, b, et c ses côtés et R le rayon du cercle circonscrit. Montrer que

$$\left(\frac{2A}{\pi}\right)^{\frac{1}{a}}\left(\frac{2B}{\pi}\right)^{\frac{1}{b}}\left(\frac{2C}{\pi}\right)^{\frac{1}{c}} \;\le\; \left(\frac{2}{3}\right)^{\frac{\sqrt{3}}{R}} .$$

Quand l'égalité a-t-elle lieu ?

2802. *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Four positive integers, a, b, c, d, are said to have property $\mathcal{PS}$ if all of $bc + cd + db$, $ac + cd + da$, $ab + bd + da$, and $ab + bc + ca$ are $\mathcal{P}$erfect $\mathcal{S}$quares.

Suppose that the positive integers m, p, q, and r satisfy $p \le q \le r$ and $pq + qr + rp = m^2$. Let $s = p + q + r + 2m$.

Prove that p, q, r, and s have property $\mathcal{PS}$.

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On dit que quatre entiers positifs, a, b, c, d, possèdent la propriété $\mathcal{CP}$ si tous les nombres $bc+cd+db$, $ac+cd+da$, $ab+bd+da$, et $ab+bc+ca$ sont des $\mathcal{C}$arrés $\mathcal{P}$arfaits.

Supposons que les entiers positifs m, p, q, et r satisfont $p \le q \le r$ et $pq + qr + rp = m^2$. Soit $s = p + q + r + 2m$.

Montrer que p, q, r, et s possèdent la propriété $\mathcal{CP}$.

2803. *Proposed by I.C. Draghicescu, Bucharest, Romania.*

Suppose that $x_1, x_2, \ldots, x_n$ $(n > 2)$ are real numbers such that the sum of any $n - 1$ of them is greater than the remaining number. Let $s = \sum\limits_{k=1}^{n} x_k$. Prove that

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \ \ge\ \frac{s}{n-2}\,.$$

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Soit x_1, x_2, $\ldots$, x_n $(n > 2)$ des nombres réels tels que la somme de $n-1$ d'entre eux est plus grande que le nombre restant. On pose $s = \sum\limits_{k=1}^{n} x_k$. Montrer que

$$\sum_{k=1}^{n} \frac{x_k^2}{s - 2x_k} \ \ge\ \frac{s}{n-2}\,.$$

2804. *Proposed by D.J. Smeenk, Zaltbommel, the Netherlands.*

Given three non-concentric circles Γ_j (M_j, R_j), let μ_j denote the power of a point P with respect to Γ_j.

Determine the locus of P if $2\mu_2 = \mu_1 + \mu_3$.

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On donne trois cercles non concentriques Γ_j (M_j, R_j) et soit μ_j la puissance d'un point P par rapport à Γ_j.

Déterminer le lieu des points P tels que $2\mu_2 = \mu_1 + \mu_3$.

2805. *Proposed by Mihály Bencze, Brasov, Romania.*

Let k be a fixed positive integer. For all positive integers n, prove that there exist positive integers a_1, a_2, …, a_n, such that $(n, a_n) = 1$ and

$$\sum_{j=1}^{n} \frac{j^k}{a_j} \;=\; 1\,.$$

. . . .

Soit k un entier positif donné. Montrer que, pour tous les entiers positifs n, il existe des entiers positifs a_1, a_2, …, a_n, tels que $(n, a_n) = 1$ et

$$\sum_{j=1}^{n} \frac{j^k}{a_j} \;=\; 1\,.$$

2806. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that x, y, $z > 0$, $\alpha \in \mathbb{R}$ and $x^\alpha + y^\alpha + z^\alpha = 1$. Prove that

(a) $x^2{+}y^2{+}z^2 \;\geq\; x^{\alpha+2}{+}y^{\alpha+2}{+}z^{\alpha+2}{+}2x^2y^2z^2\left(x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}\right)$,

(b) $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2} \;\geq\; x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+\dfrac{2\left(x^{\alpha+1}+y^{\alpha+1}+z^{\alpha+1}\right)}{xyz}$.

. . . .

Soit x, y, $z > 0$, $\alpha \in \mathbb{R}$ et $x^\alpha + y^\alpha + z^\alpha = 1$. Montrer que

(a) $x^2{+}y^2{+}z^2 \;\geq\; x^{\alpha+2}{+}y^{\alpha+2}{+}z^{\alpha+2}{+}2x^2y^2z^2\left(x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}\right)$,

(b) $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2} \;\geq\; x^{\alpha-2}+y^{\alpha-2}+z^{\alpha-2}+\dfrac{2\left(x^{\alpha+1}+y^{\alpha+1}+z^{\alpha+1}\right)}{xyz}$.

2807. *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*

In $\triangle ABC$, denote its area by $[ABC]$ (and its semi-perimeter by s). Show that

$$\min\left\{\frac{2s^4-\left(a^4+b^4+c^4\right)}{[ABC]^2}\right\} \;=\; 38\,.$$

. . . .

Soit $[ABC]$ l'aire d'un triangle ABC, et s son demi-périmètre. Montrer que

$$\min\left\{\frac{2s^4-\left(a^4+b^4+c^4\right)}{[ABC]^2}\right\} \;=\; 38\,.$$

2808. *Proposed by Aram Tangboondouangjit, student, University of Maryland, College Park, Maryland, USA.*

In $\triangle ABC$, we have $b < c$ and $a\left(3b^2 + c^2 - a^2\right) = 2b\left(c^2 - b^2\right)$. Determine the ratio $a : b : c$.

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Dans le triangle ABC, on a $b < c$ et $a\left(3b^2 + c^2 - a^2\right) = 2b\left(c^2 - b^2\right)$. Déterminer les rapports $a : b : c$.

2809. *Proposed by Mihály Bencze, Brasov, Romania.*

Suppose that $k \geq 2$ is a fixed integer. For each non-negative integer n, let x_n denote the leftmost digit of n^k.

Prove that the number $0.x_0x_1x_2\ldots x_n\ldots$ is irrational.

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Soit $k \geq 2$ un entier donné. Pour tout entier non négatif n, désignons par x_n le premier chiffre du nombre n^k.

Montrer que le nombre $0,x_0x_1x_2\ldots x_n\ldots$ est irrationnel.

2810. *Proposed by I.C. Draghicescu, Bucharest, Romania.*

Suppose that a, b and x_1, x_2, …, x_n ($n \geq 2$) are positive real numbers. Let $s = \sum\limits_{k=1}^{n} x_k$. Prove that

$$\prod_{k=1}^{n}\left(a + \frac{b}{x_k}\right) \;\geq\; \left(a + \frac{nb}{s}\right)^n .$$

. . . .

Supposons que a, b et x_1, x_2, …, x_n ($n \geq 2$) soient des nombres réels positifs et posons $s = \sum\limits_{k=1}^{n} x_k$. Montrer que

$$\prod_{k=1}^{n}\left(a + \frac{b}{x_k}\right) \;\geq\; \left(a + \frac{nb}{s}\right)^n .$$

2811. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy, for all real x,

$$f\left(x^3 + x\right) \;\leq\; x \;\leq\; f^3(x) + f(x) .$$

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Déterminer toutes les fonctions $f : \mathbb{R} \to \mathbb{R}$ satisfaisant, pour tous les x réels,

$$f\left(x^3 + x\right) \;\leq\; x \;\leq\; f^3(x) + f(x) .$$

2812. *Proposed by Mihály Bencze, Brasov, Romania.*

Determine all injective functions $f : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy

$$(2a+b)f(ax+b) \;\geq\; a\,f^2\left(\frac{1}{x}\right) + b\,f\left(\frac{1}{x}\right) + a$$

for all positive real x, where a, $b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ and $2a + b > 0$.

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Déterminer toutes les fonctions injectives $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfaisant

$$(2a+b)f(ax+b) \;\geq\; a\,f^2\left(\frac{1}{x}\right) + b\,f\left(\frac{1}{x}\right) + a$$

pour tous les x réels positifs, où a, $b \in \mathbb{R}$, $a > 0$, $a^2 + 4b > 0$ et $2a + b > 0$.

2813. *Proposed by Barry R. Monson, University of New Brunswick, Fredericton, NB and J. Chris Fisher, University of Regina, Regina, SK.*

Suppose that M is the mid-point of side AB of the square $ABCD$. Let P and Q be the points of intersection of the line MD with the circle, centre M, radius MA $(= MB)$, where P is inside the square $ABCD$ and Q is outside.

Prove that rectangle $APBQ$ is a golden rectangle; that is,

$$PB : PA = (\sqrt{5}+1) : 2\,.$$

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Soit M le point milieu du côté AB du carré $ABCD$. Soit P et Q les points d'intersection de la droite MD et du cercle de centre M, de rayon MA $(= MB)$, où P est à l'intérieur et Q à l'extérieur du carré $ABCD$.

Montrer que $APBQ$ est un rectangle d'or, c'est-à-dire

$$PB : PA = (\sqrt{5}+1) : 2\,.$$

CORRECTION : The summations in problems **2792** and **2799** require $i \neq j$.

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LA CORRECTION : Les additions dans les problèmes **2792** et **2799** nécessitent $i \neq j$.