Brahmagupta Quadrilaterals: A Description

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Introduction

Heron of Alexandria (Egypt) gave the formula $\sqrt{s(s-a)(s-b)(s-c)}$ for the area of a triangle in terms of its sides a, b, c and $s = (a+b+c)/2$. Right-angled triangles having sides and area that are integers were determined long before Heron. But to his credit he found such a triangle that is **not** a right-angled one: 13, 14, 15; 84. Because of this we honour Heron by naming triangles with integer sides and area *Heron triangles*.

The Indian mathematician Brahmagupta determined Heron triangles by adjoining two right-angled triangles along a common side. He took his principle further and gave us a construction to obtain a cyclic (inscribable in a circle) quadrilateral with integer sides, diagonals, and area. Later mathematicians were intrigued by the Brahmagupta process. But it took Kummer to demystify it. We call an inscribable quadrilateral a *Brahmagupta quadrilateral* if it has integer sides, diagonals, and area. Our present aim is to provide a description of Brahmagupta quadrilaterals via *Heron angles* (see [1], pp. 191–224, and [4]).

Background Material

An angle θ is called a Heron angle if both $\sin\theta$ and $\cos\theta$ are rational. Hence a parametrization of Heron angles is given by

$$\sin\theta = \frac{m^2-n^2}{m^2+n^2}, \quad m > n, \quad \gcd(m,n) = 1. \tag{1}$$

In (1) the integers m and n may both be odd. This enables us to obtain the Heron angle $\pi/2-\theta$ also. The reader will see the advantage of this in the proof of Theorem 2 later on.

Furthermore, we need the following well-known results from circle geometry. We refer to Figure 1. Let AB be a chord of a circle. If two angles are inscribed on the same side of AB, then they will be equal. If they are inscribed on opposite sides, then they will be supplementary. The extended Sine Rule says that

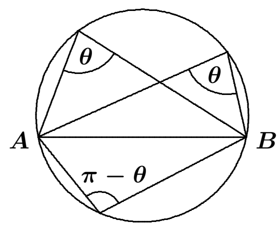

Figure 1

$$AB = (\text{diameter})(\sin\theta).$$

Here is an important observation: Suppose an inscribable qudrilateral has rational sides, diagonals and area. If all these rationals are integers, then it is a Brahmagupta quadrilateral. Otherwise we multiply all these rationals by the least common multiple of their denominators. This process yields a quadrilateral similar to the original one but with integer sides, diagonals, and area; that is, a Brahmagupta quadrilateral.

Let θ_1, θ_2, θ_3 be Heron angles. Invoking (1), we have, for appropriate pairs of natural numbers m_i, n_i ($i = 1, 2, 3$),

$$\sin\theta_i = \frac{m_i^2 - n_2^2}{m_i^2 + n_i^2} \qquad \text{and} \qquad \cos\theta_i = \frac{2m_i n_i}{m_i^2 + n_i^2}\,.$$

We will need expressions for $\sin(\theta_1 + \theta_2)$, $\cos(\theta_1 + \theta_2)$, $\sin(\theta_2 + \theta_3)$ and $\sin(\theta_1 + \theta_2 + \theta_3)$ in terms of m_i and n_i. The required expressions can be obtained very easily from the above formulas by using standard trigonometric identities. For example, $\sin(\theta_1+\theta_2) = \sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2)$ and

$$\sin(\theta_1 + \theta_2 + \theta_3) = \sin(\theta_1 + \theta_2)\cos(\theta_3) + \cos(\theta_1 + \theta_2)\sin(\theta_3)\,. \qquad (2)$$

Description of Brahmagupta Quadrilaterals.

Based on the result related to Heron angles given in (2) we state

Theorem 1 An inscribable quadrilateral is a Brahmagupta quadrilateral if and only if the sides a, b, c, d and the diagonals e, f are proportional to

$$\begin{aligned}
a &= (m_1^2 - n_1^2)(m_2^2 + n_2^2)(m_3^2 + n_3^2)\,,\\
b &= (m_1^2 + n_1^2)(m_2^2 - n_2^2)(m_3^2 + n_3^2)\,,\\
c &= (m_1^2 + n_1^2)(m_2^2 + n_2^2)(m_3^2 - n_3^2)\,,\\
d &= 4m_3n_3[m_2n_2(m_1^2 - n_1^2) + m_1n_1(m_2^2 - n_2^2)] +\\
&\qquad (m_3^2 - n_3^2)[4m_1n_1m_2n_2 - (m_1^2 - n_1^2)(m_2^2 - n_2^2)]\,,\\
e &= 2(m_3^2 + n_3^2)[m_2n_2(m_1^2 - n_1^2) + m_1n_1(m_2^2 - n_2^2)]\,,\\
f &= 2(m_1^2 + n_1^2)[m_3n_3(m_2^2 - n_2^2) + m_2n_2(m_3^2 - n_3^2)]\,,
\end{aligned}$$

where m_i, n_i are relatively prime natural numbers such that $m_i > n_i$ for $i = 1, 2, 3$ and $a, b, c, d, e, f > 0$.

Proof: Let us consider a cyclic quadrilateral in a circle of diameter 1 (see Figure 2). Let $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, $BD = f$, $\angle ADB = \theta_1$, $\angle BDC = \theta_2$, $\angle DAC = \theta_3$. Then $\angle ACD = \pi - (\theta_1 + \theta_2 + \theta_3)$. From our earlier observations we have $\angle ACB = \theta_1$, $\angle BAC = \theta_2$, $\angle DBC = \theta_3$, $a = \sin\theta_1$, $b = \sin\theta_2$, $c = \sin\theta_3$, $d = \sin(\theta_1 + \theta_2 + \theta_3)$, $e = \sin(\theta_1 + \theta_2)$, and $f = \sin(\theta_2 + \theta_3)$. We also have

$$\text{area }(ABCD) = \tfrac{1}{2}(ab + cd)\sin(\theta_1 + \theta_2)\,.$$

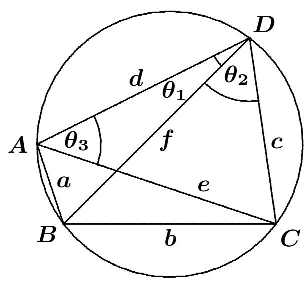

Figure 2

The lengths a, b, c, d, e, f, and the area of the quadrilateral $ABCD$ are rational if and only if the angles θ_i are Heron angles. Furthermore, as observed earlier, these rationals can be converted to integers, leading to a Brahmagupta quadrilateral. Substituting for the angles in terms of integers m_i and n_i (using our expressions from the last section), and multiplying by the least common multiple of the denominators, we obtain the expressions in the statement of Theorem 1.

We give a numerical illustration before we deduce two important theorems from Theorem 1. Suppose we set $m_1 = 2$, $n_1 = 1$, $m_2 = 3$, $n_2 = 2$, $m_3 = 4$, $n_3 = 1$. Then we have a Brahmagupta quadrilateral $ABCD$ given by

$$\begin{aligned} a &= 663\,, & b &= 425\,, & c &= 975\,, \\ d &= 943\,, & e &= 952\,, & f &= 1100\,. \end{aligned}$$

We first find $\sin(\theta_1 + \theta_2)$ from (2) and then $\frac{1}{2}(ab + cd)\sin(\theta_1 + \theta_2)$ in order to compute the area of $ABCD$. This area may also be computed by using Brahmagupta's formula (see the concluding section below).

Originally, Brahmagupta's quadrilaterals had perpendicular diagonals. Kummer derived general expressions to obtain such quadrilaterals.

Theorem 2 (Kummer) A Brahmagupta quadrilateral has perpendicular diagonals if and only if it has sides and diagonals proportional to

$$\begin{aligned} a &= (m_1^2 - n_1^2)(m_2^2 + n_2^2)\,, \quad b \;=\; (m_1^2 + n_1^2)(m_2^2 - n_2^2)\,, \\ c &= 2m_1n_1(m_2^2 + n_2^2)\,, \quad d \;=\; 2m_2n_2(m_1^2 + n_1^2)\,, \\ e &= 2[m_1n_1(m_2^2 - n_2^2) + m_2n_2(m_1^2 - n_1^2)]\,, \\ f &= 4m_1n_1m_2n_2 + (m_1^2 - n_1^2)(m_2^2 - n_2^2)\,. \end{aligned}$$

Proof: We refer to Figure 2. The diagonals AC and BD will be perpendicular to each other if and only if $\theta_3 = \pi/2 - \theta_1$. Hence, $\sin\theta_3 = \cos\theta_1$. Therefore, we put $m_3 = m_1 + n_1$ and $n_3 = m_1 - n_1$ in Theorem 1 and then divide by the greatest common divisor to get the expressions listed.

If a trapezium (trapezoid) is cyclic, then it is easy to see that it must be isosceles. Hence, Brahmagupta trapeziums are isosceles trapeziums with integer sides, diagonals, and area.

Theorem 3 An inscribable trapezium is a Brahmagupta trapezium if and only if the sides and diagonals are proportional to

$$\begin{aligned} a = c &= (m_1^2 - n_1^2)(m_1^2 + n_1^2)(m_2^2 + n_2^2)\,, \\ b &= (m_1^2 + n_1^2)^2(m_2^2 - n_2^2)\,, \\ d &= 8m_1n_1m_2n_2(m_1^2 - n_1^2) \\ &\quad + (m_2^2 - n_2^2)(2m_1n_1 + m_1^2 - n_1^2)(2m_1n_1 - m_1^2 + n_1^2)\,, \\ e = f &= 2(m_1^2 + n_1^2)[m_2n_2(m_1^2 - n_1^2) + m_1n_1(m_2^2 - n_2^2)]\,. \end{aligned}$$

Proof: The sides AD and BC are parallel if and only if $\theta_3 = \theta_1$ (see Figure 2). Hence, we set $m_3 = m_1$ and $n_3 = n_1$ in Theorem 1 to obtain the above expressions.

Conclusions:

Brahmagupta found several important results involving cyclic quadrilaterals. The reader may enjoy rediscovering some of them independently.

1. Brahmagupta's remarkable formula for the area of a cyclic quadrilateral is similar to Heron's formula for a triangle. Actually the converse holds too. Let the sides of a quadrilateral be denoted by a, b, c, d and let $s = (a + b + c + d)/2$. Prove that the quadrilateral is cyclic if and only if its area is $\Delta = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. Note that when $d = 0$, the quadrilateral degenerates into a triangle. In fact, this beautiful formula raises the general open question: Given an n–gon with sides $a_1, a_2, \ldots, a_n$, let $s = \left(\sum_{i=1}^{n} a_i\right)/2$. Determine those n–gons whose area is $\Delta_n = \sqrt{(s-a_1)(s-a_2)\cdots(s-a_n)}$. A partial solution is given in [5].

2. Express the lengths e, f of the diagonals of a cyclic quadrilateral in terms of the sides a, b, c, d.

3. Kummer has given a complex method to determine Heron quadrilaterals, the ones with integral sides, diagonals, and area. This is outlined in [1], pp. 191–224. The reader may attempt to give simpler descriptions of Heron quadrilaterals, at least of special Heron quadrilaterals such as Heron parallelograms, and Heron trapeziums.

One can find more on the life of Brahmagupta in [2], pp. 49–51. A partial solution, in terms of a special family of Heron triangles, to the general problem discussed in the present paper can be found in [3], pp. 49–52.

References:

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