SKOLIAD No. 67

Shawn Godin

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We are especially looking for solutions from high school students. Please include your name, school or other affiliation (if applicable), city, province or state, and country on any correspondence. High school students should also include their grade in school. Please send your solutions to the problems in this edition by *1 April 2003*. A copy of **MATHEMATICAL MAYHEM Vol. 1** will be presented to the pre-university reader(s) who send in the best solutions before the deadline. The decision of the editor is final.

Starting with this issue we have a slight change. Certain items in the problem sets will be marked with an asterisk ($*$). We will only print solutions to these problems if we receive them from students in grade 10 or under (or equivalent), or if we receive a unique solution or a generalization.

Our entry this issue comes from the Manitoba Mathematical Contest. My thanks go to Diane Dowling of the University of Manitoba for forwarding the material to me.

MANITOBA MATHEMATICAL CONTEST, 2002

For students in Senior 4

9:00 a.m. – 11:00 a.m. Wednesday, February 20, 2002

Sponsored by

The Actuaries' Club of Winnipeg, The Manitoba Association of Mathematics Teachers, The Canadian Mathematical Society, and The University of Manitoba

Answer as much as possible. You are not expected to complete the paper. See both sides of this sheet. Hand calculators are not permitted. Numerical answers only, without explanation, will not be given full credit.

1. (a) ($*$) Solve the equation $x^4 - 3x^2 + 2 = 0$.

($*$) Résous l'équation $x^4 - 3x^2 + 2 = 0$.

(b) ($*$) Solve the equation $\frac{4}{(x-3)^2} - \frac{4}{(x-3)} + 1 = 0$.

($*$) Résous l'équation $\frac{4}{(x-3)^2} - \frac{4}{(x-3)} + 1 = 0$.

2. (a) ($*$) Solve the equation $9x^3 - 9x^2 - 4x + 4 = 0$.

($*$) Résous l'équation $9x^3 - 9x^2 - 4x + 4 = 0$.

(b) (*) Thirty-six students took a final exam. The average score of those who passed was 60, the average score of those who failed was 42 and the average of all the scores was 53. How many students did not pass the exam?

(*) Trente-six étudiants ont passé un examen. La note moyenne de ceux qui ont réussi était 60, la note moyenne de ceux qui ont échoué était 42, et la note moyenne de toutes les notes était 53. Combien d'étudiants n'ont pas réussi à cet examen ?

3. (a) (*) The area of a rectangle is 3 and its perimeter is 7. What is the length of the diagonal of this rectangle?

(*) L'aire d'un rectangle est égale à 3 et son périmètre est égal à 7. Quelle est la longueur d'une diagonale de ce rectangle ?

(b) (*) In this problem O is the origin, A is the point $(3, 4)$ and B is a point in the first quadrant on the line joining O and A. If the length of AB is 6 what are the coordinates of B?

(*) Soit O l'origine, soit A le point $(3, 4)$, et soit B un point dans le premier quadrant sur la droite qui passe par O et A. Si la longueur de AB est égale à 6, quelles sont les coordonnées de B ?

4. (a) (*) Solve the equation $\sqrt{3-x} + \sqrt{12-4x} = \sqrt{x-1}$.

(*) Résous l'équation $\sqrt{3-x} + \sqrt{12-4x} = \sqrt{x-1}$.

(b) (*) If p, q and r are the three roots of the equation $x^3 - 7x^2 + 3x + 1 = 0$, find the value of $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$.

(*) Si p, q et r sont les trois racines de l'équation $x^3 - 7x^2 + 3x + 1 = 0$, trouve la valeur de $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$.

5. (a) (*) If a and b are real numbers such that $\sqrt{a} - \sqrt{b} = \sqrt{2}$ and $a - b = 10$, find a and b.

(*) Soient a et b des nombres réels tels que $\sqrt{a} - \sqrt{b} = \sqrt{2}$ et $a - b = 10$. Trouve les valeurs de a et b.

(b) (*) If k is a real number such that $3\left(2^{k+3}\right) - 2^{2k} = 128$, what are the possible values of k?

(*) Si k est un nombre réel tel que $3\left(2^{k+3}\right) - 2^{2k} = 128$, quelles sont les valeurs possibles de k ?

6. (a) In triangle ABC, $\angle BAC = 60°$, $\angle ACB = 90°$ and D is on BC. If AD bisects $\angle BAC$ prove that $DB = 2CD$.

Soit ABC un triangle tel que $\angle BAC = 60°$ et $\angle ACB = 90°$. Soit D un point sur le côté BC. Si AD bissecte $\angle BAC$, prouve que $DB = 2 \cdot CD$.

(b) In triangle ABC, $AC = BC = 5$ and $AB = 8$. What is the radius of the circle which passes through A, B, and C?

Soit ABC un triangle tel que $AC = BC = 5$ et $AB = 8$. Quel est le rayon du cercle qui passe par les points A, B, et C ?

7. A, B and C are points on a circle of radius 3. In triangle ABC, $\angle ACB = 30^\circ$ and $AC = 2$. Find BC.

Les points A, B et C se trouvent sur un cercle dont le rayon est égal à 3. Étant donné que $\angle ACB = 30^\circ$ et $AC = 2$ dans le triangle ABC, trouve BC.

8. If x and y are real numbers, prove that $x^3y + xy^3 \leq x^4 + y^4$.

Soient x et y des nombres réels quelconques. Prouve que $x^3y + xy^3 \leq x^4 + y^4$.

9. A, B, C, and D are points on a circle. AB is the diameter. CD is perpendicular to AB and meets AB at E. If AB and CD are integers and $AE - EB = \sqrt{7}$, find AE.

Les points A, B, C, et D se trouvent sur un cercle. AB est un diamètre de ce cercle. CD est perpendiculaire à AB et croise AB au point E. Étant donné que AB et CD sont des nombres entiers et que $AE - EB = \sqrt{7}$, trouve AE.

10. Nine points, no three of which lie on the same straight line, are located inside an equilateral triangle of side 4. Prove that some three of these points are vertices of a triangle whose area is not greater than $\sqrt{3}$.

Neuf points, dont trois quelconques ne se trouvent pas sur la même droite, sont situés à l'intérieur d'un triangle équilatéral dont la longeur des côtés est égale à 4. Prouve que parmi ces points il y en a trois qui sont les sommets d'un triangle, l'aire duquel ne surpasse pas $\sqrt{3}$.

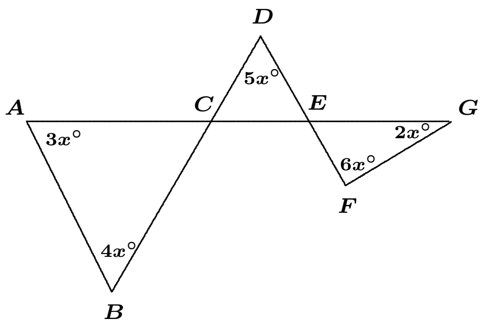
Next we give the solutions to the 2001 Canadian Open Mathematics Challenge.

2001 Canadian Open Mathematics Challenge
PART/PARTIE A

1. On définit une opération "$\triangledown$" comme suit : $a \triangledown b = a^2 + 3^b$. Quelle est la valeur de $(2 \triangledown 0) \triangledown (0 \triangledown 1)$?

Solution par Robert Bilinski, Outremont, QC.
$(2 \triangledown 0) \triangledown (0 \triangledown 1) = \left(2^2 + 3^0\right) \triangledown \left(0^2 + 3^1\right) = 5 \triangledown 3 = 5^2 + 3^3 = 52$.

2. Dans le diagramme ci-contre, quelle est la valeur de x ?



Solution par Robert Bilinski, Outremont, QC.

Dans le triangle ABC, on a $\angle ACB = 180^\circ - 3x^\circ - 4x^\circ = 180^\circ - 7x^\circ$. Ainsi $\angle DCE = \angle ACB = 180^\circ - 7x^\circ$ car ce sont des angles opposés par leurs sommets. De la même manière, en travaillant dans le triangle EFG, on obtient que $\angle DEC = \angle FEG = 180^\circ - 8x^\circ$. Puisque la somme des angles dans un triangle donne 180°, on a dans le triangle CDE :
$180^\circ - 7x^\circ + 180^\circ - 8x^\circ + 5x^\circ = 180^\circ$ ou $x = 18^\circ$.

3. Un hexagone régulier est un polygone à 6 côtés dont tous les angles sont congrus et tous les côtés sont congrus. Soient P et Q des points sur un hexagone régulier dont les côtés ont une longueur de 1. Quelle est la longueur maximale possible du segment PQ ?

Solution par Robert Bilinski, Outremont, QC.

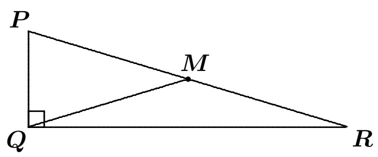
Un hexagone régulier peut se diviser en 6 triangles équilatéraux, de même côté que l'hexagone. Ainsi la grande diagonale de l'hexagone est de longueur 2 (deux fois plus long qu'un côté de triangle, donc d'hexagone). Puisque les hexagones réguliers sont inscrits, la plus longue longueur entre deux points de l'hexagone sera le diamètre du cercle, soit une grande diagonale de l'hexagone. La réponse est donc 2.

4. Résoudre l'équation suivante : $2\left(2^{2x}\right) = 4^x + 64$.

Solution par Robert Bilinski, Outremont, QC.

$$\begin{aligned} 2\left(2^{2x}\right) &= 4^x + 64 \iff 2\cdot 4^x - 4^x = 64 \\ &\iff 4^x = 4^3 \iff x = 3 \end{aligned}$$

5. Le diagramme illustre un triangle rectangle PQR, dans lequel $PQ = 14$ et $QR = 48$. M est le milieu du côté PR. Déterminer le cosinus de l'angle MQP.



Solution par Robert Bilinski, Outremont, QC.

M étant le milieu de PR, donc l'hypoténuse du triangle rectangle PQR, on a M le centre du cercle circonscrit au triangle PQR. On a donc $MP = MQ = MR$ (des rayons de ce cercle). Donc le triangle MPQ est isocèle en M. Ainsi, on a $\angle MPQ = \angle MQP$. Donc

$$\cos\angle MQP = \cos\angle MPQ = \frac{PQ}{PR} = \frac{7}{25}\,.$$

6. On définit une suite de nombres, t_1, t_2, t_3, ... , comme suit : $t_1 = 2$ et $t_{n+1} = \dfrac{t_n - 1}{t_n + 1}$, pour tout entier strictement positif n. Déterminer la valeur numérique de t_{999}.

Solution par Robert Bilinski, Outremont, QC.

On a

$$\begin{aligned} t_{n+1} &= \frac{t_n - 1}{t_n + 1} = 1 - \frac{2}{t_n + 1} = 1 - \frac{2}{\frac{t_{n-1}-1}{t_{n-1}+1} + 1} \\ &= 1 - \frac{2}{\frac{2t_{n-1}}{t_{n-1}+1}} = \frac{-1}{t_{n-1}} = t_{n-3} \end{aligned}$$

Donc la suite $\{t_n\}$ est de période 4. Il suffit donc de connaître t_1, t_2, t_3, et t_4 pour connaître toute la suite.

On a $t_1 = 2 = t_{4k+1}$, pour $k \in \mathbb{N}$. On a $t_2 = \frac{1}{3} = t_{4k+2}$, pour $k \in \mathbb{N}$. On a $t_3 = -\frac{1}{2} = t_{4k+3}$, pour $k \in \mathbb{N}$. On a $t_4 = -3 = t_{4k+4}$, pour $k \in \mathbb{N}$. Ainsi, puisque $999 = 4 \cdot 249 + 3$, nous avons que $t_{999} = -\frac{1}{2}$.

7. Sachant que a peut prendre la valeur de n'importe quel entier strictement positif et que

$$\begin{aligned} 2x + a &= y\,, \\ a + y &= x\,, \\ x + y &= z\,, \end{aligned}$$

déterminer la valeur maximale possible de l'expression $x + y + z$.

Solution par Robert Bilinski, Outremont, QC.

En prenant les deux premières équations et en réarrangeant les termes, on obtient le système de deux équations à 2 inconnues suivant (en considérant le a fixé)

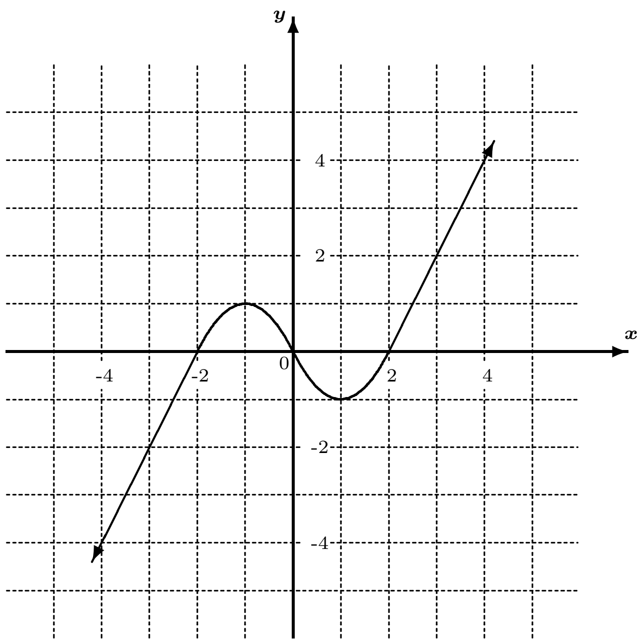
$$\begin{aligned} 2x - y &= -a \\ -x + y &= -a \end{aligned}$$

qui se résout par

$$\begin{aligned} x &= -2a \\ y &= -3a \end{aligned}$$

Par la troisième équation, on obtient facilement que $z = -5a$. Ainsi, la somme $x + y + z = -10a$. Puisque $a > 0$ par hypothèse, on obtient que le maximum de la somme $x + y + z$ est atteint lorsque $a = 1$ et ce maximum est -10.

8. The graph of the function $y = g(x)$ is shown. Determine the number of solutions of the equation $\left||g(x)| - 1\right| = \frac{1}{2}$.



Official Solution.

From the original equation $\left||g(x)| - 1\right| = \frac{1}{2}$, using the definition of absolute value, we obtain,

$$|g(x)| - 1 = \tfrac{1}{2} \quad \text{or} \quad |g(x)| - 1 = -\tfrac{1}{2}$$

$$|g(x)| = \tfrac{3}{2} \quad \text{or} \quad |g(x)| = \tfrac{1}{2}$$

$$g(x) = \pm\tfrac{3}{2} \quad \text{or} \quad g(x) = \pm\tfrac{1}{2}$$

From the original graph $g(x) = \frac{3}{2}$ has 1 solution, $g(x) = -\frac{3}{2}$ has 1 solution, $g(x) = \frac{1}{2}$ has 3 solutions and $g(x) = -\frac{1}{2}$ has 3 solutions. Therefore, $\left||g(x)| - 1\right| = \frac{1}{2}$ has 8 solutions.

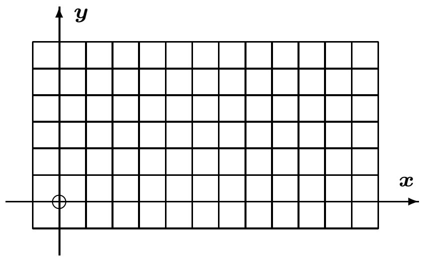
PART /PARTIE B

1. The triangular region T has its vertices determined by the intersections of the three lines: $x + 2y = 12$, $x = 2$, and $y = 1$.

(a) Determine the coordinates of the vertices of T, and show this region on the grid provided.

(b) The line $x + y = 8$ divides the triangular region T into a quadrilateral Q and a triangle R. Determine the coordinates of the vertices of the quadrilateral Q.

(c) Determine the area of the quadrilateral Q.



Solution by Robert Bilinski, Outremont, QC.

(a) To get the coordinates, we need to solve the three following systems of equations (one for each vertex):

$$\begin{aligned} x &= 2 \\ y &= 1 \end{aligned} \qquad \begin{aligned} x &= 2 \\ x + 2y &= 12 \end{aligned} \qquad \begin{aligned} x + 2y &= 12 \\ y &= 1 \end{aligned}$$

which give (in order) the vertices $A(2,1)$, $B(2,5)$ and $C(10,1)$. [*Editor's note:* This triangular region can be easily shown on the grid.]

(b) The new line crosses BC and AC, thus defining two new points with systems:

$$\begin{aligned} x + 2y &= 12 \\ x + y &= 8 \end{aligned} \qquad \begin{aligned} y &= 1 \\ x + y &= 8 \end{aligned}$$

which define, respectively, $D(4,4)$ on BC and $E(7,1)$ on AC. Hence A, B, D and E are the vertices of Q.

(c) We can split Q into two triangles, namely ABD and ADE, since their areas are easy to evaluate. We have $[ABD] = \dfrac{(4)(2)}{2} = 4$ and $[ADE] = \dfrac{(5)(3)}{2} = 7.5$. Hence, Q has area 11.5.

2. (a) Alphonse and Beryl are playing a game, starting with a pack of 7 cards. Alphonse begins by discarding at least one but not more than half of the cards in the pack. He then passes the remaining cards in the pack to Beryl. Beryl continues the game by discarding at least one but not more than half of the remaining cards in the pack. The game continues in this way with the pack being passed back and forth between the two players. The loser is the player who, at the beginning of his or her turn, receives only one card. Show, with justification, that there is always a winning strategy for Beryl.

(b) Alphonse and Beryl now play a game with the same rules as in (a), except this time they start with a pack of 52 cards, and Alphonse goes first again. As in (a), a player on his or her turn must discard at least one and not more than half of the remaining cards from the pack. Is there a strategy that Alphonse can use to be guaranteed that he will win? (Provide justification for your answer.)

Official Solution

(a) Alphonse starts with 7 cards, and so can remove 1, 2, or 3 cards, passing 6, 5, or 4 cards to Beryl. Beryl should remove 3, 2, or 1 cards, respectively, leaving 3 cards only, and pass these 3 cards back to Alphonse. Alphonse now is forced to remove 1 card only, and pass 2 back to Beryl. Beryl removes 1 card (her only option) and passes 1 back to Alphonse, who thus loses. Therefore, Beryl is guaranteed to win.

(b) Alphonse removes 21 cards from original 52, and passes 31 cards to Beryl. If Beryl removes b_1 cards with $1 \le b_1 \le 15$, then Alphonse removes $16 - b_1$ cards to reduce the pack to 15 cards. [This is always a legal move, since $2(16 - b_1) = 32 - 2b_1 \le 31 - b_1$ so $16 - b_1$ is never more than half of the pack.] If Beryl removes b_2 cards with $1 \le b_2 \le 7$, then Alphonse removes $8 - b_2$ to reduce the pack to 7 cards. [This move is always legal, by a similar argument.] Since Beryl now has 7 cards, Alphonse can adopt Beryl's strategy from (a). Thus, Alphonse has a winning strategy.

3. (a) If $f(x) = x^2 + 6x + c$, where c is an integer, prove that $f(0) + f(-1)$ is odd.

(b) Let $g(x) = x^3 + px^2 + qx + r$, where p, q and r are integers. Prove that if $g(0)$ and $g(-1)$ are both odd, then the equation $g(x) = 0$ cannot have three integer roots.

Solution by Robert Bilinski, Outremont, QC.

(a) $f(0)+f(-1) = c+1-6+c = 2c-5 = (2c-6)+1 = 2(c-3)+1$, which is always odd.

(b) $g(0) = r$ is odd. Thus, $r = 2s + 1$ for some s. Therefore,

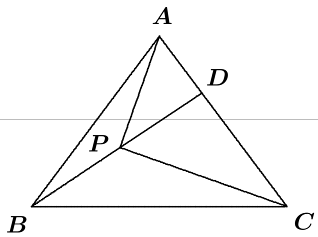
$$g(-1) \;=\; -1 + p - q + r \;=\; -1 + p - q + 2s + 1 \;=\; p - q + 2s\,.$$

Since $g(-1)$ is odd, p and q cannot be of the same parity.

Let us suppose that $g(x)$ has three integer roots, say h, i and j. Then we have $g(x) = (x-h)(x-i)(x-j) = x^3 - (h+i+j)x^2 + (hi+hj+ij)x + hij$. We see that $r = -hij$ is odd, Thus, all three of h, i and j must be odd, and $h + i + j$ is odd. But $p = -(h + i + j)$, which implies p is also odd. We also have hi, hj and ij odd. Therefore, $(hi + hj + ij) = q$ is also odd.

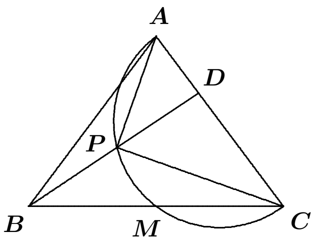
But we saw before that p and q cannot have the same parity. Hence, there is a contradiction, and it is impossible for all three roots of $g(x)$ to be integers.

4. Triangle ABC is isosceles with $AB = AC = 5$ and $BC = 6$. Point D lies on AC, and P is the point on BD so that $\angle APC = 90^\circ$. If $\angle ABP = \angle BCP$, determine the ratio $AD : DC$.



Official Solution

Draw a perpendicular from A to meet BC at M. Then, since $AB = AC$, $BM = MC = 3$ and so $AM = 4$. Let $\alpha = \angle BCP = \angle ABP$ and $\theta = \angle ACP$. Then $\angle PBC = \theta$, since $\triangle ABC$ is isosceles. Draw a circle with AC as diameter. This circle passes through both P and M since $\angle APC = \angle AMC = 90^\circ$. Join P to M.



Then $\angle PAM = \alpha$ since $\angle PAM = \angle PCM$ (subtended by the same chord). Also $\angle AMP = \theta$ for similar reasons. Therefore, $\triangle MPA$ is similar to $\triangle BPC$. Thus, $\frac{PA}{PC} = \frac{MA}{BC} = \frac{4}{6}$, whence $\tan\theta = \frac{PA}{PC} = \frac{2}{3}$.

Now we compute the length of DC. Consider $\triangle BDC$. By the Sine Law,

$$\begin{aligned} \frac{DC}{\sin\theta} &= \frac{BC}{\sin(\angle BDC)} \\ DC &= \frac{6\sin\theta}{\sin(180^\circ - \theta - \angle DCB)} = \frac{6\sin\theta}{\sin(\theta + \angle DCB)} \\ &= \frac{6\sin\theta}{\sin\theta\cos(\angle DCB) + \cos\theta\sin(\angle DCB)} \\ &= \frac{6}{\cos(\angle DCB) + \cot\theta\sin(\angle DCB)} = \frac{6}{\frac{3}{5} + \frac{3}{2}\cdot\frac{4}{5}} = \frac{10}{3}. \end{aligned}$$

Then $AD = 5 - \frac{10}{3} = \frac{5}{3}$, and thus $AD : DC = 1 : 2$.

That ends another Skoliad Corner. Send us your problems and solutions. Remember, we have past volumes of MAYHEM for pre-university solvers.