

# THE OLYMPIAD CORNER

No. 227

R.E. Woodrow

*All communications about this column should be sent to Professor R.E. Woodrow, Department of Mathematics and Statistics, University of Calgary, Calgary, AB, Canada. T2N 1N4.*

Thank you to our contributors and to Joanne Longworth, who so capably changes my scrawl into readable type.

Mohammed Aassila	Murray S. Klamkin
Miguel Amengual Covas	Raul A. Simon Lamb
Jean Claude Andrieux	Elena Lee
Ed Barbeau	Hojoo Lee
Michel Bataille	Robert Barrington Leigh
Robert Bilinski	Brett Link
Pierre Bornsztein	Andy Liu
Christopher J. Bradley	David Loeffler
Bob Cao	Pavlos Maragoudakis
Babis Stergiou Chalkida	Greg Martin
Adrian Chan	Stewart Metchette
Peter de Caux	Vedula N. Murty
Peter Du	Moubinool Omarjee
George Evagelopoulos	Toshio Seimiya
Dora Ho	Achilleas Sinefakopoulos
Oleg Ivrii	D.J. Smeenk
Geoffrey A. Kandall	Panos E. Tsaoussoglou
April Keller	Edward T.H. Wang

To start off the new year we give the problems of the 2000 Korean Mathematical Olympiad. Thanks go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

## 2000 KOREAN MATHEMATICAL OLYMPIAD

**1.** Prove that for any prime  $p$ , there exist integers  $x$ ,  $y$ ,  $z$ , and  $w$  such that  $x^2 + y^2 + z^2 - wp = 0$  and  $0 < w < p$ .

**2.** Determine all functions  $f$  from the set of real numbers to itself such that for every  $x$  and  $y$ ,

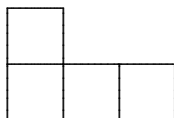
$$f(x^2 - y^2) = (x - y)(f(x) + f(y)).$$

**3.** A rectangle  $ABCD$  is inscribed in a circle with centre  $O$ . The exterior bisectors of  $\angle ABD$  and  $\angle ADB$  intersect at  $P$ ; those of  $\angle DAB$  and  $\angle DBA$  intersect at  $Q$ ; those of  $\angle ACD$  and  $\angle ADC$  intersect at  $R$ ; and those of  $\angle DAC$  and  $\angle DCA$  intersect at  $S$ . Prove that  $P, Q, R,$  and  $S$  are concyclic.

**4.** Let  $p$  be a prime such that  $p \equiv 1 \pmod{4}$ . Evaluate

$$\sum_{k=1}^{p-1} \left( \left\lfloor \frac{2k^2}{p} \right\rfloor \right) - 2 \left( \left\lfloor \frac{k^2}{p} \right\rfloor \right).$$

**5.** Prove that an  $m \times n$  rectangle can be constructed using copies of the following shape if and only if  $mn$  is a multiple of 8.



**6.** The real numbers  $a, b, c, x, y,$  and  $z$  are such that  $a > b > c > 0$  and  $x > y > z > 0$ . Prove that

$$\frac{a^2 x^2}{(by + cz)(bz + cy)} + \frac{b^2 y^2}{(cz + ax)(cx + az)} + \frac{c^2 z^2}{(ax + by)(ay + bx)} \geq \frac{3}{4}.$$

As a second set we give the problems of the 2000 Bulgarian Mathematical Olympiad. Thanks again go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

## 2000 BULGARIAN MATHEMATICAL OLYMPIAD

**1.** In the Cartesian plane, a set of 2000 points  $M_i(x_i, y_i)$  is said to be *good* if  $0 \leq x_i \leq 83$  and  $0 \leq y_i \leq 1$  for  $i = 1, 2, \dots, 2000$  and  $x_i \neq x_j$  for  $i \neq j$ . Find all positive integers  $n$  such that for any good set, some  $n$  of its points lie in a square of side length 1, but there exists a good set such that no  $n + 1$  of its points lie in a square of side length 1. (A point on a side of a square lies in the square).

**2.** Let  $ABC$  be an acute triangle.

(a) Prove that there exist unique points  $A', B',$  and  $C'$ , on  $BC, CA,$  and  $AB$ , respectively, such that  $A'$  is the mid-point of the orthogonal projection of  $B'C'$  onto  $BC$ ,  $B'$  is the mid-point of the orthogonal projection of  $C'A'$  onto  $CA$ , and  $C'$  is the mid-point of the orthogonal projection of  $A'B'$  onto  $AB$ .

(b) Prove that  $A'B'C'$  is similar to the triangle formed by the medians of  $ABC$ .

**3.** Let  $p$  be an odd prime and  $a_1, a_2, \dots, a_{p-2}$  be a sequence of positive integers such that for all  $k = 2, 3, \dots, p-2$ , the prime  $p$  does not divide both  $a_k$  and  $a_{k-1}$ . Prove that the product of some elements of this sequence is congruent to 2 modulo  $p$ .

**4.** Find all polynomials  $P(x)$  with real coefficients such that we have  $P(x)P(x+1) = P(x^2)$  for all real  $x$ .

**5.** In triangle  $ABC$ , we have  $CA = CB$ . Let  $D$  be the mid-point of  $AB$  and  $E$  an arbitrary point on  $AB$ . Let  $O$  be the circumcentre of  $\triangle ACE$ . Prove that the line through  $D$  perpendicular to  $DO$ , the line through  $E$  perpendicular to  $BC$ , and the line through  $B$  parallel to  $AC$  are concurrent.

**6.** Let  $\mathcal{A}$  be the set of all binary sequences of length  $n$ , and let  $\mathbf{0} \in \mathcal{A}$  be the sequence all terms of which are zeroes. The sequence  $c = \langle c_1, c_2, \dots, c_n \rangle$  is called the sum of  $a = \langle a_1, a_2, \dots, a_n \rangle$  and  $b = \langle b_1, b_2, \dots, b_n \rangle$  if  $c_i = 0$  when  $a_i = b_i$  and  $c_i = 1$  when  $a_i \neq b_i$ . Let  $f : \mathcal{A} \rightarrow \mathcal{A}$  be a function such that  $f(\mathbf{0}) = \mathbf{0}$  and if the sequences  $a$  and  $b$  differ in exactly  $k$  terms then the sequences  $f(a)$  and  $f(b)$  differ also exactly in  $k$  terms. Prove that if  $a, b$ , and  $c$  are sequences from  $\mathcal{A}$  such that  $a + b + c = \mathbf{0}$ , then  $f(a) + f(b) + f(c) = \mathbf{0}$ .

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As a final set of puzzles for this number we give the problems of the 2000 Vietnamese Mathematical Olympiad. Again, thanks go to Andy Liu, Canadian Team Leader to the IMO in Korea, for collecting them.

## 2000 VIETNAMESE MATHEMATICAL OLYMPIAD

**1.** Given a real number  $c > 0$  and an initial value  $x_0$  where  $0 < x_0 < c$ , a sequence  $\{x_n\}$  of real numbers is defined by  $x_{n+1} = \sqrt{c - \sqrt{c + x_n}}$  for  $n \geq 0$ . Find all positive real numbers  $c$  such that for each initial value  $x_0$  in  $(0, c)$ , the sequence  $\{x_n\}$  is defined for all  $n$  and has a finite limit.

**2.** Two circles  $\Omega_1$  and  $\Omega_2$  with respective centres  $O_1$  and  $O_2$  are given on the plane. Let  $M_1$  and  $M_2$  be two points on  $\Omega_1$  and  $\Omega_2$ , respectively, such that the lines  $O_1M_1$  and  $O_2M_2$  intersect at  $Q$ . Starting simultaneously from these positions, the points  $M_1$  and  $M_2$  move clockwise, each on its own circle, with the same angular velocity.

(a) Determine the locus of the mid-point of  $M_1M_2$ .

(b) Prove that the circumcircle of the triangle  $M_1QM_2$  passes through a fixed point.

**3.** Consider the polynomial  $P(x) = x^3 + 153x^2 - 111x + 38$ .

(a) Prove that the closed interval  $[1, 3^{2000}]$  contains at least 9 integers  $a$  for which  $P(a)$  is divisible by  $3^{2000}$ .

(b) Determine the number of integers  $a$  in the closed interval  $[1, 3^{2000}]$  for which  $P(a)$  is divisible by  $3^{2000}$ .

**4.** For every integer  $n \geq 3$  and any given angle  $\alpha$  in  $(0, \pi)$ , let  $P_n(x) = x^n \sin \alpha - x \sin n\alpha + \sin(n-1)\alpha$ .

(a) Prove that there is only one polynomial of the form  $f(x) = x^2 + ax + b$  such that for every  $n \geq 3$ ,  $P_n(x)$  is divisible by  $f(x)$ .

(b) Prove that there does not exist a polynomial  $g(x)$  of the form  $g(x) = x + c$  such that for every  $n \geq 3$ ,  $P_n(x)$  is divisible by  $g(x)$ .

**5.** Determine all integers  $n \geq 3$  such that there exists  $n$  points  $A_1, A_2, \dots, A_n$  in space, with no three on a line and no four on a circle, such that all the circumcircles of the triangles  $A_i A_j A_k$  are congruent.

**6.** Let  $P(x)$  be a non-zero polynomial such that, for all real numbers  $x$ , we have  $P(x^2 - 1) = P(x)P(-x)$ . Determine the maximal number of roots of  $P(x)$ .

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Next we have some tidying up and apologies to make.

First, my apologies to Robert Bilinski, Outremont, QC, who sent in a batch of problems, only one of which was cited (and the other two misfiled with that month's "solved problems"). He should have been listed as a solver of problem #2 of the 47<sup>th</sup> Latvian Mathematical Olympiad [2000 : 324; 2002 : 425], as well as one of the solvers of Problem #1 of the XXIII All Russian Olympiad [2000 : 388–389; 2002 : 493–494].

Continuing in this vein, I have also received two solutions from Pavlos Maragoudakis, Pireas, Greece which were not cited last year. He sent in a solution to problem #3 of the 47<sup>th</sup> Latvian Mathematical Olympiad [2000 : 324; 2002 : 425–426], as well as a different solution to problem #1 of the XXIII All Russian Olympiad [2000 : 388–389; 2002 : 493–494]. We give his alternate solution here.

**1.** Solve, in integers, the equation

$$(x^2 - y^2)^2 = 1 + 16y.$$

*Alternate solution by Pavlos Maragoudakis, Pireas, Greece.*

If  $(x, y)$  is a solution then  $1 + 16y \geq 0$ , so  $y \geq 0$ . Also, if  $(x, y)$  is solution,  $(-x, y)$  is also a solution. We find the solutions  $(x, y)$  with  $x, y \geq 0$ .

- If  $x = 0$ , then  $y^4 - 16y - 1 = 0$ , which has no integer solutions.

- If  $y = 0$ , then  $x = 1$  or  $x = -1$ .
- If  $x, y \geq 1$ , then

$$2(x - y)^2 \leq (x + y)(x - y)^2 = \frac{1 + 16y}{x + y} < \frac{16x + 16y}{x + y} = 16,$$

so  $(x - y)^2 < 8 \implies |x - y| < 3$ .

Since  $1 + 16y$  is odd and  $x - y$  divides  $1 + 16y$ , we get that  $x - y$  is odd. So  $x - y = 1$  or  $-1$ .

If  $x - y = 1$ , then  $(2y + 1)^2 = 1 + 16y \iff y = 0$  or  $y = 3$ .

If  $x - y = -1$ , then  $(2y - 1)^2 = 1 + 16y \iff y = 0$  or  $y = 5$ .

All solutions are:  $(1, 0)$ ,  $(-1, 0)$ ,  $(4, 3)$ ,  $(-4, 3)$ ,  $(4, 5)$ ,  $(-4, 5)$ .

While going through files, I noticed that somehow we managed to skip over the solutions we had received to some of the problems from the 20<sup>th</sup> Austrian-Polish Mathematical Competition and Selected Problems from the Israel Mathematical Olympiads that normally should have appeared in the October 2002 number. Interested readers must have wondered where their solutions had gone astray. We start to set this straight by giving readers' solutions to problems 6 through 9 of the 20<sup>th</sup> Austrian-Polish Mathematical Competition [2000 : 197–199].

**6.** Prove that there does not exist a function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  such that  $f(x + f(y)) = f(x) - y$  for all integers  $x$  and  $y$ .

*Solved by Michel Bataille, Rouen, France; and Pierre Bornshtein, Pontoise, France. We give Bataille's write-up.*

Suppose that such a function exists. We denote by  $(R)$  the identity  $f(x + f(y)) = f(x) - y$ . Let  $a = f(0)$ . Suitable choices for  $x$  and  $y$  in  $(R)$  yield successively:

$$\begin{aligned} f(a) &= a \quad [x = y = 0] \\ f(2a) &= 0 \quad [x = y = a] \\ a &= 0 \quad [x = 0, y = 2a]; \quad \text{that is, } f(0) = 0. \end{aligned}$$

Taking  $x = 0$  in  $(R)$ , we obtain the identity

$$f(f(y)) = -y, \tag{R'}$$

valid for all integers  $y$ . From this,  $-f(y) = f(f(f(y))) = f(-y)$ , so that  $f$  is an odd function.

With the help of  $(R)$  and  $(R')$ , it follows that

$$f(u + v) = f(u + f(-f(v))) = f(u) + f(v)$$

for all integers  $u$  and  $v$  and, using induction, that  $f(m) = mf(1)$  for all integers  $m$ . Taking  $m = f(1)$ , we get  $f(f(1)) = (f(1))^2$  while  $(R')$  yields  $f(f(1)) = -1$ , a clear contradiction. The result follows.

**7.** (a) Prove that for all real numbers  $p$  and  $q$  the inequality  $p^2 + q^2 + 1 > p(q + 1)$  holds.

(b) Determine the greatest real number  $b$  such that for all real numbers  $p$  and  $q$  the inequality  $p^2 + q^2 + 1 > bp(q + 1)$  holds.

(c) Determine the greatest real number  $c$  such that for all integers  $p$  and  $q$  the inequality  $p^2 + q^2 + 1 > cp(q + 1)$  holds.

*Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztejn, Pontoise, France; and Murray S. Klamkin, University of Alberta, Edmonton, AB. We give Aassila's write-up.*

[*Editor's Note:* Unless we replace the strict inequality  $>$  by  $\geq$ , we cannot find a *greatest* real number for (b) or a *greatest* integer for (c).]

(a)–(b) It is easy to see that we can suppose  $p, q \geq 0$ . We have

$$p^2 + q^2 + 1 \geq p^2 + 2\left(\frac{q+1}{2}\right)^2 \geq 2\sqrt{\frac{p^2(q+1)^2}{2}} = \sqrt{2}p(q+1);$$

equality holds for  $p = \sqrt{2}, q = 1$ . Hence  $b = \sqrt{2}$ .

(c) If  $p = q = 1$ , we have  $1 + 1 + 1 = \frac{3}{2}1(1 + 1)$ , whence  $\frac{3}{2}$  is an upper bound for the maximum. In fact, it is the maximum, since:

- if  $q = 0$ , then  $p^2 + 1 > \frac{3}{2}p$ , true.
- if  $q = 1$ , then  $p^2 + 2 \geq 3p$ ; that is,  $(p - 1)(p - 2) \geq 0$ , true.
- if  $q = 2$ , then  $p^2 + 5 \geq \frac{9}{2}p$ ; that is,  $(p - 2)(p - \frac{5}{2}) \geq 0$ , true.
- if  $q \geq 3$ , then  $p(3 - q) - 2 < 0 \leq 2(p - q)^2$ ; hence

$$p^2 + q^2 + 1 > \frac{3}{2}p(q + 1).$$

**8.** Let  $n$  be a natural number and let  $M$  be a set with  $n$  elements. Find the biggest integer  $k$  with the property: there exists a  $k$ -element family  $K$  of three-element subsets of  $M$  such that any two sets from  $K$  are non-disjoint.

*Comment by Pierre Bornsztejn, Pontoise, France.*

This is a special case ( $p = 3$ ) of a well-known theorem of Erdős, Ko and Rado: "The largest size of an intersecting  $p$ -family in an  $n$ -set is  $\binom{n-1}{p-1}$ ."

**Reference:**

[1] M. Aigner, G.M. Ziegler, *Proofs from The Book*, Springer, p. 136–137.

**9.** Let  $P$  be a parallelepiped, let  $V$  be its volume,  $S$  its surface area, and  $L$  the sum of the lengths of the edges of  $P$ . For  $t \geq 0$  let  $P_t$  be the solid

consisting of points having distance from  $P$  not greater than  $t$ . Prove that the volume of  $P_t$  is equal to

$$V + St + \frac{\pi}{4}Lt^2 + \frac{4}{3}\pi t^3.$$

*Comment by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

It is a known result [1] that the volume  $V(P_t)$  of a parallel body of a convex polyhedron  $P$  at a distance  $t$  from  $P$  is the sum of

(a)  $V(P)$ ,

(b) the volumes of right prisms of height  $t$  whose bases are the faces of  $P$ , altogether a volume  $St$ , where  $S$  is the total surface area of  $P$ ,

(c) the volumes of the cylindrical segments whose heights are the lengths  $e_i$  of the edges and whose bases are circular sectors of radius  $t$  and centre angles  $\alpha_i$  equal to the angles between the normals to the faces intersecting in  $e_i$ , and

(d) the volumes of the spherical sectors at the vertices of  $P$ , which altogether equal the volume of one spherical ball of radius  $t$ .

Thus,

$$V(P_t) = V(P) + St + \left( \sum \frac{e_i \alpha_i}{2} \right) t^2 + \frac{4\pi t^3}{3}.$$

For the special case where  $P$  is a rectangular parallelepiped,  $\alpha_i = \frac{\pi}{2}$ , and the given result follows. The result for general parallelepipeds is not valid.

*Reference:*

[1] H. Guggenheimer, *Applicable Geometry*, R. Krieger Publishing Co., N.Y., 1977, p. 116.

Next we give solutions from the readers to Selected Problems from Israel Mathematical Olympiads, given [2000 : 199].

**1.** Prove that there are at most 3 primes between 10 and  $10^{10}$  all of whose digits in base ten are 1 (for example, 11).

*Solved by Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up and comment.*

In number theory, a positive integer each digit of which in base 10 is 1 is called a *repunit*. We show that, in fact, 11 is the *only* repunit between 10 and  $10^{10}$  which is also a prime.

Let  $n = 111 \dots 11$  be a repunit with  $k$  digits where  $2 \leq k \leq 10$ .

• If  $k = 3, 6,$  or  $9,$  then clearly  $n > 3$  and is divisible by  $3.$  Hence it is not a prime.

• If  $k = 4, 8,$  or  $10,$  then by a well-known test for divisibility,  $n$  is divisible by  $11.$  Since  $n > 11,$  it is not a prime.

• If  $k = 5,$  then  $n = 11111 = 41 \times 271,$  which is composite.

• If  $k = 7,$  then  $n = 1111111 = 239 \times 4649,$  which is composite.

Finally, since  $n = 11$  is a prime, our proof is complete.

*Comment.* This problem can be found as Exercise Number 14 on page 164 of the book *Elementary Number Theory and its Applications* by Kenneth H. Rosen, 3rd edition.

**2.** Is there a planar polygon whose vertices have integer coordinates, whose area is  $\frac{1}{2},$  such that this polygon is

(a) a triangle with at least two sides longer than 1000?

(b) a triangle whose sides are all longer than 1000?

(c) a quadrangle?

*Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB.*

(a) Yes. Just consider the one with vertices  $(0, 0), (1, 0),$  and  $(1001, 1).$

(b) Yes. Consider a triangle with integer coordinates  $(0, 0), (a, b),$  and  $(c, d)$  where the three sides are  $> 1000.$  The area is given by  $|ad - bc|/2.$  Thus, we need  $ad - bc = \pm 1.$  For instance, let  $(a, b) = (1000, 999)$  and  $(c, d) = (10001, 9991).$  We can even make all the sides arbitrarily large. We first would choose  $a$  and  $b$  to be arbitrarily large and relatively prime. Then we can find arbitrarily large  $c$  and  $d$  satisfying  $|ad - bc| = 1.$  For example, for the choice of  $(a, b)$  above,  $(c, d) = (1000m + 1, 999m + 1)$  for large  $m.$

(c) No. By Pick's Theorem, the area of such a quadrangle is  $I + \frac{B}{2} - 1$  where  $I$  is the number of interior lattice points and  $B$  is the number of boundary lattice points. Since  $B$  is at least 4, the area of any quadrangle is at least 1.

**3.** Find all real solutions of

$$\sqrt[4]{13+x} + \sqrt[4]{4-x} = 3.$$

*Solved by Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give Wang's write-up.*

Let  $u = \sqrt[4]{13+x}$  and  $v = \sqrt[4]{4-x}.$  Then  $u + v = 3$  and  $u^4 + v^4 = 17.$  Thus,

$$17 + 2u^2v^2 = (u^2 + v^2)^2 = ((u + v)^2 - 2uv)^2 = (9 - 2uv)^2.$$

Simplifying, we get  $u^2v^2 - 18uv + 32 = 0,$  whence  $(uv - 2)(uv - 16) = 0.$



If  $uv = 2$ , then  $(13 + x)(4 - x) = 16$  yields  $x^2 + 9x - 36 = 0$  or  $(x - 3)(x + 12) = 0$ . Hence  $x = 3$  or  $-12$ .

If  $uv = 16$ , then  $(13 + x)(4 - x) = 16^4$  yields  $x^2 + 9x + 65484 = 0$  which clearly has no real solutions.

Since it is easy to see that both  $x = 3$  and  $x = -12$  satisfy the given equation, we conclude that the only real solutions are  $x = 3$  and  $x = -12$ .

**4.** Prove that if two altitudes of a tetrahedron intersect, then so do the other two altitudes.

*Solved by Michel Bataille, Rouen, France; and Murray S. Klamkin, University of Alberta, Edmonton, Alberta. First we give Bataille's write-up and comment.*

Suppose that the altitudes  $AH$  and  $BK$  of tetrahedron  $ABCD$  concur, say at  $U$  (with  $H$  and  $K$  in the planes  $(BCD)$  and  $(ACD)$ , respectively). Since  $CD$  is orthogonal to  $AH$  and  $BK$ ,  $CD$  is orthogonal to the plane  $(ABU)$ , and hence to  $AB$ .

Conversely, suppose  $AB \perp CD$  and let  $B'$  be the foot of the altitude from  $B$  in  $\triangle BCD$ . Then  $CD$  is orthogonal to the plane  $(ABB')$ , hence to  $AB'$ , so that  $B'$  is also the foot of the altitude from  $A$  in  $\triangle ACD$ . Thus,  $AH$  and  $BK$  are both contained in the plane  $(ABB')$  and, as such, are concurrent (obviously they cannot be parallel).

It follows that the concurrency of  $AH$  and  $BK$  is *equivalent* to the condition  $AB \perp CD$ , and therefore also to the concurrency of the altitudes issued from  $C$  and  $D$ . This completes the proof.

*Comment.* With the hypotheses above,  $UB'$  is the third altitude in  $\triangle ABB'$  (the first two being  $AH$  and  $BK$ ) so that  $UB' \perp AB$  and  $UB' \perp CD$ . Reasoning similarly with the intersection  $V$  of the altitudes from  $C$  and  $D$ , we may conclude that, when  $V \neq U$ , the line  $UV$  is the common perpendicular to the orthogonal lines  $AB$  and  $CD$ .

*Next we give Klamkin's solution.*

This is a known result and is included in the following results from [1].

**203 Theorem.** If a pair of opposite edges of a tetrahedron are orthogonal, the two altitudes of the tetrahedron issued from the ends of each of these two edges are coplanar.

*Proof.* If the two opposite edges  $BC$ ,  $AD$  of the tetrahedron  $ABCD$  are orthogonal, it is possible to draw through  $BC$  a plane  $BMC$  orthogonal to  $AD$  at  $M$ . This plane is orthogonal to the two planes  $ADB$ ,  $ADC$  passing through  $AD$ . Hence, the altitude from  $C$  to  $ABD$  and the altitude from  $B$  to  $ADC$  both lie in the plane  $BCM$ . Similarly, for the altitudes issued from the ends  $A$ ,  $D$  of  $AD$ .

**204 Converse Theorem.** If two altitudes of a tetrahedron are coplanar, the edge joining the two vertices from which these altitudes issue is orthogonal

to the opposite edge of the tetrahedron.

*Proof.* If the altitudes issued from  $B$  and  $C$  meet in a point  $H$ , the plane  $BHC$  is orthogonal to the planes  $ADC$ ,  $ADB$  and, therefore, also to their intersection  $AD$ .

**205 Corollary I.** If two altitudes of a tetrahedron intersect, the remaining two intersect also.

*Comment.* We leave it as a related exercise to show that if one altitude intersects two other altitudes, then the four altitudes are concurrent.

*Reference:*

[1] N. Altshiller-Court, *Modern Pure Solid Geometry*, Macmillan, N.Y., 1935, pp. 61–62.

Now we move to readers' solutions to problems from the Estonian Mathematical Olympiads 1996–1997, Final Round of the National Olympiad [2000 : 452–453].

**1.** On a square table of size  $3n \times 3n$  each unit square is coloured either red or blue. Each red square not lying on the edge of the table has exactly five blue squares among its eight neighbours. Each blue square not lying on the edge of the table has exactly four red squares among its eight neighbours. Find the number of red and blue squares on the table.

*Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztejn, Pontoise, France; and Tobias Reuter, student, Wilnsdorf, Germany. We give Bornsztejn's solution.*

We will show that there are exactly  $5n^2$  blue squares and  $4n^2$  red squares. Divide the square table of size  $3n \times 3n$  into  $n^2$  pairwise disjoint square tables of size  $3 \times 3$ . For each of these squares:

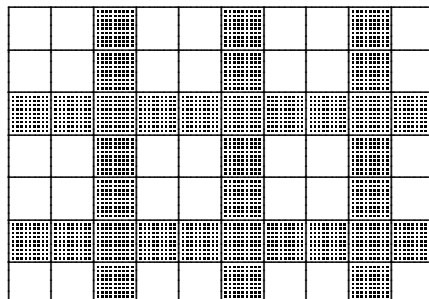
- If the unit square at the centre is red, then there are exactly 5 blue unit squares on the edge and, thus, exactly 3 red unit squares on the edge.
- If the unit square at the centre is blue, then there are exactly 4 red unit squares on the edge and, thus, exactly 4 blue unit squares on the edge.

Then, in each case, there are exactly 5 blue unit squares and 4 red unit squares in a square table of size  $3 \times 3$ . Thus, there are  $5n^2$  blue squares and  $4n^2$  red squares in the whole  $3n \times 3n$  square table, as claimed.

It remains to show that such a colouring is possible. It suffices to use  $n^2$  square tables of size  $3 \times 3$  of the type:

R	R	B
R	R	B
B	B	B

to obtain



## 2. Find

(a) all quadruples of positive integers  $(a, k, l, m)$  for which the equality  $a^k = a^l + a^m$  holds;

(b) all 5-tuples of positive integers  $(a, k, l, m, n)$  for which the equality  $a^k = a^l + a^m + a^n$  holds.

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Tobias Reuter, student, Wilnsdorf, Germany. We give the solution of Maragoudakis.*

(a) Obviously  $a \geq 2$  and  $k \geq l, m$ . If  $l \neq m$ , for example,  $l > m$ , then  $a^{k-m} = a^{l-m} + 1$  with  $k - m, l - m \geq 1$ . Therefore,  $a$  divides  $a^{k-m} - a^{l-m} = 1$ , a contradiction.

Thus,  $l = m$  and  $a^k = 2 \cdot a^m \iff a^{k-m} = 2 \iff a = 2, k - m = 1$ . All quadruples are  $(2, m + 1, m, m)$ ,  $m = 1, 2, 3, \dots$

(b) Obviously  $a \geq 2, k \geq l, m, n$ . Without loss of generality, we suppose that  $l \geq m \geq n$ . As above, we see that  $m = n$ , so the equation becomes  $a^{k-m} = a^{l-m} + 2$ .

- If  $l = m$ , then  $a^{k-m} = 3$ , then  $a = 3, k - m = 1$ .
- If  $l > m$ , then  $a|2$ . Thus,  $a = 2$  and  $2^{k-m} = 2^{l-m} + 2$ . By (a),  $k - m = 2, l - m = 1$ .

All 5-tuples are  $(3, m + 1, m, m, m)$ ,  $(2, m + 2, m + 1, m, m)$ , where  $m = 1, 2, 3, \dots$ , with appropriate permutations.

**3.** Prove that, for any real numbers  $x$  and  $y$ , the following inequality holds:

$$x^2 + y^2 + 1 > x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Tobias Reuter, student, Wilnsdorf, Germany; D.J. Smeenk, Zaltbommel, the*

Netherlands; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. Many found similarly nice solutions. We give Aassila's write-up as an example.

**First Solution.** We have  $(x - \sqrt{y^2 + 1})^2 \geq 0$ ,  $(y - \sqrt{x^2 + 1})^2 \geq 0$ . Hence,

$$(x - \sqrt{y^2 + 1})^2 + (y - \sqrt{x^2 + 1})^2 \geq 0.$$

That is,

$$x^2 + y^2 + 1 \geq x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

The equality holds if and only if  $x^2 = y^2 + 1$  and  $y^2 = x^2 + 1$ , a system which has no solutions. Consequently,

$$x^2 + y^2 + 1 > x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

**Second Solution.** By the AM-GM inequality we have

$$\frac{x^2 + (y^2 + 1)}{2} \geq \sqrt{x^2(y^2 + 1)} \geq x\sqrt{y^2 + 1}$$

$$\frac{(x^2 + 1) + y^2}{2} \geq \sqrt{y^2(x^2 + 1)} \geq y\sqrt{x^2 + 1}.$$

Hence

$$x^2 + y^2 + 1 \geq x\sqrt{y^2 + 1} + y\sqrt{x^2 + 1}.$$

The rest of the proof is the same as the first solution.

*We also give Klamkin's generalization.*

We show more generally that for all real  $x, y, z$ ,

$$x^2 + y^2 + z^2 > x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}.$$

By Cauchy's Inequality,

$$(x^2 + y^2) [(y^2 + z^2) + (x^2 + z^2)] \geq [x\sqrt{y^2 + z^2} + y\sqrt{x^2 + z^2}]^2.$$

Hence, it suffices to prove the stronger inequality

$$(x^2 + y^2 + z^2)^2 \geq (x^2 + y^2) [(y^2 + z^2) + (x^2 + z^2)],$$

which, on expanding out, reduces to  $z^4 \geq 0$ .

*Comment.* It also follows in a similar fashion that

$$\begin{aligned} & \sqrt{2}(x^2 + y^2 + z^2 + w^2) \\ & \geq x\sqrt{y^2 + z^2 + w^2} + y\sqrt{z^2 + x^2 + w^2} + z\sqrt{x^2 + y^2 + w^2}. \end{aligned}$$

Here, using Cauchy's Inequality as before, we have the stronger inequality

$$2(x^2 + y^2 + z^2 + w^2)^2 \geq (x^2 + y^2 + z^2)[(y^2 + z^2 + w^2) + (z^2 + x^2 + w^2) + (x^2 + y^2 + w^2)],$$

with equality if and only if  $w = 0$ .

**4.** In a triangle  $ABC$  the values of  $\tan \angle A$ ,  $\tan \angle B$  and  $\tan \angle C$  relate to each other as  $1 : 2 : 3$ . Find the ratio of the lengths of the sides  $AC$  and  $AB$ .

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztejn, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; and Panos E. Tsaoussoglou, Athens, Greece. We give Bataille's solution.*

From  $\tan B = 2 \tan A$ ,  $\tan C = 3 \tan A$  and the well-known relation  $\tan A + \tan B + \tan C = \tan A \tan B \tan C$ , we obtain  $6 \tan A = 6(\tan A)^3$ .

Certainly  $\tan A \neq 0$ ; hence,  $\tan A = 1$  or  $\tan A = -1$ . But the latter cannot hold since otherwise all the angles of  $\triangle ABC$  would be obtuse. Thus,  $\tan A = 1$  and  $A = 45^\circ$ .

Now, let  $H$  be the foot of the altitude from  $C$ . We have

$$\frac{CH}{AH} = \tan A = 1, \quad \frac{CH}{BH} = \tan B = 2,$$

and  $H \in [AB]$  (since  $A, B$  are acute). It follows that

$$AB = AH + HB = \frac{3}{2}CH.$$

Since we clearly have  $AC = CH\sqrt{2}$ , we finally get  $\frac{AC}{AB} = \frac{2\sqrt{2}}{3}$ .

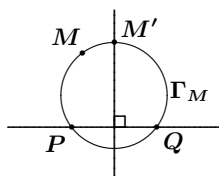
**5.** There are  $n$  points ( $n \geq 3$ ) in the plane, no three of which are collinear. Is it always possible to draw a circle through three of these points so that it has no other given points

(a) in its interior?

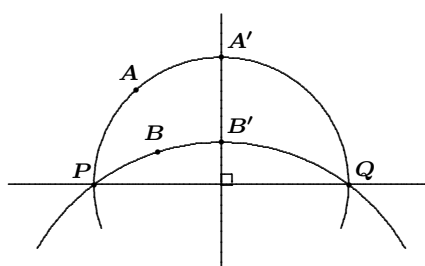
(b) in its interior nor on the circle?

*Solved by Mohammed Aassila, Strasbourg, France; Pierre Bornsztejn, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; and Tobias Reuter, student, Wilnsdorf, Germany. We give Bornsztejn's solution.*

(a) Yes. Let  $E$  be a set of  $n \geq 3$  points in the plane, no three of which are collinear. Let  $P, Q$  be two adjacent vertices of the convex hull of  $E$ . We then have  $P, Q \in E$ . For each point  $M \in E - \{P, Q\}$ , let  $\Gamma_M$  denote the circle through  $M, P, Q$ ; and let  $M'$  denote the intersection of  $\Gamma_M$  and the perpendicular bisector of  $[PQ]$  which is on the same side of  $(PQ)$  as  $M$ .



Since three points in  $E$  are never collinear, and from the choice of  $P$  and  $Q$ , all the points in  $E - \{P, Q\}$  are on the same side of  $(PQ)$ . Since  $E$  is finite, we may consider a point  $A \in E$  such that the distance from  $A'$  to  $(PQ)$  is minimal.



Suppose that  $B \in E$  is interior to  $\Gamma_A$ . Then the arc of  $\Gamma_B$  whose endpoints are  $P$  and  $Q$ , and which contains  $B$  (and  $B'$ ), is in the interior of  $\Gamma_A$  ( $P$  and  $Q$  excepted). Thus,

$$0 < d(B', (PQ)) < d(A', (PQ)),$$

which contradicts the minimality of  $A'$ . Then  $\Gamma_A$  has no other point of  $E$  in its interior, and we are done.

(b) For  $n = 3$ , the answer is obviously yes. For  $n \geq 4$ , the answer is no. It suffices to choose the  $n$  points on a given circle.

**6.** For positive integers  $m, n$  denote  $T(m, n) = \gcd\left(m, \frac{n}{\gcd(m, n)}\right)$ .

(a) Prove that there exist infinitely many pairs of integers  $(m, n)$  such that  $T(m, n) > 1$  and  $T(n, m) > 1$ .

(b) Does there exist a pair of integers  $(m, n)$  such that  $T(m, n) = T(n, m) > 1$ ?

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Pierre Bornsztein, Pontoise, France; Pavlos Maragoudakis, Pireas, Greece; Tobias Reuter, student, Wilnsdorf, Germany; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We give the solution by Reuter.*

In what follows we simply denote  $\gcd(a, b)$  by  $(a, b)$ .

(a) Let  $(m, n) = d$ ,  $m = dm_1$ ,  $n = dn_1$ ,  $(m_1, n_1) = 1$ . Now we have

$$T(m, n) = (dm_1, n_1) > 1 \quad \text{if and only if} \quad (d, n_1) > 1.$$

Similarly, we obtain

$$T(n, m) = (dn_1, m_1) > 1 \quad \text{if and only if } (d, m_1) > 1.$$

It is clear that there are infinitely many pairs  $(m, n)$  which satisfy these two conditions: for example, if we set  $d = 2^p 3^q$ ,  $m_1 = 2$  and  $n_1 = 3$ , we obtain infinitely many such pairs, since there are infinitely many pairs  $(p, q)$  with natural numbers  $p, q$ .

(b) If we have the required relation, then using the notation from (a) we must have

$$(dn_1, m_1) = (dm_1, n_1) > 1 \iff (d, m_1) = (d, n_1) = d_1.$$

It follows that  $d_1 = 1$ , since otherwise we would have  $(n_1, m_1) > 1$ . Hence, there are no integers which satisfy the conditions.

**7.** A function  $f$  satisfies the condition

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n)$$

for any positive integer  $n$ . Given that  $f(1) = 999$ , find  $f(1997)$ .

*Solved by Mohammed Aassila, Strasbourg, France; Michel Bataille, Rouen, France; Robert Bilinski, Outremont, QC; Pierre Bornsztejn, Pontoise, France; Murray S. Klamkin, University of Alberta, Edmonton, AB; Pavlos Maragoudakis, Pireas, Greece; Tobias Reuter, student, Wilnsdorf, Germany; and Edward T.H. Wang, Wilfrid Laurier University, Waterloo, ON. We use Bilinski's solution.*

We have:

$$\begin{aligned} f(1) + \cdots + f(n) &= n^2 f(n), \\ f(1) + \cdots + f(n-1) &= (n-1)^2 f(n-1). \end{aligned}$$

Subtracting and isolating  $f(n)$ , we get (for  $n \geq 2$ )

$$f(n) = \frac{(n-1)^2}{n^2-1} f(n-1) = \frac{n-1}{n+1} f(n-1).$$

By telescoping this formula, we obtain

$$f(n) = 2 \frac{(n-1)!}{(n+1)!} f(1) = \frac{2f(1)}{(n+1)n}.$$

Since  $f(1) = 999$ , we have

$$f(1997) = \frac{2 \cdot 999}{(1998)(1997)} = \frac{1}{1997}.$$

That completes this number of the *Corner*. As we enter Olympiad season, remember to send me contests and of course, your nice solutions and generalizations.