MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 975 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7**. The electronic address is

`mayhem-editors@cms.math.ca`

The Mayhem Editor is Shawn Godin (Cairine Wilson Secondary School). The Assistant Mayhem Editor is John Grant McLoughlin (University of New Brunswick). The other staff members are Paul Ottaway (Dalhousie University) and Larry Rice (University of Waterloo).

Mayhem Editorial

Shawn Godin

Welcome to another year with MAYHEM! There are a few things that we should mention at the start of the year.

First, it is with sadness that I bid farewell to CHRIS CAPPADOCIA, my assistant editor for the last two years. Chris has been invaluable to me as I took over the helm from NAOKI SATO. Unfortunately, the demands of school (Chris is an undergraduate at the University of Waterloo) are becoming too great for him to keep helping me. I wish Chris all the best. He is an exceptional young man with great potential, and it has been my pleasure to have him as a student, assistant and friend. We will miss you, Chris.

Next, we have two welcomes. First to my new Editor-in-Chief, JIM TOTTEN. Jim has been involved with Crux longer than I have, and checking my back issues, I see that he was involved every year that I have been a subscriber (this will be my 10^{th} year). I first met Jim yesterday (as it would be), and I feel confident that ***CRUX with MAYHEM*** is in good hands. I look forward to working with Jim over the next x years. By the way, Bruce always allowed me to hand in my material several days late . . .

One more welcome, to another name familiar to Crux: JOHN GRANT McLOUGHLIN has agreed to take over from Chris as my assistant editor. John will take over the problem proposals, and we will be searching for some new problem editors to help with selecting solutions. John has been involved with many activities that will be valuable to us and to you as a reader. He has taught at a number of faculties of education, written a problem column in the *Ontario Mathematics Gazette* a few years ago, is working with the Canadian Mathematics Competitions and other local mathematics competitions as a problem setter, and has numerous other related experiences. I feel John will make a very positive contribution to Mayhem, and I think the two of us will make a good team.

A few minor changes are being made. In SKOLIAD, we will now ask for solutions to only some of the problems. You may send solutions to the other problems, but we will print a solution only if we receive one from a "junior" student (grade 10 and equivalent or earlier), a unique solution, or a nice generalization. We will continue to give out past copies of MAYHEM to some pre-university solvers.

The MAYHEM TAUNT is officially over. A couple of solutions to these problems appear in this issue. Over the rest of the volume the other TAUNT solutions will appear. Also, starting in the next issue, we will publish the names of the winners of the prizes for each section.

We are hoping to continue giving some sort of prizes to our pre-university solvers, but we are still brainstorming as to what form they will take. Details will follow in later issues. Keep sending us your solutions.

I hope you enjoy your 2003 volume of Mayhem. Keep sending us your problems, solutions, comments and articles. We are *your* journal, and its success is a function of the involvement of you, the readers.

Je vous souhaite la bienvenue en cette nouvelle année du MAYHEM ! Je profite de l'occasion pour vous transmettre quelques bribes d'information.

En premier lieu, c'est avec tristesse que je fais mes adieux à CHRIS CAPPADOCIA, mon rédacteur adjoint des deux dernières années. Chris m'a été d'un grand soutien à partir du moment où j'ai pris la relève de NAOKI SATO. Malheureusement, une charge d'étude devenue trop lourde (Chris est un étudiant de premier cycle à l'Université de Waterloo) l'empêche de continuer à m'aider. Je lui souhaite une bonne continuation. Chris est un jeune homme exceptionnel qui a beaucoup de potentiel, et je suis enchanté de l'avoir eu comme étudiant, comme assistant et comme ami. Tu nous manqueras !

En second lieu, c'est avec joie que nous accueillons deux nouvelles recrues. D'abord notre nouveau rédacteur en chef, JIM TOTTEN. Jim participe aux activités du Crux depuis plus longtemps que moi, et il m'a suffi de consulter mes anciens numéros pour constater qu'il a contribué à la revue au moins une fois par année depuis que je suis abonné (et j'entreprends ma 10^e année). J'ai rencontré Jim pour la première fois tout récemment, et je sais que le ***CRUX with MAYHEM*** est entre bonnes mains avec lui. Je serai heureux de travailler avec Jim au cours des prochaines années. À propos, Bruce me permettait toujours de remettre mes textes avec plusieurs jours de retard...

Nous souhaitons également la bienvenue à une autre personne qui n'est pas étrangère au Crux. JOHN GRANT McLOUGHLIN a accepté de remplacer Chris comme rédacteur adjoint du Mayhem. John reprendra le dossier des propositions de problèmes, et nous chercherons de nouveaux rédacteurs de problèmes qui nous aideront à choisir des solutions. La feuille de route de John sera un atout pour notre équipe, ainsi que pour vous, en tant que lecteurs. John a enseigné dans plusieurs facultés d'éducation, il a signé une chronique de résolution de problèmes dans l'*Ontario Mathematics Gazette* il y a quelques années, il participe à la rédaction des problèmes du Concours canadien de mathématiques et d'autres concours locaux, et il a bien d'autres cordes à son arc. Je crois que John sera un excellent atout pour le Mayhem et que nous formerons une bonne équipe.

Signalons également quelques changements. Dans la chronique « SKOLIAD », nous ne solliciterons plus des solutions à tous les problèmes. Vous pourrez tout de même résoudre les autres problémes, mais nous ne publierons que les solutions venant d'un « jeune » élève (équivalent de la 10^{e} année ou moins), les solutions uniques ou les généralisations élégantes. Nous conserverons notre habitude de remettre d'anciens numéros du MAYHEM aux élèves de niveau préuniversitaire qui nous font parvenir de bonnes solutions.

C'est maintenant officiel, nous ne relancerons plus le DÉFI MAYHEM. Nous publions quelques solutions ce mois-ci, et d'autres paraîtront au cours de l'année. À partir du prochain numéro, nous publierons le nom des gagnants dans chaque section.

Nous espérons être en mesure de continuer à offrir des prix à nos élèves de niveau préuniversitaire, mais nous n'avons pas encore déterminé quelle en sera la nature. Nous vous tiendrons au courant dans les mois à venir. Entre-temps, continuez à nous envoyer vos solutions !

J'espère que vous apprécierez le volume 2003 du Mayhem. Nous attendons vos problèmes, solutions, commentaires et articles avec impatience. Le Mayhem est **votre** revue, et une partie de son succès dépend de **votre** participation.

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, 2191 Saturn Crescent, Orleans, Ontario, K4A 3T6** or emailed to

`mayhem-editors@cms.math.ca`

Please include in all correspondence your name, school, grade, city, province or state, and country. We are especially looking for solutions from high school students.

Please send your solutions to the problems in this edition by *1 August 2003*. Solutions received after this time will be considered only if there is time before publication of the solutions.

M76. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Two buildings A and B are twenty feet apart. A ladder thirty feet long has its lower end at the base of building A and its upper end against building B. Another ladder forty feet long has its lower end at the base of building B and its upper end against building A.

How high above the ground is the point where the ladders intersect?

. . . .

Deux bâtiments A et B sont distants de vingt pieds. Une échelle, longue de trente pieds, a son extrémité inférieure à la base de A et est appuyée contre le bâtiment B. Une autre échelle, de longueur de quarante pieds celle-là, a son extrémité inférieure à la base de B et est appuyée contre le bâtiment A.

À quelle hauteur au-dessus du sol se trouve le point d'intersection des deux échelles ?

M77. *Proposed by Richard Hoshino, Dalhousie University, Halifax, NS.*

Find *all* ordered pairs of integers (a, b) such that the equation $x^2 + |y^2 - 6ay + b| = b - a^2 + 6$ has *exactly* 2001 solutions in positive integers (x, y).

. . . .

Trouver *toutes* les paires ordonnées d'entiers (a, b) telles que l' équation $x^2 + |y^2 - 6ay + b| = b - a^2 + 6$ possède *exactement* 2001 solutions en entiers positifs (x, y).

M78. *Proposed by K.R.S. Sastry, Bangalore, India.*

In a right-angled triangle we consider the two vertices at the two acute angles and draw medians from them to the opposite sides. Determine the maximum (acute) angle between these medians.

. . . .

Dans un triangle rectangle, on considère les deux sommets d'angles aigus, et les médianes abaissées sur le côté opposé. Déterminer l'angle (aigu) maximal entre ces médianes.

M79. *Proposed by the Mayhem Staff.*

Three people play the following game. N marbles are placed in a bowl and the players, in turn, remove 1, 2, or 3 marbles from the bowl. The person who removes the last marble loses. For what values of N can the first and third player work together to force the second player to lose? (Inspired by a recent problem on the Canadian Open Mathematics Challenge.)

. . . .

Trois personnes jouent le jeu suivant. A tour de rôle, chaque joueur retire 1, 2, ou 3 billes d'une urne qui en contient N. La personne qui retire la dernière bille a perdu. Pour quelles valeurs de N le premier et le troisième joueur peuvent-ils collaborer pour forcer le second joueur à perdre ? (Inspiré par un récent problème du Défi Ouvert Canadien de Mathématiques.)

M80. *Proposed by J. Walter Lynch, Athens, GA, USA.*

Compute the number of ways that 4 tires can be rotated so that each tire is relocated. (*Editor's note:* "rotating" a car's tires means changing their position on the car so that they can wear more evenly.)

. . . .

Trouver le nombre de rotations qu'on peut effectuer sur les 4 pneus d'une voiture pour qu'ils se trouvent dans une autre position. (*Note de l'éditeur :* "rotation" signifie ici : changement de position pour assurer une usure uniforme des pneus.)

M81. *Proposed by K.R.S. Sastry, Bangalore, India.*

Let $a \neq 0$, b, c be integers and $\sin\theta$, $\cos\theta$ be the rational roots of the equation $ax^2 + bx + c = 0$. Show that $a \pm 2c$ are perfect squares.

. . . .

Soit $a \neq 0$, b, c des entiers et $\sin\theta$, $\cos\theta$ les racines rationnelles de l'équation $ax^2 + bx + c = 0$. Montrer que $a \pm 2c$ sont des carrés parfaits.

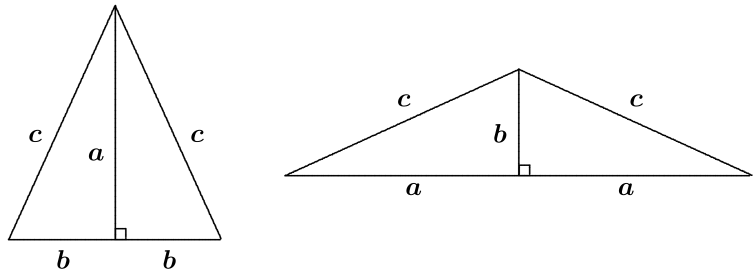
Mayhem Solutions

M26. *Proposed by the Mayhem staff.*

Find two isosceles triangles, with two sides 106 units long and the other side an integer, that have the same area.

Solution by Geneviève Lalonde, Massey, ON.

Two different isosceles triangles can be created that have the same length for their double side and the same area by creating them from two scalene right-angled triangles as in the diagram below, where (a, b, c) is a Pythagorean triple.



For our problem we need a Pythagorean triple $(a, b, 106)$. If we can find two positive integers x and y with $x > y$, such that $x^2 + y^2 = 106$, then we can let $a = x^2 - y^2$ and $b = 2xy$. Since $106 = 9^2 + 5^2$, we have $a = 9^2 - 5^2 = 56$ and $b = 2 \cdot 9 \cdot 5 = 90$. Thus, the triangles with sides 106, 106, 112 and 106, 106, 180 both have area $ab = 5040$.

M27. *Proposed by the Mayhem staff.*

Find $\sqrt{ab+1}$ where $a = \overbrace{111\cdots11}^{2002\text{ 1's}}$ and $b = 1\overbrace{00\cdots00}^{2001\text{ 0's}}5$.

Solution by Mihály Bencze, Brasov, Romania.

In general, if $a = \overbrace{111\cdots11}^{n\text{ 1's}} = \dfrac{10^n - 1}{9}$ and $b = 1\overbrace{00\cdots00}^{n-1\text{ 0's}}5 = 10^n + 5$, then

$$\sqrt{ab+1} = \sqrt{\left(\frac{10^n+2}{3}\right)^2} = \frac{10^n+2}{3}.$$

Thus, for this problem the result is $\dfrac{10^{2002}+2}{3} = \overbrace{333\cdots3}^{2001\text{ 3's}}4$.

Also solved by Gustavo Krimker, Universidad CAECE, Argentina. One incomplete solution was received.

M28. *Proposed by the Mayhem staff.*

Shawn tosses 2001 fair coins and Bruce tosses 2002 fair coins. What is the probability that Bruce gets more heads than Shawn?

Solution by Geneviève Lalonde, Massey, ON.

To start out, let Shawn toss his 2001 coins and Bruce toss 2001 of his. By symmetry $P(\text{Bruce more heads}) = P(\text{Shawn more heads}) = \frac{1-p}{2}$ where p is the probability that they have the same number of heads. If Bruce has more heads, he has already won and the last toss is immaterial. If they have the same number of heads, then Bruce can get more by flipping heads, but if Shawn already has more heads, Bruce cannot get more with his last toss. Thus, the probability that Bruce will have more heads after his 2002^{nd} toss is

$$\frac{1-p}{2} + \frac{1}{2}p = \frac{1}{2}.$$

Also solved by José L. Díaz–Barrero and Juan J. Egozcue, UPC, Barcelona, Spain.

M29. *Proposed by the Mayhem staff.*

Define the "silly product" of two numbers as the sum of the product of all the corresponding digits. So $235 \times_s 718 = 2 \times 7 + 3 \times 1 + 5 \times 8 = 57$. Find two numbers A and B so that $A \times_s B = 2002$ and $A + B$ is a minimum.

Solution by Antonio Lei, year 12, Colchester Royal Grammar School, Colchester, UK.

A and B should have the same number of digits. Otherwise, some digits would be multiplied by zero which has no contribution to the "silly product". It contradicts the condition that $A + B$ is minimum.

Let $A = a_n a_{n-1} \cdots a_1$, and $B = b_n b_{n-1} \cdots b_1$ where the a_i and b_i are the digits of A and B. Then

$$A \times_s B \;=\; a_n b_n + a_{n-1} b_{n-1} + \cdots + a_1 b_1 \;\leq\; 9 \times 9 + 9 \times 9 + \cdots + 9 \times 9\,,$$

whence $2002 \leq 81n$. Thus, $n \geq \frac{2002}{81}$, but since n is an integer we must have $n \geq 25$.

In order to keep $A + B$ minimum, we want the least number of digits. Therefore, $a_i b_i$ must be as great as possible. But $a_i b_i$ is at most 81 when $a_i = b_i = 9$, and $25 \times 81 = 2025 > 2002$. Hence, only the least significant 24 digits can be 9. Hence, there remains $2002 - 24 \times 81 = 58 = 2 \times 29$. Since 29 is prime, we cannot make 58 with the silly product of two one-digit numbers. Thus, we need two more digits for our number, say a_n, a_{n+1} for A and b_n, b_{n+1} for B. Hence, $a_{n+1} b_{n+1} + a_n b_n = 58$ and a_{n+1} and b_{n+1} should be kept as small as possible. The smallest occurs for $1 \times 2 + 7 \times 8 = 58$. Thus, the two possibilities for A and B that keep the sum $A + B$ a minimum are $1899 \ldots 9$, $2799 \ldots 9$ and $1799 \ldots 9$, $2899 \ldots 9$ (there are twenty-four 9's in each number).

Also solved by Robert Bilinski, Outremont, PQ; Jack Gu, grade 11, Rachel Li, grade 12, Alvin Miao, grade 10, Molly Yan, grade 11, and Corey Zhou, grade 12, Dalian Maple Leaf International School, Dalian, China. Three incorrect solutions were received.

M30. *Proposed by Haralampy Steryion, Chalkis, Greece.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property

$$f(x + y) \;=\; f(x) e^{f(y) - 1} \qquad \text{for every } x,\, y \in \mathbb{R}.$$

Solution by Shien Jin Ong, MIT, USA.

Claim: $f(x) = 1$ or $f(x) = 0$ for all $x \in \mathbb{R}$.

Note that the function $f(x) = 0$ for all $x \in \mathbb{R}$ is a solution. Now assume that $f(x) \neq 0$ for some $x \in \mathbb{R}$, say for $x = x_1$. Substitute $y = 0$ and $x = x_1$ into the equation. We conclude that $f(0) = 1$. Next, substitute $x = 0$ into the same equation. We get $f(y) = e^{f(y)-1}$ for all $y \in \mathbb{R}$. Let $g(t) = t - e^{t-1}$. Note that $g'(t) < 0$ if $t > 1$, $g'(t) > 0$ if $t < 1$, and $g(1) = 0$. Hence, $g(t)$ has only one root, at $t = 1$. This means that the only solution to $f(y) = e^{f(y)-1}$ is $f(y) = 1$ for all $y \in \mathbb{R}$. A simple check verifies that the solution indeed fits into the given equation.

Also solved by Jack Gu, grade 11, Rachel Li, grade 12, and Corey Zhou, grade 12, Dalian Maple Leaf International School, Dalian, China. One incorrect solution was received.

Pólya's Paragon

Paul Ottaway

In this installment of Pólya's Paragon, we will examine the *Extreme Principle*. Here is the basic premise:

If possible, assume that the elements of your problem are "in order". Focus on the "largest" and "smallest" elements, as they may be constrained in interesting ways.

Using this simple idea, we will be able to provide elegant solutions to difficult problems. Let's illustrate this with several examples.

Problem 1: On the plane, we colour a *finite* number of points either black or white. We choose the points and their colours so that every line segment which joins two points of the same colour contains a point of the other colour. Prove that all the points must lie on a single line segment.

Solution: Suppose that the points do not all lie on a single line segment. Then there must exist at least one set of three points that form a triangle (that is, these three points are not all on the same line). Of all such triangles that can be formed, consider the triangle ABC of *smallest* area. We will obtain a contradiction by finding a triangle whose area is smaller than the area of $\triangle ABC$.

Each of the points A, B, and C are coloured black or white. So at least two of the points must be coloured the same. Without loss of generality, suppose that B and C are both coloured white. Then, there must be a black point D somewhere between B and C. Then $\triangle ABD$ is a triangle whose area is strictly smaller than the area of $\triangle ABC$, which contradicts the fact that $\triangle ABC$ was the triangle of smallest area.

Since we have a contradiction, we conclude that all the points must lie on a single line segment.

Problem 2: Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Determine all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$xf(y) + yf(x) = (x + y)f(x^2 + y^2)$$

for all x and y in $\mathbb{N}$.

Solution: This problem appeared as the last question of the 2002 Canadian Mathematical Olympiad. The proposer's solution was purely number-theoretic and spanned several pages. It was fully expected that very few people would solve this problem, and that the only correct solutions would be similar to the highly technical approach obtained by the proposer.

To the CMO committee's surprise (and delight!), the following solution was submitted by David Han from Woburn Collegiate Institute, who was the winner of the 2002 CMO Contest.

We claim that f is a constant function. Suppose, for a contradiction, that there exist x and y with $f(x) < f(y)$. Choose x and y such that $f(y) - f(x) = d > 0$ is minimal. Then,

$$f(x) = \frac{xf(x) + yf(x)}{x+y} < \frac{xf(y) + yf(x)}{x+y} < \frac{xf(y) + yf(y)}{x+y} = f(y).$$

Letting $z = x^2 + y^2$, we have shown that $f(x) < f(z) < f(y)$. Hence, the integers x and z satisfy $0 < f(z) - f(x) < d$, which contradicts the minimality of d. Therefore, no such x and y exist, and so we conclude that f must be a constant function. [*Editor's Note:* The above argument requires $x \neq 0$ and $y \neq 0$. To complete the proof, we need only show that $f(0) = f(x)$ for some non-zero $x \in \mathbb{N}$. Now, select $y = 0$ and $x = 1$ in the given functional equation, and we are done.]

We quickly see that for all $c \in \mathbb{N}$, the function $f(x) = c$ satisfies the given functional equation. Therefore, we have solved the problem.

Problem 3: Prove that the equation $x^4 + y^4 = z^4$ has no solutions in positive integers x, y, z.

Solution: This is the most famous application of the Extreme Principle. This is the $n = 4$ case of Fermat's Last Theorem.

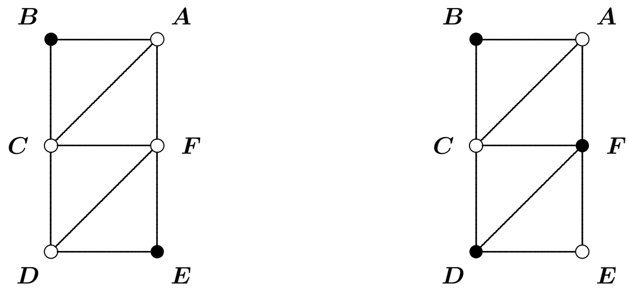
We outline the proof here, and we invite you to fill in the details!

1. Define (a, b, c) to be a *special triple* if a, b, c are positive integers for which $a^4 + b^4 = c^2$.
2. Suppose that a special triple (a, b, c) exists. Consider the *smallest* one; that is, one where c is minimized.
3. We have $a^4 + b^4 = c^2$, so (a^2, b^2, c) is a Pythagorean Triple. Explain why there must exist positive integers p and q for which $a^2 = p^2 - q^2$, $b^2 = 2pq$, and $c = p^2 + q^2$.
4. Show that there exist positive integers d and e for which $a = d^2 - e^2$, $q = 2de$, and $p = d^2 + e^2$.
5. Explain why d, e, and $d^2 + e^2$ are all perfect squares. Conclude that there must exist a special triple (r, s, t) with $t < c$, which gives you the desired contradiction.
6. Explain why no special triples (r, s, t) exist, and thus conclude that the equation $x^4 + y^4 = z^4$ has no solutions in positive integers x, y, z.

We conclude this article by providing some more questions where the Extreme Principle may be used.

1. Imagine an infinite chessboard that contains a positive integer in each square. If the value in each square is equal to the average of its four neighbours to the north, south, west, and east, prove that the values in all the squares are equal.

2. Consider a graph with finitely many points, some of which are joined to one another by lines. We shall colour each point either black or white, and call the graph "integrated" if each white point has at least as many black as white neighbours, and vice versa. The example below shows two different colourings of the same graph. The one on the left is not integrated, because point A has two white neighbours (C and F), and only one black neighbour (B). The graph on the right is integrated.



Given *any* graph, can we colour the points so that the graph is integrated?

3. Prove that the equation $x^2 + y^2 = 3z^2$ has no solutions in positive integers (x, y, z).

4. On a large flat field, n people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and, at a given signal, fires and hits the person who is closest. When n is odd, show that there is at least one person left dry. Is this always true when n is even?
(1987 CMO, Question 4.)

5. Consider finitely many points in the plane such that, if we choose any three points A, B, C among them, the area of triangle ABC is always less than 1. Prove that all of these points lie within the interior or on the boundary of a triangle with area less than 4.
(1995 Korean Mathematical Olympiad.)