MATHEMATICAL MAYHEM

Mathematical Mayhem began in 1988 as **a Mathematical Journal for and by High School and University Students**. It continues, with the same emphasis, as an integral part of ***Crux Mathematicorum with Mathematical Mayhem***.

All material intended for inclusion in this section should be sent to **Mathematical Mayhem, Cairine Wilson Secondary School, 977 Orleans Blvd., Gloucester, Ontario, Canada. K1C 2Z7 (NEW!)**. The electronic address is

`mayhem-editors@cms.math.ca`

The Assistant Mayhem Editor is Chris Cappadocia (University of Waterloo). The other staff member is Paul Ottaway (Dalhousie University).

Editorial

We have a change in the staff here at MAYHEM. Jimmy Chui, author of The Problem of The Month, has left us. Jimmy will be entering fourth year at the University of Toronto and wishes to have a bit more time to concentrate on his undergraduate thesis. We will all miss Jimmy here at MAYHEM, and appreciate the fine work he has done over the years. Best of luck Jimmy!

Joining us, and taking over the column Polya's Paragon, is Paul Ottaway. Paul is starting a Masters program at Dalhousie University in Halifax this fall. He has just completed concurrent B.Math. and B.Ed. degrees at the University of Waterloo and Queen's University, respectively. He comes to us with a lot of experience from participating in mathematics competitions since grade 7, up to working with the CMC on their annual summer problem seminars and working on contest problem setting committees. We welcome Paul to MAYHEM, and know that he is going to bring our readers some interesting material while he is here with us.

Mayhem Problems

Proposals and solutions may be sent to **Mathematical Mayhem, c/o Faculty of Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1**, or emailed to

`mayhem-editors@cms.math.ca`

Please include in all correspondence your name, school, grade, city, province or state and country. We are especially looking for solutions from high school students. Please send your solutions to the problems in this edition by *1 March 2003*. Solutions received after this time will be considered only if there is time before publication of the solutions.

To be eligible for this month's MAYHEM TAUNT, solutions must be postmarked *before* 1 January 2003.

NOTE: We will also accept entries to the MAYHEM TAUNT that have been handwritten, scanned, and emailed as long as they are legible.

M51. Proposed by the Mayhem Staff.

You have a deck with cards numbered 1 through 25. You perform the following operations on the deck:

- you place the top card on the bottom of the deck.
- you place the new top card on the bottom of the deck.
- you flip the new top card face up on the table.

You continue this process until all cards are face up on the table. Find the order of the cards in the deck if, when the process is performed, the cards get laid out on the table in the order 1, 2, 3, $\ldots$, 25.

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Sur une pile de cartes numérotées de 1 à 25, on effectue les opérations suivantes :

– on place la carte du dessus en-dessous de la pile.

– on place la nouvelle carte du dessus en-dessous de la pile.

– on retourne la nouvelle carte du dessus et on la pose sur la table.

On continue ainsi de suite jusqu'à ce que toutes les cartes soient retournées sur la table. Trouver l'ordre des cartes dans la pile si, une fois le processus terminé, les cartes qu'on a retournées sont dans l'ordre 1, 2, 3, $\ldots$, 25.

M52. Proposed by J. Walter Lynch, Athens, GA, USA.

You have two coins. One is a normal half dollar and the other is a fake half dollar with a head on both sides. You randomly toss one of the coins into a drawer and the other coin into another drawer. A man comes into the room and opens one of the drawers. He looks in and sees a head. Question: What is the probability that he is seeing the coin with two heads?

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On a deux pièces de monnaie. L'une est une pièce d'un dollar normale et l'autre une fausse pièce d'un dollar, avec deux faces. On jette au hasard chacune des pièces dans deux tiroirs différents. Quelqu'un entre dans la chambre et ouvre un des tiroirs et aperçoît une pièce, côté face. Quelle est la probabilité que cette pièce soit celle à deux faces ?

M53. Proposed by the Mayhem Staff.

A circular path is surrounded by 17 stepping stones numbered 0, 1, 2, $\dots$, 16. Sally starts on stone 0 and moves 1 step to stone 1, then 4 steps to stone 5, then 9 steps to step 14 and continues in the following pattern until at last she moves 2002^2 steps and stops (to rest). What stone is Sally standing on while she rests?

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Un sentier circulaire est entouré de 17 marches numérotées 0, 1, 2, $\dots$, 16. Sophie commence sur la marche 0 et fait 1 pas jusqu'à la marche 1, puis 4 pas jusqu'à la marche 5, puis 9 pas jusqu'à la marche 14 et ainsi de suite, jusqu'à ce qu'elle se déplace de 2002^2 pas et s'arrête (pour se reposer). Quel est le numéro de la marche sur laquelle Sophie se repose?

M54. Proposed by Gary Tupper, Pedagoguery Software Inc., Terrace, BC.

An ellipse with major axis AB and foci F and F' is inscribed in a circle with diameter AB and centre C. P is a point on the ellipse and D is a point on the circle so that radius CD bisects FP. Show that line DP is tangent to the ellipse.

Pedagoguery Software has offered a copy of their software GrafEq *to the first correct solution received by the MAYHEM problems editor.*

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Une ellipse de grand axe AB et de foyers F et F' est inscrite dans un cercle de diamètre AB et de centre C. Soit P un point sur l'ellipse et D un point sur le cercle de sorte que le rayon CD coupe FP en son milieu. Montrer que la droite DP est tangente à l'ellipse.

Pedagoguery Software a offert une copie de leur logiciel GrafEq *à l'auteur de la première solution correcte reçue par l'éditeur des problèmes du MAYHEM.*

M55. Proposed by the Mayhem Staff.

Find the sum of the first 2002 terms in the following sequence

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ \dots .$$

. . . .

Trouver la somme des 2002 premiers termes de la suite suivante

$$1,\ 2,\ 2,\ 3,\ 3,\ 3,\ 4,\ 4,\ 4,\ 4,\ 5,\ 5,\ 5,\ 5,\ 5,\ \dots .$$

M56. Proposed by Vedula N. Murty, Dover, PA, USA.
Prove the identity

$$\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 \;=\; 4\cos A \cos B \cos C\,,$$

where the sums are cyclic and $A + B + C = \pi$.

Démontrer l'identité

$$\left(\sum \sin A\right)^2 - \left(1 + \sum \cos A\right)^2 \;=\; 4\cos A \cos B \cos C\,,$$

où les sommes sont cycliques et $A + B + C = \pi$.

Mayhem Problem Solutions

The solutions in this issue are to Australian Mathematics Trust Questions.

M1. [2001 : 322] Four singers take part in a musical round of 4 equal lines, each finishing after singing the round through three times. The second singer begins when the first singer begins the second line, the third singer begins when the first singer begins the third line, the fourth singer begins when the first singer begins the fourth line. Find the fraction of the total singing time that all four are singing at the same time.

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

Let x be the time it takes to sing each line. Then the total time spent singing is:

$$\begin{array}{c} \text{Time when the} \\ \text{last singer is} \\ \text{not singing} \end{array} + \begin{array}{c} \text{Time when the} \\ \text{last singer} \\ \text{is singing} \end{array} = 3x + 12x$$
$$= 15x\,.$$

The time for which all four singers are singing is:

$$\begin{array}{c} \text{Time when the} \\ \text{first singer} \\ \text{stops singing} \end{array} - \begin{array}{c} \text{Time when the} \\ \text{last singer} \\ \text{begins singing} \end{array} = 12x - 3x$$
$$= 9x\,.$$

Therefore, the fraction of the total singing time that all four are singing at the same time is $\frac{9x}{15x} = \frac{3}{5}$.

M2. [2001 : 322] When 5 new classrooms were built for Wingecarribee School the average class size was reduced by 6. When another 5 classrooms were built, the average class size reduced by another 4. If the number of students remained the same throughout the changes, how many students were there at the school?

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

Let y be the original number of classrooms, and let x be the number of students.

The addition of the first five new classrooms gives $\frac{x}{y} = \frac{x}{y+5} + 6$. Multiplying both sides first by $y(y+5)$ gives

$$\begin{aligned} x(y+5) &= xy + 6y(y+5) \\ 5x &= 6y(y+5) \,. \end{aligned} \qquad (1)$$

The second addition of five new classrooms gives $\frac{x}{y+5} = \frac{x}{y+10} + 4$, which simplifies to

$$5x = 4(y+5)(y+10) \,. \qquad (2)$$

Equating the righthand sides of (1) and (2), we get:

$$\begin{aligned} 6y(y+5) &= 4(y+5)(y+10) \\ 6y &= 4y + 40 \\ 2y &= 40 \\ y &= 20 \,. \end{aligned}$$

Substituting $y = 20$ into (1) or (2) gives $x = 600$, and so there were 600 students at the school.

One incorrect solution was received.

M3. [2001 : 323] How many years in the 21^{st} century will have the property that, dividing their year number by each of 2, 3, 5, and 7 always leaves a remainder of 1?

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

Suppose that n has the desired property. Then $n-1$ is divisible by 2, 3, 5, and 7, and hence, by $2\times3\times5\times7 = 210$. Hence, x is one greater than some multiple of 210. But 2101 satisfies this condition, and thus, no number from 2001 to 2100 inclusive will also satisfy the condition. Therefore, no years in the 21^{st} century will have the stated property.

M4. [2001 : 323] We write down all the numbers 2, 3, $\ldots$, 100, together with all their products taken two at a time, their products taken three at a time, and so on up to and including the product of all 99 of them. Find the sum of the reciprocals of all the numbers written down.

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

When adding the reciprocals of all the numbers, we use the common denominator $2 \times 3 \times \cdots \times 100$. The numerator of the sum of the reciprocals is the straightforward sum of all the numbers written down plus 1 (excluding the entire product). Add and subtract 1 to this sum (keeping the sum unchanged), making sure to write the first 1 as $\frac{2\times 3\times\cdots\times 100}{2\times 3\times\cdots\times 100}$. Then, the numerator factors and we obtain the following expression for the sum:

$$\frac{(2+1)(3+1)\dots(100+1)}{2\times 3\times\cdots\times 100} - 1 = \frac{2+1}{2}\times\frac{3+1}{3}\times\cdots\times\frac{100+1}{100} - 1$$
$$= \frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{101}{100} - 1 = \frac{101}{2} - 1 = \frac{99}{2}.$$

M5. [2001 : 323] The ratio of the speeds of two trains is equal to the ratio of the time they take to pass each other going in the same direction to the time they take to pass each other going in the opposite directions. Find the ratio of the speeds of the two trains.

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

Let x be the speed of the faster train, and let y be the speed of the slower train. Now, the ratio of the time taken for the two trains to pass each other is equal to the ratio of their relative speeds as they pass each other, since in both cases the faster train covers the same distance relative to the slower train. Thus,

$$\frac{x}{y} = \frac{x+y}{x-y}, \qquad x(x-y) = y(x+y),$$
$$x^2 - 2xy = y^2, \qquad x^2 - 2xy + y^2 = 2y^2,$$
$$(x-y) = \sqrt{2}y, \qquad x = (1+\sqrt{2})y;$$

whereupon, $\frac{y}{x} = \sqrt{2} - 1$.

M6. A city railway network has for sale one-way tickets for travel from one station to another station. Each ticket specifies the origin and destination. Several new stations were added to the network, and an additional 76 different ticket types had to be printed. How many new stations were added to the network?

Solution by Paul Jeffries, student, Berkhamsted Collegiate School, UK.

If x is the initial number of stations, then we start with $x(x-1)$ tickets. Then, if y new stations are built, there are $(x+y)(x+y-1)$ tickets in all. Thus, there are $2xy + y^2 - y$ new tickets, and we get $y(2x+y-1) = 76$, so that y must divide 76. Because $x > 0$, we have $y < 2x + y - 1$; thus, y is the smaller factor in the equation $y(2x+y-1) = 76$. Hence, y is 1, 2, or 4. But y cannot be 1 since several new stations were added. $y = 2$ forces x to be a fraction, which is not possible. $y = 4$ gives $x = 8$, so that 4 new stations were added to the network.