SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

We inadvertantly omitted Murray S. Klamkin, University of Alberta, Edmonton, Alberta, from the list of solvers of 2649, and Michel Bataille, Rouen, France, from the list for 2626. We also erred in the name of Marcelo R. de Souza in the list of solvers for 2645. Sorry, Murray, Michel and Marcelo.

2572. [2000 : 374, 2001 : 473, 2002 : 57] *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let a, b, c be positive real numbers. Prove that

$$a^b b^c c^a \;\le\; \left(\frac{a+b+c}{3}\right)^{a+b+c} .$$

[Compare problem **2394** [1999 : 524], note by V.N. Murty on the generalization.]

IV. Murray S. Klamkin has pointed out that in Walter Janous's remarks, he proves the inequality:

$$(x_1+x_2+\cdots+x_n)^2 \;\ge\; 4\left(x_1x_2+x_2x_3+\cdots+x_nx_1\right) .$$

This inequality from the 1984 Moscow Olympiad appears with solution as #15 in [1985 : 288–289].

2651★. [2001 : 335] *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta*. Dedicated to Professor M.V. Subbarao on the occasion of his 80th birthday. (Professor Klamkin offers a prize of $100 for the first correct solution received by the Editor-in-Chief.)

Let P be a non-exterior point of a regular n–dimensional simplex $A_0A_1A_2\ldots A_n$ of edge length e. If

$$F \;=\; \sum_{k=0}^{n} PA_k + \min_{0\le k\le n} PA_k\,, \qquad F' \;=\; \sum_{k=0}^{n} PA_k + \max_{0\le k\le n} PA_k\,,$$

determine the maximum and minimum values of F and F'.

This problem was suggested by problem 2594 for a general triangle, and the proposer was trying to obtain a stronger inequality by finding the maximum of F.

Solution by Fedor Petrov, Saint Petersburg, Russian Federation; revised by Rudolf Fritsch, Munich, Germany

The crucial notion for this solution is the notion of a *convex function* $f : \mathcal{P} \to \mathbb{R}$; that is, a function f whose domain is a compact convex subset of some Euclidean space $\mathbb{R}^n$ and which satisfies

$$f\left(\frac{M+N}{2}\right) \;<\; \frac{f(M)+f(N)}{2}$$

for all pairs of points M, $N \in \mathcal{P}$, $M \neq N$. The usual definition of a convex function is slightly different, but in the following just the given property is needed. In our case the domain will be either the simplex $\mathcal{T}$ with the vertices A_0, A_1, $\ldots$, A_n or a compact, convex subset of $\mathcal{T}$. For a convex function f (with a convex domain) there is at most one place where f takes a minimal value. Indeed, if there were two different places M, N with this property, then the mid-point of the line segment $[MN]$ would yield a smaller value in contradiction to the assumption. On the other hand, a maximal place of a convex function must be a vertex of the domain $\mathcal{P}$. Recall that a point P is a *vertex* or *extreme point* of a compact convex subset $\mathcal{P}$ of some Euclidean space $\mathbb{R}^n$, if it is not an interior point of a line segment contained in $\mathcal{P}$. Indeed, assume $f : \mathcal{P} \to \mathbb{R}$ to be a convex function and consider a point $P \in \mathcal{P}$ which is not a vertex. Then, P is an interior point of a line segment contained in $\mathcal{P}$ and we can find a smaller line segment $[MN]$ having P as mid-point. By convexity we get

$$f(P) \;<\; \frac{f(M)+f(N)}{2} \;\leq\; \frac{2\cdot\max\{f(M),f(N)\}}{2} \;=\; \max\{f(M),f(N)\}\,;$$

thus, the point P cannot be a maximal place. Note that the vertices of a simplex are the extreme points of the simplex in the sense just described.

Now we turn to the given problem. To begin with, observe that the triangle inequality

$$(M+N)A \;\leq\; MA+NA$$

yields by means of division by 2:

$$KA \;\leq\; \frac{MA+NA}{2}$$

for all triples of points A, M, $N \in \mathbb{R}^n$, $M \neq N$, where K denotes the mid-point of the line segment $[MN]$. Equality holds if and only if the point A belongs to line MN but is not an interior point of the line segment $[MN]$.

From this we conclude that the function $F' : \mathcal{T} \to \mathbb{R}$ under consideration is convex. In fact, given a pair of points M, $N \in \mathbb{R}^n$, $M \neq N$, note first

$$KA_j \;\leq\; \frac{MA_j+NA_j}{2} \;=\; \frac{MA_j}{2}+\frac{NA_j}{2} \;\leq\; \max_{0\leq i\leq n}\frac{MA_i}{2}+\max_{0\leq i\leq n}\frac{NA_i}{2}$$

for all $j \in \{0, 1, \ldots, n\}$, implying that

$$\max_{0\le i\le n} KA_i \;\le\; \max_{0\le i\le n} \frac{MA_i}{2} \;+\; \max_{0\le i\le n} \frac{NA_i}{2} i\,.$$

Secondly, there is at least one vertex A_i which is not on the line MN and thus,

$$\begin{aligned} F'(K) &= \sum_{i=0}^{n} KA_i + \max_{0\le i\le n} KA_i \\ &< \sum_{i=0}^{n} \frac{MA_i + NA_i}{2} + \max_{0\le i\le n} \frac{MA_i}{2} + \max_{0\le i\le n} \frac{NA_i}{2} i \\ &= \frac{F'(M) + F'(N)}{2}\,. \end{aligned}$$

The function $F' : \mathcal{T} \to \mathbb{R}$ has a compact domain and is continuous; thus, it has a maximal and a minimal value. Let M_1' denote a point where F' takes the maximal value and M_2' the unique point where F' takes the minimal value.

By the convexity of F' the point M_1' must be a vertex of the simplex $\mathcal{T}$; that is, there is an index $j \in \{0, 1, \ldots, n\}$ such that

$$F'(M_1') \;=\; F'(A_j) = (n+1)\cdot e\,.$$

Therefore,

$$(n+1)\cdot e$$

is the maximal value of the function F'.

By definition, the function F' is invariant under a permutation of the vertices $A_0, A_1, \ldots, A_n$. If M_2' were different from the circumcentre of $\mathcal{T}$, then a point M_2'' obtained by a permutation of the barycentric coordinates of M_2 would be also a minimal place, in contradiction to the uniqueness of the minimal place of a convex function. Thus, the point M_2' is the circumcentre C of T having equal distances from all vertices of the simplex $\mathcal{T}$. Since the circumradius of a regular n–simplex with edge length e is computed to be

$$\rho_n \;=\; e\cdot\sqrt{\frac{n}{2(n+1)}}$$

the minimal value of F' is

$$(n+2)\cdot CA_0 \;=\; (n+2)\cdot e\cdot\sqrt{\frac{n}{2(n+1)}}\,.$$

Now we discuss the function F. For $j \in \{0, 1, \ldots, n\}$ define

$$\mathcal{T}_j \;=\; \{P \in \mathcal{T} \mid PA_j \;=\; \min_{0\le i\le n} PA_i\}\,.$$

Since by definition the function F is invariant under a permutation of the vertices $A_0, A_1, \dots, A_n$, it is sufficient to compute the extreme values of the restriction $F_0 = F|\mathcal{T}_0$. The proof of the convexity of the function F' given above shows that the function F_0 is convex. But F_0 is also continuous, so that it has at least one maximal place and a unique minimal place M_1.

We claim that M_1 must belong to the line segment $[A_0C]$ where C, as above, denotes the circumcentre of the simplex $\mathcal{T}$. An arbitrary point $M \in \mathcal{T}_0$ has a unique representation of the form

$$M = \sum_{i=0}^{n} r_i A_i$$

with $r_i \in [0,1]$ for all $i \in \{0, 1, \dots, n\}$, such that $\sum_{i=0}^{n} r_i = 1$, and $r_0 = \max\{r_i | i \in \{0, 1, \dots, n\}\}$; it belongs to the line segment $[A_0C]$ if and only if $r_1 = r_2 = \cdots = r_n$. The scalars r_i are the so-called barycentric coordinates of the point M. Now consider a point M not belonging to the line segment $[A_0C]$. Then there are indices j, k with $1 \leq j < k \leq n$ and $r_j \neq r_k$. Form the point

$$N = \sum_{i=0}^{n} \tilde{r}_i A_i$$

by taking

$$\tilde{r}_i = \begin{cases} r_i, & j \neq i \neq k, \\ r_k, & i = j, \\ r_j, & i = k. \end{cases}$$

The points M, N and K, where K denotes, as before, the mid-point of the line segment $[MN]$, all belong to $\mathcal{T}_0$. We have

$$F_0(M) = F(M) = F(N) = F_0(N)$$

and by convexity

$$F_0(K) = F(K) < F(M) = F_0(M).$$

Thus, M cannot be a minimal place of the function F_0.

It follows that the minimal place M_1 of the function F_0 has a representation of the form

$$M_1 = (1-r)A_0 + rC$$

with $r \in [0,1]$. Denote by h_n the altitude of the regular simplex $\mathcal{T}$ (that is,

$$h_n = \frac{n+1}{n} \cdot \rho_n = e \cdot \sqrt{\frac{n+1}{2n}} = \sqrt{\frac{n+1}{n-1}} \cdot \rho_{n-1},)$$

and by s the distance of M_1 from the circumcentre of the face of $\mathcal{T}$ opposite to the vertex A_0 (that is,

$$s \;=\; h_n - M_1A_0 \;=\; h_n - r\cdot\rho_n)\,.$$

We compute

$$\begin{aligned} M_1A_1 &= \sqrt{s^2+\rho_{n-1}^2} \;=\; M_1A_2 \;=\; M_1A_3 \;=\; \cdots \;=\; M_1A_n\,,\\ F_0(M_1) &= F(M_1) \;=\; 2\cdot M_1A_0 + n\cdot M_1A_1\\ &= 2\cdot(h_n-s)+n\cdot\sqrt{s^2+\rho_{n-1}^2}\,. \end{aligned}$$

Thus, s is a place in the interval $[h_n-\rho_n, h_n]$ where the function

$$g:[h_n-\rho_n,h_n]\to\mathbb{R}\,,\; t\;\mapsto\; 2\cdot(h_n-t)+n\cdot\sqrt{t^2+\rho_{n-1}^2}$$

takes its absolute minimal value. To see that this condition determines s uniquely and to compute the corresponding value of s, we form the derivative

$$g'(t) \;=\; -2+\frac{n\cdot t}{\sqrt{t^2+\rho_{n-1}^2}}\,.$$

Now we have two cases.

1. If $n=2$, then g' is negative on the interval under consideration. Thus, the function g is monotonic decreasing and takes its minimal value at the upper end of the interval; that is, for $t=h_n$. Then, $M_1=A_0$ and

   $$F_0(M_1) \;=\; 2e$$

   is the minimal value of F.

2. If $n>2$, we find the unique minimal place

   $$s \;=\; \frac{2\rho_{n-1}}{\sqrt{n^2-4}}$$

   and thus, the minimal value of F is

   $$\begin{aligned} F_0(M_1) &= \rho_{n-1}\left(2\sqrt{\frac{n+1}{n-1}}+\sqrt{n^2-4}\right)\\ &= \frac{e}{\sqrt{2n}}\cdot\left(2\sqrt{n+1}+\sqrt{(n-1)(n-2)(n+2)}\right). \end{aligned}$$

Finally, we are looking for the maximal value of F. Let M_2 be a maximal place for the function F_0. It is a vertex of the convex polytope $\mathcal{T}_0$. The vertices of this polytope are the vertex A_0 of the simplex $\mathcal{T}$ and the circumcentres of the faces of the simplex $\mathcal{T}$ containing the vertex A_0 (to be proved in the appendix). If C_k is a circumcentre of a k–dimensional face of this sort, then it has $k+1$ barycentric coordinates with value $\frac{1}{k+1}$ and the remaining $n-k$ barycentric coordinates vanish. Without loss of generality it suffices to consider

$$C_k \;=\; \frac{1}{k+1}\sum_{i=0}^{k} A_i$$

including $C_0 = A_0$ and $\rho_0 = 0$ and to compute

$$\begin{aligned} C_kA_0 &= \rho_k \;=\; C_kA_1 \;=\; \cdots \;=\; C_kA_k\,, \\ C_kA_{k+1} &= e\cdot\sqrt{\frac{k+2}{2(k+1)}} \;=\; C_kA_{k+2} \;=\; \cdots \;=\; C_kA_n\,, \\ F_0(C_k) &= F(C_k) \;=\; (k+2)\cdot C_kA_0 + (n-k)\cdot C_kA_{k+1} \\ &= \frac{e}{\sqrt{2(k+1)}}\cdot\left((k+2)\sqrt{k} + (n-k)\sqrt{k+2}\right). \end{aligned}$$

Thus, it remains to check, given n fixed, for which $k \in \{0,\, 1,\, \ldots,\, n\}$ the value $F_0(C_k)$ is maximal. There are several cases to distinguish.

1. In the case $n \geq 5$, we claim

$$F_0(C_k) \;\leq\; F_0(C_0) \;=\; n\cdot e$$

for all $k \in \{0,\, 1,\, \ldots,\, n\}$. This implies that $n \cdot e$ is the maximal value of the function F in the case $n \geq 5$. To see this, transform the desired inequality to

$$(k+2)\sqrt{k} + (n-k)\sqrt{k+2} \;\leq\; n\cdot\sqrt{2(k+1)}\,.$$

For $k = 1$ this inequality becomes

$$3 + (n-1)\sqrt{3} \;\leq\; n\cdot 2\,,$$

which is equivalent to $n \geq 1$. Thus, we can restrict our attention to the cases with $2 \leq k \leq n$ and transform the inequality into

$$(k+2)\sqrt{k} - k\sqrt{k+2} \;\leq\; n\cdot\left(\sqrt{2(k+1)} - \sqrt{k+2}\right)\,.$$

Multiplication by $\sqrt{2(k+1)} + \sqrt{k+2}$ and division by k yield

$$\left(\frac{k+2}{\sqrt{k}} - \sqrt{k+2}\right)\cdot\left(\sqrt{2(k+1)} + \sqrt{k+2}\right) \;\leq\; n\,.$$

Next consider the following transformations and estimations:

$$\begin{aligned} \frac{k+2}{\sqrt{k}} - \sqrt{k+2} &= \sqrt{1+\frac{2}{k}}\cdot\left(\sqrt{k+2} - \sqrt{k}\right)\,, \\ \sqrt{2(k+1)} &< \sqrt{2}\sqrt{k+2}\,, \\ \sqrt{k+2} &\leq \sqrt{2}\sqrt{k} \qquad \text{(in view of } k \geq 2)\,, \\ \sqrt{2(k+1)} + \sqrt{k+2} &< \sqrt{2}\left(\sqrt{k+2} + \sqrt{k}\right)\,. \end{aligned}$$

These yield

$$\left(\frac{k+2}{\sqrt{k}} - \sqrt{k+2}\right)\cdot\left(\sqrt{2(k+1)} + \sqrt{k+2}\right) \ < \ 2\cdot\sqrt{2+\frac{4}{k}} \ \le \ 4 \ < \ n\,,$$

as desired.

2. For the remaining cases we compute $\frac{F_0(C_k)}{e}$ in the following table where $n \in \{1, 2, 3, 4\}$ is the row index and $k \in \{0, 1, 2, 3, 4\}$ is the column index.

	0	1	2	3	4
1	1	$\frac{3}{2}$			
2	2	$\frac{3+\sqrt{3}}{2} \approx 2.37$	$\frac{4\sqrt{3}}{3} \approx 2.31$		
3	3	$\frac{3}{2}+\sqrt{3} \approx 3.23$	$\frac{4+\sqrt{2}}{\sqrt{3}} \approx 3.13$	$\frac{5\sqrt{3}}{2\sqrt{2}} \approx 3.06$	
4	4	$\frac{3+3\sqrt{3}}{2} \approx 4.10$	$\frac{4+2\sqrt{2}}{\sqrt{3}} \approx 3.94$	$\frac{5\sqrt{3}+\sqrt{5}}{2\sqrt{2}} \approx 3.85$	$\frac{6\sqrt{2}}{\sqrt{5}} \approx 3.79$.

This shows that, for regular simplices up to dimension 4 (that is, for line segments, equilateral triangles, regular tetrahedra and regular 4–simplices), the function F takes its maximal values at the mid-points of one-dimensional edges. These maximal values are

$$\frac{3}{2}e\,,\quad \left(\frac{3+\sqrt{3}}{2}\right)e\,,\quad \left(\frac{3}{2}+\sqrt{3}\right)e\,,\quad \left(\frac{3+3\sqrt{3}}{2}\right)e\,.$$

Appendix

To make the presentation self-contained we add a proof for the fact that the vertices of the compact convex set $\mathcal{T}_0$ are just the circumcentres of the faces of the simplex $\mathcal{T}$ containing the vertex A_0. Before doing this, note that this fact depends on the regularity of the simplex $\mathcal{T}$. For general simplices one must take the centroids instead of the circumcentres. The circumcentres are characterized by the fact that their barycentric coordinates have at most two values, 0 and $\frac{1}{k+1}$ if dealing with a face of dimension k; in particular the 0^{th} barycentric coordinate has the value $\frac{1}{k+1}$.

As noted above, the points

$$M \ = \ \sum_{i=0}^{n} r_i A_i$$

of $\mathcal{T}_0$ are characterized among all points of the simplex $\mathcal{T}$ by the condition

$$r_0 \ = \ \max\{r_i | i \in \{0, 1, \ldots, n\}\}\,.$$

If such a point is not a circumcentre, then there is an index $j \in \{1, 2, \dots, n\}$ with $0 < r_j < r_0$. We choose μ such that r_μ is the largest barycentric coordinate of M which is smaller than r_0. Further, let $\mathcal{L}$ denote the set of indices l with $r_l = r_0$, $k+1$ the number of elements of $\mathcal{L}$ and define

$$C_k \;=\; \frac{1}{k+1}\sum_{i\in\mathcal{L}} A_i\,.$$

Note that the presence of r_μ implies $r_0 < \dfrac{1}{k+1}$. Consider now the ray emanating from C_k and passing through M. Its points are represented in the form

$$\sum_{i\in\mathcal{L}}\left(\frac{1-t}{k+1}+tr_i\right)A_i + t\sum_{i\notin\mathcal{L}} r_iA_i\,.$$

Specializing to

$$t \;=\; \frac{1}{1-(k+1)(r_0-r_\mu)} \;>\; 1\,,$$

we obtain the point

$$N \;=\; tr_\mu\sum_{i\in\mathcal{L}} A_i + t\sum_{i\notin\mathcal{L}} r_iA_i$$

on this ray which still belongs to $\mathcal{T}_0$ and therefore, M is an interior point of the line segment $[C_kN]$. Thus, this point M is not a vertex of $\mathcal{T}_0$.

On the other hand we show that the points C_k of the form

$$C_k \;=\; \frac{1}{k+1}\sum_{i\in\mathcal{L}} A_i\,,$$

(where $\mathcal{L}$ denotes a set of $k+1$ indices) are vertices. To this end, assume that M, $N \in \mathcal{T}_0$ are different with C_k belonging to the line segment $[MN]$. We shall show that either $C_k = M$ or $C_k = N$, which proves that C_k is a vertex. There is a $t \in [0,1]$ such that $C_k = (1-t)M + tN$. Let us fix the barycentric coordinates of M and N:

$$\begin{aligned} M &= \sum_{i=0}^{n} r_iA_i\,,\\ N &= \sum_{i=0}^{n} s_iA_i\,. \end{aligned}$$

which implies

$$(1-t)\cdot r_i + ts_i \;=\; \begin{cases} \dfrac{1}{k+1}\,, & i\in\mathcal{L}\,,\\ 0\,, & \text{otherwise.}\end{cases}$$

We have two cases.

1. If $k < n$ then there is an index $j \in \{1, 2, \ldots, n\} \backslash \mathcal{L}$ and thus, 0 belongs to the interval $[r_j, s_j]$ with non-negative end-points. Thus, either r_j or s_j must vanish. If $r_j = 0 \neq s_j$ it follows $t = 0$ and thus, $C_k = M$; if $r_j \neq 0 = s_j$ we get $t = 1$ and $C_k = N$.

   It remains for us to consider the case in which $r_j = 0 = s_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \mathcal{L}$. From M, $N \in \mathcal{T}_0$ it follows that $r_0, s_0 \geq \frac{1}{k+1}$, but we have either $r_0 \leq \frac{1}{k+1} \leq s_0$ or $s_0 \leq \frac{1}{k+1} \leq r_0$. Thus, we get three possibilities:

   (a) $r_0 = \frac{1}{k+1} < s_0$: In this case we have $t = 0$ and $C_k = M$.

   (b) $s_0 = \frac{1}{k+1} < r_0$: In this case we have $t = 1$ and $C_k = N$.

   (c) $s_0 = \frac{1}{k+1} = r_0$: From $0 \leq r_i, s_i \leq \frac{1}{k+1}$ for all $i \in \mathcal{L}$ and $\sum\limits_{i \in \mathcal{L}} r_i = \sum\limits_{i \in \mathcal{L}} s_i = 1$, we obtain $r_i = s_i = \frac{1}{k+1}$ for all $i \in \mathcal{L}$ and thus, the contradiction $M = N$. This case does not occur.

2. The case $k = n$ follows in the same way as the last part of the previous case.

This finishes the proof.

Also solved by EMILIO FERNÁNDEZ MORAL, I.B. Sagasta, Logroño, Spain; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

2652★. [2001 : 336] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

Let d, e and f be the sides of the triangle determined by the three points at which the internal angle-bisectors of given $\triangle ABC$ meet the opposite sides. Prove that

$$d^2 + e^2 + f^2 \;\leq\; \frac{s^2}{3},$$

where s is the semiperimeter of $\triangle ABC$.

Show also that equality occurs if and only if the triangle is equilateral.

Solution by David Loeffler, student, Trinity College, Cambridge, UK.

Let the internal bisectors of $\angle BAC$, $\angle CBA$ and $\angle ACB$ meet BC, CA and AB at D, E and F, respectively, and let s be the semiperimeter of $\triangle ABC$. It is possible to obtain a proof of Janous' inequality by adapting Bataille's proof of the weaker inequality $d + e + f \leq s$ [2001 : 53–54]. In the proof, Bataille obtains the inequality

$$DF^2 \;\leq\; \frac{ab^2c}{(a+b)(b+c)}.$$

Adding the corresponding inequalities for FE and ED and applying the AM–GM inequality, we obtain

$$\begin{aligned} d^2+e^2+f^2 &\leq \frac{abc[b(a+c)+a(b+c)+c(a+b)]}{(a+b)(b+c)(c+a)} \\ &= \frac{2abc(ab+bc+ca)}{(a+b)(b+c)(c+a)} \\ &\leq \frac{2abc(ab+bc+ca)}{(2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca})} \\ &= \tfrac{1}{4}(ab+bc+ca) \;\leq\; \tfrac{1}{12}(a+b+c)^2\,. \end{aligned}$$

(The last inequality follows from $2\left((a+b+c)^2-3(ab+bc+ca)\right) = (a-b)^2+(b-c)^2+(c-a)^2 \geq 0$.) Therefore, $d^2+e^2+f^2 \leq \frac{s^2}{3}$. Equality occurs if and only if $\triangle ABC$ is equilateral.

Also solved by GEORGE BALOGLOU, SUNY Oswego, Oswego, NY, USA; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; D. KIPP JOHNSON, Beaverton, OR, USA; MURRAY S. KLAMKIN, University of Alberta, Edmonton, Alberta; KEE-WAI LAU, Hong Kong; TODOR MITEV, University of Rousse, Rousse, Bulgaria; C.R. PRANESACHAR, Department of Mathematics, Indian Institute of Science, Bangalore, India; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; and PETER Y. WOO, Biola University, La Mirada, CA, USA.

2653. [2001 : 336] *Proposed by Walther Janous, Ursulinengymnasium, Innsbruck, Austria.*

For whole numbers $n \geq 0$ and $N \geq 1$, evaluate the (combinatorial) sum

$$S_N(n) := \sum_{k\geq n} \binom{N}{2k}\binom{k}{n}\,.$$

I. *Comment by Michel Bataille, Rouen, France,* (*expanded slightly by the editor*).

This problem is not new. It was proposed by Frank Gerrish as problem 78F in the July, 1994 issue of the Mathematical Gazette [Math. Gaz. 78 (482) 199]. Two solutions appeared in the March, 1995 issue [ibid. 79 (484) 129–132]. The first one was a fairly complicated combinatorial argument by J.K.R. Barnett and the second one was a more straightforward proof given by Nick Lord which used Newton's Generalized Binomial Theorem. Later on, a second and shorter combinatorial proof given by Chris Norman appeared in the November, 1995 issue [ibid. 79 (486) 587–588].

[Ed: Below we present a solution which is different from the ones that have appeared before and is also interesting and self-contained.]

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA, (modified slightly by the editor).*

Clearly, $S_N(n) = 0$, if $N < 2n$. We shall show that

$$S_N(n) \;=\; T_N(n) \quad \text{if} \quad N \geq 2n\,, \tag{1}$$

where $T_N(n) = \dfrac{2^{N-2n-1}N}{N-n}\dbinom{N-n}{n}$.

When $n = 0$, we have $S_N(0) = \sum\limits_{k\geq 0}\binom{N}{2k} = 2^{N-1} = T_N(0)$.

If $N = 1$, then $n = 0$. Hence (1) holds for $N = 1$. Note also that $S_{2n}(n) = 1 = T_{2n}(n)$ for all $n \geq 1$ and thus, in particular, (1) holds for $N = 2$.

We claim that for $N \geq 3$ and $n \geq 1$, $S_N(n)$ satisfies the recurrence relation:

$$S_N(n) \;=\; 2S_{N-1}(n) + S_{N-2}(n-1)\,. \tag{2}$$

Indeed, since

$$\begin{aligned}
2\binom{N-1}{2k} &= \binom{N}{2k} - \binom{N-1}{2k-1} + \binom{N-1}{2k} \\
&= \binom{N}{2k} - \left(\binom{N-2}{2k-2} + \binom{N-2}{2k-1}\right) + \left(\binom{N-2}{2k} + \binom{N-2}{2k-1}\right) \\
&= \binom{N}{2k} - \binom{N-2}{2k-2} + \binom{N-2}{2k}\,,
\end{aligned}$$

[Ed: Note that $\binom{N-1}{2k-1} = \binom{N-2}{2k-2} = \binom{N-2}{2k-1} = 0$ if $k = 0$, by convention.]

we have

$$\begin{aligned}
&2S_{N-1}(n) + S_{N-2}(n-1) \\
&= \sum_{k\geq 0}\binom{N}{2k}\binom{k}{n} - \sum_{k\geq 1}\binom{N-2}{2k-2}\binom{k}{n} + \sum_{k\geq 0}\binom{N-2}{2k}\left(\binom{k}{n} + \binom{k}{n-1}\right) \\
&= S_N(n) - \sum_{j\geq 0}\binom{N-2}{2j}\binom{j+1}{n} + \sum_{k\geq 0}\binom{N-2}{2k}\binom{k+1}{n} = S_N(n)\,,
\end{aligned}$$

which establishes (2). Since $S_N(n)$ and $T_N(n)$ have the same initial values, it suffices to show that $T_N(n)$ also satisfies the same recurrence relation. Indeed,

$$\begin{aligned}
&2T_{N-1}(n) + T_{N-2}(n-1) \\
&= \frac{2^{N-2n-1}}{N-n-1}\left[\binom{N-n-1}{n}(N-1) + \binom{N-n-1}{n-1}(N-2)\right]
\end{aligned}$$

$$= \frac{2^{N-2n-1}}{N-n-1}\left[\left(\binom{N-n}{n} - \binom{N-n-1}{n-1}\right)(N-1) + \binom{N-n-1}{n-1}(N-2)\right]$$
$$= \frac{2^{N-2n-1}}{N-n-1}\left[\binom{N-n}{n}(N-1) - \binom{N-n-1}{n-1}\right]$$
$$= \frac{2^{N-2n-1}}{N-n-1}\binom{N-n}{n}\left[N-1-\frac{n}{N-n}\right]$$
$$= \frac{2^{N-2n-1}}{N-n-1}\binom{N-n}{n}\frac{N(N-n-1)}{N-n}$$
$$= \frac{2^{N-2n-1}N}{N-n}\binom{N-n}{n} = T_N(n)\,,$$

and our proof is complete.

Also solved by NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; and the proposer. There was also one incomplete solution.

Though it is obvious that $S_N(n) = 0$ for $N < 2n$, only Guersenzvaig, Seiffert and the proposer pointed this out explicitly in their solutions. Checking the lists of the solvers of the original problem in the Gazette and the current problem reveals that there is exactly one person in the intersection. Since this solver did not mention anything about the original problem, the editor can only assume that after eight years, he has completely forgotten about it.

2654. [2001 : 336] *Proposed by Christopher J. Bradley, Clifton College, Bristol, UK.*

Suppose that $\triangle ABC$ has medians AD, BE and CF. Suppose that L, M and N are points on the sides BC, CA and AB, respectively.

Prove that the line through L parallel to AD, the line through M parallel to BE and the line through N parallel to CF are concurrent if and only if

$$\frac{BL^2}{BC^2} + \frac{CM^2}{CA^2} + \frac{AN^2}{AB^2} = \frac{LC^2}{BC^2} + \frac{MA^2}{CA^2} + \frac{NB^2}{AB^2}\,.$$

I. *Solution by David Loeffler, student, Trinity College, Cambridge, UK.*

We apply a linear transformation to $\triangle ABC$, transforming it into an equilateral triangle $A'B'C'$. Clearly, the original lines concur if and only if their images do so. Specifically, the lines through L', M' and N' are parallel to the images of the medians of ABC, which are the medians of $A'B'C'$ (since the transformation preserves ratios of distances). These are now perpendicular to the sides, so the lines concur if and only if

$$B'L'^2 + C'M'^2 + A'N'^2 = L'C'^2 + M'A'^2 + N'B'^2$$

by Carnot's theorem. Since $A'B' = B'C' = C'A'$ we may write this as

$$\frac{B'L'^2}{B'C'^2} + \frac{C'M'^2}{C'A'^2} + \frac{A'N'^2}{A'B'^2} = \frac{L'C'^2}{B'C'^2} + \frac{M'A'^2}{C'A'^2} + \frac{N'B'^2}{A'B'^2}$$

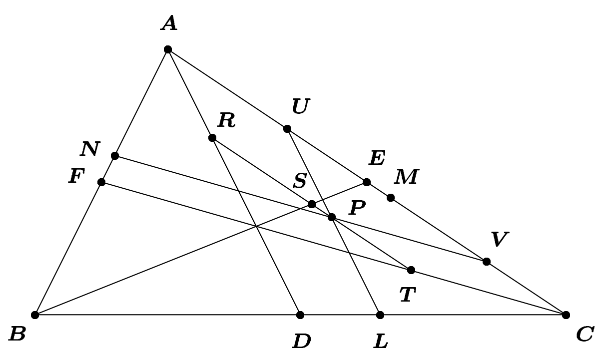
However, we have $\frac{B'L'^2}{B'C'^2} = \frac{BL^2}{BC^2}$, etc., since the transformation preserves ratios of distances. Hence, concurrency occurs if and only if

$$\frac{BL^2}{BC^2} + \frac{CM^2}{CA^2} + \frac{AN^2}{AB^2} = \frac{LC^2}{BC^2} + \frac{MA^2}{CA^2} + \frac{NB^2}{AB^2}$$

as required

II. *Solution by Li Zhou, Polk Community College, Winter Haven, FL, USA.*

For concreteness, we give the solution for the configuration in the accompanying figure. The proof for other positions of P is similar. [*Editor's comment:* in fact, the proof is valid for all P if one uses directed distances.]



First, we note that the given equation is equivalent to

$$\frac{BL - LC}{BC} + \frac{CM - MA}{CA} + \frac{AN - NB}{AB} = 0\,.$$

Since D is the mid-point of BC, $\frac{BL - LC}{BC} = \frac{DL}{DC} = \frac{AU}{AC}$, and likewise $\frac{CM - MA}{CA} = \frac{-2EM}{AC}$ and $\frac{AN - NB}{AB} = \frac{-VC}{AC}$. Suppose that PR is parallel to AC. If the three lines concur at P, then

$$AU - 2EM - VC = RP - 2SP - PT = RS - ST = 0$$

as desired. Conversely, if the given equation holds then $EM = SP$, which implies that MP is parallel to BE.

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; HENRY LIU, student, University of Memphis, Memphis, TN, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; and the proposer.

2655. [2001 : 336] *Proposed by Vedula N. Murty, Dover, PA, USA.*

Let a, b and c be the sides of $\triangle ABC$ and let s be its semiperimeter. Given that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} \;=\; s\,,$$

show that $\triangle ABC$ is equilateral.

I. *Solution by Richard Eden, Ateneo de Manila University, Philippines.*

By the AM–HM inequality,

$$\frac{x+y}{2} \;\geq\; \frac{2}{\frac{1}{x}+\frac{1}{y}}$$

if x, $y > 0$, with equality if and only if $x = y$. Thus,

$$\begin{aligned} a+b+c \;=\; \frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2} &\;\geq\; \frac{2}{\frac{1}{a}+\frac{1}{b}}+\frac{2}{\frac{1}{b}+\frac{1}{c}}+\frac{2}{\frac{1}{c}+\frac{1}{a}} \\ &\;=\; \frac{2ab}{a+b}+\frac{2bc}{b+c}+\frac{2ca}{c+a} \\ \text{or}\qquad s \;=\; \frac{a+b+c}{2} &\;\geq\; \frac{ab}{a+b}+\frac{bc}{b+c}+\frac{ca}{c+a} \end{aligned}$$

with equality if and only if $a = b = c$, when the triangle is equilateral.

II. *Solution by D.J. Smeenk, Zaltbommel, the Netherlands.*

Let us denote $b+c = 2p$, $c+a = 2q$, and $a+b = 2r$. Then

$$\begin{aligned} a+b+c &= p+q+r \\ a &= -p+q+r \\ b &= p-q+r \\ c &= p+q-r\,. \end{aligned}$$

The given hypothesis now successively implies:

$$\begin{aligned} \frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b} &= \frac{1}{2}(a+b+c) \\ \frac{p^2-(q-r)^2}{2p}+\frac{q^2-(r-p)^2}{2q}+\frac{r^2-(p-q)^2}{2r} &= \frac{1}{2}(p+q+r) \\ \frac{(q-r)^2}{p}+\frac{(r-p)^2}{q}+\frac{(p-q)^2}{r} &= 0\,. \end{aligned}$$

This forces $p = q = r$, which means that $a = b = c$.

III. *Solution by Panos E. Tsaoussoglou, Athens, Greece.*

$$\begin{aligned}\frac{a+b+c}{2} = \frac{2(a+b+c)}{2\times 2} &= \frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\\ \frac{a+b}{4}-\frac{ab}{a+b}+\frac{b+c}{4}-\frac{bc}{b+c}+\frac{c+a}{4}-\frac{ca}{c+a} &= 0\\ \frac{(a+b)^2-4ab}{4(a+b)}+\frac{(b+c)^2-4bc}{4(b+c)}+\frac{(c+a)^2-4ca}{4(c+a)} &= 0\\ \frac{(a-b)^2}{4(a+b)}+\frac{(b-c)^2}{4(b+c)}+\frac{(c-a)^2}{4(c+a)} &= 0\,.\end{aligned}$$

Therefore, $a=b=c$.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 proofs); MICHEL BATAILLE, Rouen, France; MIHÁLY BENCZE, Brasov, Romania; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; JAMES T. BRUENING, Southeast Missouri State University, Cape Girardeau, MO, USA(2 proofs); CON AMORE PROBLEM GROUP, Royal Danish School of Educational Studies, Copenhagen, Denmark; JOSÉ LUIS DÍAZ-BARRERO and JUAN JOSÉ EGOZCUE, Universitat Politècnica de Catalunya, Barcelona, Spain; CHARLES R. DIMINNIE, Angelo State University, San Angelo, TX, USA; RUSSELL EULER and JAWAD SADEK, NW Missouri State University, Maryville, MO, USA; VINAYAK GANESHAN, University of Waterloo, Waterloo, Ontario; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; NATALIO H. GUERSENZVAIG, Universidad CAECE, Buenos Aires, Argentina.; ZELJKO HANJS, University of Zagreb, Zagreb, Croatia; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFREYS, student, Berkhamsted Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA (3 proofs); KEE-WAI LAU, Hong Kong; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA (2 proofs); DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ROBERT McGREGOR, Auburn, Alabama, USA; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria; STANLEY RABINOWITZ, Westford, MA, USA; JUAN-BOSCO ROMERO MÁRQUEZ, Universidad de Valladolid, Valladolid, Spain; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; TOSHIO SEIMIYA, Kawasaki, Japan; ANDREI SIMION, student, Cornell University, Ithaca, NY, USA; TREY SMITH, Angelo State University, San Angelo, TX, USA; ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There was one incorrect solution.

Bencze established a nice generalization: Let a_k, $k=1, 2, \ldots, n$ be the sides of the convex polygon $A_1A_2\cdots A_n$ and let s be the semiperimeter. Given that

$$\frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}}+\frac{1}{\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n}}+\cdots+\frac{1}{\frac{1}{a_n}+\frac{1}{a_1}+\cdots+\frac{1}{a_{n-2}}} = \frac{2s}{n-1}\,,$$

show that $A_1A_2\cdots A_n$ is equilateral. We will let the interested reader prove it.

2658. [2001 : 337] *Proposed by Juan-Bosco Romero Márquez, Universidad de Valladolid, Valladolid, Spain.*

Let $\triangle ABC$ have $\angle BCA = 90^\circ$. Squares $ACDE$ and $CBGF$ are drawn externally to the triangle. Suppose that AG and BE intersect at M. Show that M lies on the altitude CN.

I. *Solution by John G. Heuver, Grande Prairie Composite High School, Grande Prairie, Alberta.*

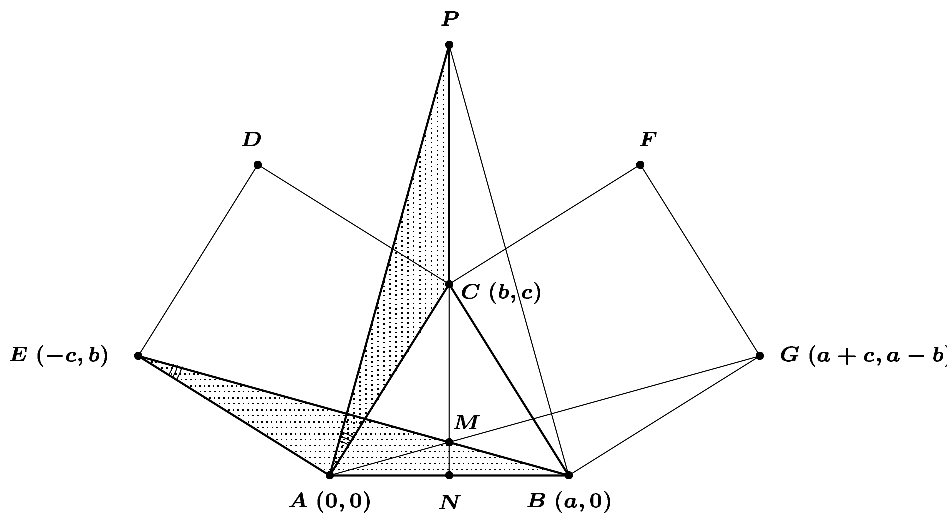
We do not need the condition that $\angle BCA = 90^\circ$ — M lies on the altitude CN for any angle at C. Let $A = (0,0)$, $B = (a,0)$, $C = (b,c)$ be the coordinates of the triangle's vertices. Then $G = (a+c, a-b)$ and $E = (-c, b)$. The equations of AG and BE are

$$y \;=\; \frac{x(a-b)}{a+c} \quad \text{and} \quad y \;=\; \frac{b(x-a)}{-c-a}\,.$$

The coordinates of M are

$$\left(b, \frac{b(a-b)}{a+c}\right)\,.$$

Thus, M lies on the altitude CN.



II. *Solution by Toshio Seimiya, Kawasaki, Japan.*

Even more generally, the squares on the sides CA and CB can be replaced by similar rectangles — *we assume that* $\angle BCA$ *is arbitrary and that* $ACDE$ *and* $CBGF$ *are rectangles such that* $AC : AE = BC : BG$.

Let P be a point on the altitude CN produced beyond C such that

$$CP : AB \;=\; AC : AE \;=\; BC : BG\,. \qquad (1)$$

Since $\angle PCA$ is an exterior angle of $\triangle CNA$ we have

$$\begin{aligned} \angle PCA &= \angle CAN + \angle CNA \;=\; \angle CAN + 90^\circ \\ &= \angle CAN + \angle CAE \;=\; \angle NAE \;=\; \angle BAE\,. \qquad (2) \end{aligned}$$

From (1) and (2), we have $\triangle PCA \sim \triangle BAE$, so that $\angle PAC = \angle BEA$. Thus,

$$\angle BEA + \angle PAE \;=\; \angle PAC + \angle PAE \;=\; \angle CAE \;=\; 90^\circ\,.$$

We therefore deduce that $PA \perp BE$. Similary, $PB \perp AG$. Therefore the three altitudes BE, AG, and PN are concurrent at the orthocentre of $\triangle PAB$. This implies that M lies on CN.

Also solved by MIGUEL AMENGUAL COVAS, Cala Figuera, Mallorca, Spain; ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (2 solutions); ∗MICHEL BATAILLE, Rouen, France; FRANCISCO BELLOT ROSADO, I.B. Emilio Ferrari, Valladolid, Spain; CHRISTOPHER J. BRADLEY, Clifton College, Bristol, UK; RICHARD B. EDEN, Philippines; VINAYAK GANESHAN, Waterloo, ON; IAN JUNE L. GARCES, Ateneo de Manila University, The Philippines; RICHARD I. HESS, Rancho Palos Verdes, CA, USA; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; PAUL JEFFERYS, student, Berkhamsted Collegiate School, UK; D. KIPP JOHNSON, Beaverton, OR, USA; GERRY LEVERSHA, St. Paul's School, London, England; ∗HENRY LIU, student, University of Memphis, Memphis, TN, USA; DAVID LOEFFLER, student, Trinity College, Cambridge, UK; ∗WOLFGANG LUDWICKI, Stendal Germany, and ECKARD SPECHT, Otto-von-Guericke University, Magdeburg, Germany; ROBERT McGREGOR, Auburn AL; THEOKLITOS PARAGIOU, Limassol, Cyprus, Greece; GOTTFRIED PERZ, Pestalozzigymnasium, Graz, Austria (multiple solutions); ∗STANLEY RABINOWITZ, Westford, MA, USA; HEINZ-JÜRGEN SEIFFERT, Berlin, Germany; D.J. SMEENK, Zaltbommel, the Netherlands; PETER Y. WOO, Biola University, La Mirada, CA, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. An asterisk indicates that the solver used an arbitrary angle at C.

Noting the many possible elementary approaches to the problem, two readers said that they intended to use the 90° version as an exercise in their classes. Evidently the problem has been popular for a long time — it has appeared in journals over the past 200 years, both with and without the condition requiring the 90° angle at C. Bataille came across it in [3]*, where it is proposed as a problem set by Dr. Porson (1759–1808). Rabinowitz found it in* [1]*. Further 19*th *century references are provided in* [2]*; the proof that appears there dates from 1879 and resembles our solution II, but without Seimiya's generalization to rectangles.*

The configuration of squares on the sides of a triangle has other nice properties that have been featured before in **CRUX with MAYHEM**. *See "One problem — Six Solutions" by Georg Gunther* [1998 : 81–87]*, and also* #1493 [1991 : 52–53] *and* #1496 [1991 : 56–57]. *Among other things, the altitude of* $\triangle ABC$ *turns out to be the median of* $\triangle CFD$.

References

[1] M.N. Aref and William Wernick, *Problems and Solutions in Euclidean Geometry*. Dover Publications 1968. Page 21, problem 1.27.

[2] F.G.-M., *Exercices de Géométrie*. Page 225, section 446.

[3] *Math. Gazette* **76** n^o 477 (Nov.1992) p. 455, and **77** n^o 479 (July 1993) pp. 295–296.

2660. [2001 : 337] . *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Terrassa, Spain.*

Let z_1, z_2, ..., z_n be distinct non-zero complex numbers. Prove that

$$\sum_{j=1}^{n} z_j^{n-1} \left(1 + \prod_{\substack{k=1 \\ k \neq j}}^{n} z_k \right) \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{1}{z_k - z_j}$$

is a real number, and determine its value.

Composite of solutions by David Loeffler, student, Trinity College, Cambridge, UK and Heinz-Jürgen Seiffert, Berlin, Germany (modified slightly by the editor).

Let L_n denote the given sum. We show that $L_n = (-1)^{n-1}$ for all $n \geq 2$. [Ed: for $n = 1$ both products in L_n are "empty" and thus the value of L_n is subject to interpretation.] The claim follows immediately from Lambrou's general result given in his solution to problem # 2487 [Ed: Also by the same proposer of the present problem.] [1999 : 431; 2000 : 512]. Using his notation, let

$$S_n(m) = \sum_{j=1}^{n} \frac{z_j^m}{\prod\limits_{\substack{k=1 \\ k\neq j}}^{n} (z_j - z_k)} .$$

Lambrou proved that

$$S_n(m) = \begin{cases} 0 & \text{if } 0 \leq m \leq n-2, \\ 1 & \text{if } m = n-1, \\ \sum\limits_{j=1}^{n} z_j & \text{if } m = n. \end{cases}$$

Since

$$\begin{aligned} L_n &= \sum_{j=1}^{n} \frac{z_j^{n-1}}{\prod\limits_{\substack{k=1 \\ k\neq j}}^{n} (z_k - z_j)} + \sum_{j=1}^{n} \frac{z_j^{n-2}\left(\prod\limits_{k=1}^{n} z_k\right)}{\prod\limits_{\substack{k=1 \\ k\neq j}}^{n} (z_k - z_j)} \\ &= (-1)^{n-1} S_n(n-1) + \left(\prod_{k=1}^{n} z_k\right) S_n(n-2) , \end{aligned}$$

it follows immediately that $L_n = (-1)^{n-1}$ as claimed.

Also solved by ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina; MICHEL BATAILLE, Rouen, France; PAUL DEIERMANN, Southeast Missouri State University, Cape Girardeau, MO, USA; JOHN G. HEUVER, Grande Prairie Composite High School, Grande Prairie, Alberta; GERRY LEVERSHA, St. Paul's School, London, England; HENRY LIU, student, University of Memphis, Memphis, TN, USA; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer. There were two incorrect solutions.

Both Deiermann and the proposer used complex integration and the theory of residue to obtain the result.