Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following two conditions.

- Each rectangle has the same number of white squares and black squares.
- No two rectangles have the same number of squares.

Find the maximum value of p for which such a decomposition is possible. For this maximum value of p, determine all corresponding decompositions of the chessboard into p rectangles.

Lastly, here are the official solutions to the 2001 Maritime Mathematics Contest from the December 2001 issue [2001 : 521].

2001 Maritime Mathematics Contest

1. Alice and Bob were comparing their stacks of pennies. Alice said "If you gave me a certain number of pennies from your stack, then I'd have six times as many as you, but if I gave you that number, you'd have one-third as many as me." What is the smallest number of pennies that Alice could have had?

Solution: Let a and b be, respectively, the number of pennies that Alice and Bob had, and let x be the certain number of pennies. From the given information, we obtain the following two equations.

$$\begin{aligned} a + x &= 6(b - x)\,, \\ a - x &= 3(b + x)\,. \end{aligned}$$

From the first equation, $a = 6b - 7x$, and, from the second equation, $a = 3b + 4x$. Therefore, $6b - 7x = 3b + 4x$, so that $b = \frac{11}{3}x$.

Since a, b, and x are required to be positive integers, the smallest possible value for x is 3. Then $b = 11$ and $a = 6(11) - 7(3) = 45$. Therefore, 45 is the smallest number of pennies that Alice could have had.

2. The infinite sequence

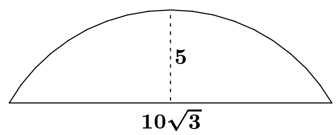
$$1\,2\,3\,4\,5\,6\,7\,8\,9\,1\,0\,1\,1\,1\,2\,1\,3\,1\,4\,1\,5\,1\,6\,1\,7\,1\,8\,1\,9\,2\,0\,2\,1\,2\,2\,2\,3\ldots$$

is obtained by writing the positive integers in order. What is the 2001^{st} digit in this sequence?

Solution: The digits 1, 2, $\ldots$, 9 occupy 9 positions, and the digits in the numbers 10, 11, $\ldots$, 99 occupy $2 \times 90 = 180$ positions. Further, the digits in the numbers 100, 101, $\ldots$, 199 occupy $100 \times 3 = 300$ positions. Similarly, 300 positions are required for the numbers 200 to 299, *et cetera*.

Therefore, the digits in the numbers up to and including 699 occupy the first $9 + 180 + (6 \times 300) = 1989$ positions. A further 12 positions are required to write 700, 701, 702, 703, so that the 2001^{st} digit is the "3" in 703.

3. The maximum height of a railway tunnel is 5 metres and the width of the tunnel is $10\sqrt{3}$ metres. The outline of the tunnel is in the form of a segment of a circle as shown below. Determine the area of a cross-section of the tunnel.



Solution: Consider a circle with centre O and let AB be a chord (which is not a diameter) of the circle. Let P be the point on the circumference of the circle such that OP is the perpendicular bisector of AB. Finally, let X be the point of intersection of AB and OP. Suppose that $|XP| = 5$ and $|AX| = |BX| = 5\sqrt{3}$. The chord AB divides the circle into two sections; the problem is to determine the area of the smaller section. Let r be the radius of the circle so that $|OB| = r$ and $|OX| = r - 5$. Applying the Pythagorean Theorem to $\triangle OXB$, we obtain $(r-5)^2 + \left(5\sqrt{3}\right)^2 = r^2$, from which we get $r = 10$.

Therefore, the lengths of the sides of $\triangle OXB$ are 5, $5\sqrt{3}$, and 10; that is, in the ratio $1 : \sqrt{3} : 2$. Therefore, $\angle XOB = 60^\circ$, so that the area of the sector OAB of the circle is

$$\frac{1}{3} \times \text{area of the entire circle} = \frac{1}{3}\pi(10)^2 = \frac{100\pi}{3}.$$

The area of $\triangle AOB$ is

$$\frac{1}{2} \times |AB| \times |OX| = \frac{1}{2}\left(10\sqrt{3}\right)(5) = 25\sqrt{3},$$

so that the required area is

$$\frac{100\pi}{3} - 25\sqrt{3} \text{ square units.}$$

4. Which of the following numbers is greater?

$$A = \frac{2.0000004}{(1.0000004)^2 + 2.0000004} \quad \text{or} \quad B = \frac{2.0000002}{(1.0000002)^2 + 2.0000002}$$

Solution: Consider the numbers

$$a = \frac{2+2x}{(1+2x)^2 + (2+2x)} = \frac{2+2x}{3+6x+4x^2} \quad \text{and}$$

$$b = \frac{2+x}{(1+x)^2 + (2+x)} = \frac{2+x}{3+3x+x^2},$$

where $x > 0$. Now $a < b$ is equivalent to

$$(2+2x)(3+3x+x^2) \;<\; (2+x)(3+6x+4x^2)\,.$$

Expanding both sides, we obtain

$$6+12x+8x^2+2x^3 \;<\; 6+15x+14x^2+4x^3\,;$$

that is, $2x^3+6x^2+3x > 0$. This inequality is true for any $x > 0$ so, for all such x, $a < b$. Setting $x = 0.0000002$, we have $a = A$ and $b = B$, so that $A < B$; that is, B is greater.

5. Alice and Bob play the following game with a pile of 2001 beans. A move consists of removing one, two, or three beans from the pile. The players move alternately, beginning with Alice. The person who takes the last bean in the pile is the winner. Which player has a winning strategy for this game and what is that strategy?

Solution: Alice has the following winning strategy. On her first move, she takes one bean. On subsequent moves, Alice removes $4-x$ beans, where x is the number of beans removed by Bob on the preceding turn.

We now prove that the above strategy guarantees a win for Alice. After Alice's first move the pile contains 2000 beans. Moreover, after every pair of moves, a move by Bob followed by a move by Alice, the pile decreases by exactly 4 beans. Therefore, after every move by Alice the number of beans in the pile is a multiple of 4. Eventually, after a move by Alice, there will be 4 beans left in the pile. After Bob removes one, two, or three beans, Alice takes the remainder and wins the game.

6. Show that, regardless of what integers are substituted for x and y, the expression

$$x^5 - x^4y - 13x^3y^2 + 13x^2y^3 + 36xy^4 - 36y^5$$

is never equal to 77.

Solution: The given expression may be factored as follows.

$$\begin{aligned} N &= x^5 - x^4y - 13x^3y^2 + 13x^2y^3 + 36xy^4 - 36y^5 \\ &= x^4(x-y) - 13x^2y^2(x-y) + 36y^4(x-y) \\ &= (x-y)(x^4 - 13x^2y^2 + 36y^4) \\ &= (x-y)(x^2-4y^2)(x^2-9y^2) \\ &= (x-y)(x+2y)(x-2y)(x+3y)(x-3y)\,. \end{aligned}$$

If $y = 0$ then $N = x^5$ which is not equal to 77 for any integer x. On the other hand, if $y \neq 0$ then the five factors of N are all distinct. However, any expression of 77 as a product of distinct integers contains at most four factors, specifically as $(1)(-1)(7)(-11)$ or $(1)(-1)(-7)(11)$. Therefore, for any choice of x and y, N is never equal to 77.