Some generalizations of an inequality from IMO 2001

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The purpose of this paper is to consider some natural generalizations of Problem 2 from IMO 2001 which states:

Prove that

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}} \;\geq\; 1\,,$$

where a, b and c are arbitrary positive numbers.

Many different proofs of this inequality were given during the Olympiad and it was also shown by the first author that

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ac}}+\frac{c}{\sqrt{c^2+\lambda ab}} \;\geq\; \frac{3}{\sqrt{1+\lambda}}$$

for arbitrary a, b, $c > 0$ and $\lambda \geq 8$. It is easy to see that the latter inequality is not true for $0 < \lambda < 8$. Moreover, it can be shown that in this case

$$\frac{a}{\sqrt{a^2+\lambda bc}}+\frac{b}{\sqrt{b^2+\lambda ac}}+\frac{c}{\sqrt{c^2+\lambda ab}} \;>\; 1\,,$$

and the lower bound is sharp.

We now prove a general inequality that encompasses all of these results.

Proposition 1. For any positive integers n and m, and any positive numbers x_1, x_2, $\dots$, x_n with $x_1x_2\dots x_n = \lambda^n$ ($\lambda > 0$), we have the following sharp inequality:

$$\sum_{i=1}^{n} \frac{1}{(1+x_i)^{\frac{1}{m}}} \;\geq\; \min\left(1, \frac{n}{(1+\lambda)^{\frac{1}{m}}}\right) . \qquad (1)$$

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Proof. Set

$$d \;=\; \min\left(1, \frac{n}{(1+\lambda)^{\frac{1}{m}}}\right) .$$

Multiplying both sides of (1) by $\prod_{i=1}^{n}(1+x_i)^{\frac{1}{m}}$ and then taking the m^{th} power we see that (1) is equivalent to the inequality

$$\sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k)+T \;\geq\; d^m\prod_{i=1}^{n}(1+x_i)\,, \tag{2}$$

where

$$T \;=\; \left[\sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k)^{\frac{1}{m}}\right]^m - \sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k)\,.$$

Denote by $\sigma_1, \sigma_2, \ldots, \sigma_n$, the elementary symmetric functions of the x_i and set $\sigma_0 = 1$. Then it is easy to check that

$$\prod_{i=1}^{n}(1+x_i) \;=\; \sum_{i=0}^{n}\sigma_i \quad \text{and} \quad \sum_{i=1}^{n}\prod_{k=1,k\neq i}^{n}(1+x_k) \;=\; \sum_{i=0}^{n-1}(n-i)\sigma_i\,.$$

Hence, (2) can be rewritten as

$$\sum_{i=0}^{n-1}(n-i-d^m)\sigma_i + T \;\geq\; d^m\sigma_n$$

By the AM-GM inequality we have

$$\sigma_i \geq \binom{n}{i}(\sigma_n)^{\frac{i}{n}} \;=\; \binom{n}{i}\lambda^i\,, \quad 0 \leq i \leq n\,, \tag{3}$$

and, therefore,

$$\prod_{i=1}^{n}(1+x_i) \;=\; \sum_{i=0}^{n}\sigma_i \geq \sum_{i=0}^{n}\binom{n}{i}\lambda^i = (1+\lambda)^n\,. \tag{4}$$

To estimate the term T we use the following inequality

$$\left(\sum_{i=1}^{n}a_i\right)^m \;\geq\; \sum_{i=1}^{n}a_i^m + (n^m-n)\left(\prod_{i=1}^{n}a_i\right)^{\frac{m}{n}} \quad \text{for} \quad a_i > 0\,, \tag{5}$$

which follows easily by induction on m. Setting

$$a_i \;=\; \prod_{k=1,k\neq i}^{n}(1+x_k)^{\frac{1}{m}}$$

in (5) gives

$$T \ \geq \ (n^m - n) \prod_{i=1}^{n} (1 + x_i)^{\frac{n-1}{n}} \, ,$$

and, therefore, (4) implies that

$$T \ \geq \ (n^m - n)(1 + \lambda)^{n-1} \, . \tag{6}$$

In view of (3), (6) and the fact that $d \leq 1$, to prove (2), it is sufficient to show that

$$d^m \lambda^n - (n^m - n)(1 + \lambda)^{n-1} - \sum_{i=0}^{n-1} (n - i - d^m) \binom{n}{i} \lambda^i \ \leq \ 0 \, . \tag{7}$$

But the left hand side of (7) is equal to $(1 + \lambda)^{n-1}(d^m(1 + \lambda) - n^m)$ (this can be seen, for example, by comparing the coefficients of the powers of λ in both expressions) and the inequality (7) follows since

$$d \ \leq \ \frac{n}{(1 + \lambda)^{\frac{1}{m}}} \, .$$

Note that, if $\lambda \geq n^m - 1$, then $d = n(1 + \lambda)^{-\frac{1}{m}}$ and (1) tells us that

$$\sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\frac{1}{m}}} \ \geq \ \frac{n}{(1 + \lambda)^{\frac{1}{m}}}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n = \lambda$. On the other hand, if $\lambda < n^m - 1$, then $d = 1$ and (1) takes the form

$$\sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\frac{1}{m}}} \ > \ 1 \, .$$

To see that the latter inequality is sharp, set $x_1 = x_2 = \cdots = x_{n-1} = \frac{1}{t}$ and $x_n = t^{n-1}\lambda^n$, where $t \to 0$.

Now, we shall show that the inequality (1) still holds if we replace the power $\frac{1}{m}$ by any real number $\alpha \in (0, 1]$. In this case, however, it is not possible to proceed as in the proof of Proposition 1, since inequality (5) is not true for any real number $m > 1$ and any positive integer n (take, for example, $m = \frac{3}{2}$, $n = 2$, $x_1 = 1$, $x_2 = \frac{1}{16}$). Instead, we shall use the powerful Lagrange multiplier criterion.

Proposition 2 . For any $\alpha \in (0, 1]$ and any positive numbers $x_1, x_2, \ldots, x_n$ with $x_1 x_2 \ldots x_n = \lambda^n$ $(\lambda > 0)$, we have the following sharp inequality:

$$\sum_{i=1}^{n} \frac{1}{(1 + x_i)^{\alpha}} \ \geq \ \min\left(1, \frac{n}{(1 + \lambda)^{\alpha}}\right) \, . \tag{8}$$

Proof. Denote by d the infimum of the function

$$f(x_1, x_2, \dots, x_n) \;=\; \sum_{i=1}^{n} \frac{1}{(1+x_i)^\alpha}$$

on the set

$$A \;=\; \{\ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 x_2 \dots x_n = \lambda^n\ , x_1, x_2, \dots, x_n > 0\ \}\,.$$

Suppose first that this infimum is not attained at a point of A. Then, $d \;=\; \lim\limits_{k\to\infty} f(x_1^{(k)}, \dots, x_n^{(k)})$, where, for example, $\lim\limits_{k\to\infty} x_n^{(k)} \;=\; 0$ or $+\infty$. Then, for example, $\lim\limits_{k\to\infty} x_1^{(k)} \;=\; +\infty$ or 0 and, in both cases, we see that $d \geq 1$. Note that if $\lim\limits_{k\to\infty} x_s^{(k)} \;=\; +\infty$ for $s = 1, 2, \dots, n-1$ and $\lim\limits_{k\to\infty} x_n^{(k)} \;=\; 0$, then $\lim\limits_{k\to\infty} f(x_1^{(k)}, \dots, x_n^{(k)}) \;=\; 1$, which shows that $d = 1$. Now, let d be attained at a point of A. Consider the function

$$F(x_1, x_2, \dots, x_n, \mu) \;=\; f(x_1, x_2, \dots, x_n) + \mu(x_1 x_2 \dots x_n - \lambda^n)\,.$$

Then the Lagrange multiplier criterion says that d is attained at a point $(x_1, x_2, \dots, x_n) \in A$ such that

$$\frac{\partial F}{\partial x_i} \;=\; -\frac{\alpha}{(1+x_i)^{\alpha+1}} + \frac{\mu x_1 \dots x_n}{x_i} \;=\; 0\,;$$

that is, when

$$\frac{x_i}{(1+x_i)^{\alpha+1}} \;=\; \frac{x_j}{(1+x_j)^{\alpha+1}}\,,\ 1 \leq i, j \leq n\,. \tag{9}$$

Consider the function

$$g(x) \;=\; \frac{x}{(1+x)^{\alpha+1}}\,.$$

Then,

$$g'(x) \;=\; \frac{1-\alpha x}{(1+x)^{\alpha+2}}\,,$$

and, therefore, $g(x)$ takes each of its values at most twice. Hence, (9) shows that $x_1 = \dots = x_k = x$ and $x_{k+1} = \dots = x_n = y$ for some $1 \leq k \leq n$. If $k = n$, then $x_1 = x_2 = \dots = x_n = \lambda$ and

$$f(x_1, x_2, \dots, x_n) \;=\; \frac{n}{(1+\lambda)^\alpha}\,.$$

If $k < n$, then

$$f(x_1, x_2, \dots, x_n) \;=\; \frac{k}{(1+x)^\alpha} + \frac{n-k}{(1+y)^\alpha} \;\geq\; \frac{1}{(1+x)^\alpha} + \frac{1}{(1+y)^\alpha}\,.$$

To prove Proposition 2 it is sufficient to show that

$$\frac{1}{(1+x)^{\alpha}} + \frac{1}{(1+y)^{\alpha}} \; > \; 1 \tag{10}$$

provided that

$$\frac{x}{(1+x)^{\alpha+1}} \; = \; \frac{y}{(1+y)^{\alpha+1}}\,,\; x \neq y\,. \tag{11}$$

Set $\beta = 1/\alpha \geq 1$, $z = (1+x)^{\alpha}$ and $t = (1+y)^{\alpha}$. Then (10) and (11) can be written, respectively, as $z + t > zt$ and

$$(zt)^{\beta} \; = \; \frac{z^{\beta+1} - t^{\beta+1}}{z - t}\,.$$

Thus, we have to prove that

$$(z+t)^{\beta} \; > \; \frac{z^{\beta+1} - t^{\beta+1}}{z - t}\,. \tag{12}$$

Assume that $z < t$ and set $u = z/t < 1$. Applying Bernoulli's inequality twice, we obtain

$$(1+u)^{\beta} \; \geq \; 1 + \beta u \; > \; \frac{1 - u^{\beta+1}}{1 - u}\,,$$

which is just the inequality (12).

Remark. Using similar arguments to the ones used in the proof of Proposition 2, one can show that the inequality (8) holds also in the case $\alpha > 1$ and $n \geq \alpha + 1$. Note that if $\alpha > 1$ but $n < \alpha + 1$, then this inequality is not true in general (take, for example, $\alpha = n = 2$, $x_1 = 8$, $x_2 = \frac{1}{50}$).

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